Rotationally symmetric p-harmonic flows from D^2 to S^2: local well-posedness and finite time blow-up

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Abstract

We study the p-harmonic flow from the unit disk D^2 to the unit sphere S^2 under rotational symmetry. We show that the Dirichlet problem with constant boundary conditions is locally well-posed in the class of classical solutions and we also give a sufficient criterion, in terms of the boundary condition, for the derivative of the solutions to blow-up in finite time.

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1 Introduction and main results

In this paper we will focus on the following system of PDEs

$$\mathbf{u}_t = \mathrm{div}(|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}) + \mathbf{u}|\nabla \mathbf{u}|^p, \tag{1.1}$$

where $p > 1$, and $\mathbf{u} : \Omega \mapsto S^N$, where Ω is an open domain in $\mathbb{R}^N$ and S^N is the unit sphere of $\mathbb{R}^{N+1}$. This system of PDEs is called the *p-harmonic flow* and it corresponds to the gradient flow in $L^2(\Omega)$ of the following energy functional:

$$E_p(\mathbf{u}) := \frac{1}{p}\int_\Omega |\nabla \mathbf{u}|^p \, dx. \tag{1.2}$$

Apart from the interest on the results of the competition between the two effects appearing in the system (diffusion and reaction), (1.1) arises from a number of applications to other sciences and engineering: ferromagnetism [7], theory of liquid crystals [13], multigrain problems [16] and image processing [19].

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We study the Dirichlet problem for (1.1) with boundary condition $\mathbf{u}(t,x) = \mathbf{u}_0(x)$ for $(t,x) \in \partial Q$, where $Q := (0,\infty) \times \Omega$. Related to this problem, there are several previous results for the semilinear diffusion case $p = 2$, as in [2, 3], and also for $p > 1$ [5, 8, 14, 18]. A special case in the analysis, with some particular mathematical features due to lack of sufficient regularity, is the limit case $p = 1$, intensively studied recently; indeed, local well-posedness, steady states and either the appearance of finite time blow-up or the existence of global solutions were studied under the hypothesis of rotational symmetry in [6, 11, 12]; in particular, the most interesting feature that has been found is the occurence of blow-up in finite time of the derivative. The Neumann problem for $p = 1$ has been investigated recently in [9, 10].

In a previous work [15], the authors studied the rotationally symmetric steady states corresponding to the system for $1 < p < \infty$, $M = S^2$, the unit sphere in $\mathbb{R}^3$, $\Omega = D^2$, the unit disk in $\mathbb{R}^2$. A rotationally symmetric solution to (1.1) has the general form

$$\mathbf{u}(t,x) = \left(\frac{x_1}{r}\sin\, h(t,r), \frac{x_2}{r}\sin\, h(t,r), \cos\, h(t,r)\right), \quad r = |x|,\ x = (x_1,x_2) \in D^2, \tag{1.3}$$

and the Dirichlet boundary condition $h(t,1) = h(0,1) = l$. The energy functional E_p in (1.2) becomes

$$E_p(\mathbf{u}) = \int_0^1 L_p(r,h,h_r)\, dr, \tag{1.4}$$

with a non-coercive Lagrangian given by

$$L_p(r,s,\xi) = \left(r^{2/p}h_r^2 + r^{(2-2p)/p}\sin^2\, h\right)^{p/2}.$$

By replacing $\mathbf{u}$ in (1.1) by its special form (1.3), the system (1.1) becomes

$$\begin{aligned} h_t = \left(h_r^2 + \frac{sin^2\, h}{r^2}\right)^{(p-4)/2} \Bigg[(p-1)h_r^2 h_{rr} + (p-3)\left(\frac{h_r^2 \sin\, h\cos\, h}{r^2} - \frac{h_r \sin^2\, h}{r^3}\right) \\ + \frac{h_r^3}{r} + \frac{h_{rr}\sin^2\, h}{r^2} - \frac{\sin^3\, h\cos\, h}{r^4}\Bigg], \end{aligned} \tag{1.5}$$

see [15] for details. In our previous work, we studied and classified the steady states for (1.5), for all $1 < p < \infty$.

In the present paper we continue the study of the rotationally symmetric flows begun in [15], and we devote it to some qualitative properties of the solution $\mathbf{u}$ to (1.1), or equivalently, h solution to (1.5), such as existence and well-posedness for short times, regularity of solutions and conditions for the phenomenon of blow-up in finite time to occur. We restrict ourselves to the fast diffusion range $1 < p < 2$, since, for the range $p \geq 2$, it has been shown in [8] that strong solutions exist globally for any initial datum, they are Hölder continuous and they converge to a steady-state as $t \to +\infty$. A similar result for the range $1 < p < 2$ was also obtained in [8] but with a condition of small energy on the initial datum. We point out that our approach and results are totally different to those in [8]. First of all, we obtain local existence of classical solutions for any initial datum and secondly, we give a sufficient condition on the boundary constraint for classical solutions ceasing to exist in finite time.

For convenience, we denote

$$J(h) := h_r^2 + \frac{\sin^2 h}{r^2},$$

and

$$A(h) := (p-1)h_r^2 h_{rr} + (p-3)\left(\frac{h_r^2 \sin h \cos h}{r^2} - \frac{h_r \sin^2 h}{r^3}\right) + \frac{h_r^3}{r} + \frac{h_{rr}\sin^2 h}{r^2} - \frac{\sin^3 h \cos h}{r^4},$$

thus (1.5) together with the initial and boundary conditions can be written in the form of the following Dirichlet problem:

$$\begin{cases} h_t = F(h) := J(h)^{(p-4)/2} A(h), & (t,r) \in (0,T)\times(0,1), \\ h(0,r) = h_0(r), & r \in (0,1), \\ h(t,1) = h_0(1), & t \in [0,T). \end{cases} \tag{1.6}$$

We introduce the concept of solution that we will use throughout the paper.

Definition 1.1. *A classical solution to the Dirichlet problem* (1.6) *in* $(0,T)$ *with initial datum* h_0 *is a function* $h \in C^{1,2}([0,T]\times[0,1])$ *which satisfies* (1.6) *pointwisely in* $[0,T]\times[0,1]$.

Arguing similarly as in the proof of [12, Lemma 2.1], one can easily get that $\mathbf{u}$ is a rotationally symmetric solution to the Dirichlet problem associated to the system (1.1) if and only if $\mathbf{u}$ has the form (1.3) with h being a classical solution to the Dirichlet problem (1.6) such that $h(t,0) = 0$ for all $t \in [0,T)$. Our study is divided into two main parts: a first one devoted to the local well-posedness of the Dirichlet problem (1.6), and a second, more specialized one dealing with the phenomenon of blow-up in finite time. We point out that the results in this paper and the strategy to prove them are analogous to those of the case $p = 1$ provided in [11]. However, due to the special nonlinear term $(p-1)h_r^2 h_{rr}$ in (1.5), all the proofs are in the present case much more involved than those in [11].

Local well-posedness. We notice that, if $h(t,r) = k\pi$ for some $r \in (0,1]$ and $k \in \mathbb{Z}$, then the Dirichlet problem (1.6) becomes degenerate around (t,r). In order to avoid this extra difficulty, we assume that the initial condition h_0 satisfies the following conditions

$$h_0 \in C^2([0,1]), \quad h_0(0) = 0, \quad h_0(r) \in (0,\pi), \text{ for all } r \in (0,1], \quad h_0'(0) > 0. \tag{1.7}$$

In this way, we avoid the degeneracy at $t = 0$. However, these properties are preserved along the evolution. More precisely, we have the following uniqueness result.

Theorem 1.2. *Let* $p \in (1,2)$ *and assume that* h_0 *satisfies* (1.7). *Then there exists at most one classical solution* h *defined in a maximal interval* $(0,T)$ *to the Dirichlet problem* (1.6) *with initial condition* h_0. *Moreover, the solution* h *satisfies the following*
(a) There exists $\delta = \delta(h_0) \in (0,1)$, *such that*

$$h(t,0) = 0, \quad \delta r \le h(t,r) \le \pi - \delta r, \quad h_r(t,0) \ge \delta, \tag{1.8}$$

for all $r \in (0,1]$, $t \in (0,T)$.
(b) Furthermore,

$$h_{rr}(t,0) = 0, \quad F(h(t,\cdot))|_{r=1} = 0, \tag{1.9}$$

for all $t \ge 0$.

The proof of Theorem 1.2 is given in Subsection 3.1 and follows as a consequence of a more general comparison principle that will be proved in Section 2 and of the availability of explicit sub- and supersolutions.

In order to prove existence of solutions and continuous dependence on the initial data, we need to impose some further conditions on the initial datum.

Theorem 1.3. *Let h_0 be a function that satisfies* (1.7) *and,*

$$h_{0rr}(0) = 0, \quad F(h_0)|_{r=1} = 0, \quad r \mapsto \frac{h_0(r)}{r} \in C^2([0,1]). \tag{1.10}$$

Then there exists a classical solution h to (1.6) *with initial condition h_0, defined in a maximal interval $[0,T)$. Moreover, the problem is well-posed (that is, the solution depends continuously on h_0) and we have further regularity results:*

(a)

$$(t,r) \mapsto \frac{h(t,r)}{r} \in C^{1,2}([0,T) \times [0,1]),$$

(b)

$$h_r \in C^{1,2}((0,T] \times (0,1]).$$

The proof is quite complex and relies on the application of a general theory of sectorial operators, that is developed in an abstract framework in the book [17], after noticing that the linearization of the (fully nonlinear) elliptic operator associated to our equation is indeed sectorial.

Finite time blow-up. This is the more specialized part of the paper, where we prove that, if $h_0(1)$ is sufficiently large, then the solution to (1.6) is not global; more precisely, its first derivative blows up in finite time. The critical value for $h_0(1)$ is related to the non-existence of stationary solutions. The stationary solutions to (1.5) for $p \in (1,2)$, that is, solutions to $A(h) = 0$ in our notations, were completely classified by the authors in [15, Theorem 2], which we recall below in a shortened version which is useful for our aims.

Proposition 1.4. *There exists a unique global stationary solution $h_* \in C^\infty((0,\infty)) \cap C^1([0,\infty))$ to Eq.* (1.5) *(that is, $A(h_*) = 0$), which satisfies the following properties:*
(a) $h_*(0) = 0$, $\lim_{r\to\infty} h_*(r) = \pi/2$;
(b) There exists an increasing sequence of critical points r_n of h such that $r_n \to \infty$ as $n \to \infty$, $h_*(r_{2n}) \in (\pi/2, \pi)$ and are local maxima, $h_*(r_{2n+1}) \in (0, \pi/2)$ and are local minima. Moreover, there exists some positive integer n_0 sufficiently large such that $|h_*(r_n) - \pi/2|$ is decreasing for $n \geq n_0$, and

$$\left|h_*(r_0) - \frac{\pi}{2}\right| = \max\left\{\left|h_*(r_n) - \frac{\pi}{2}\right| : n \geq 0\right\};$$

(c) Any non-constant finite stationary global solution h to Eq. (1.5) *has the form $h(r) = k\pi \pm h_*(\alpha r)$, for some $\alpha > 0$ and $k \in \mathbb{Z}$.*

Let $H := h_*(r_0) = \max\{h_*(r) : r > 0\}$. Then $H \in (\pi/2, \pi)$ and, from the above proposition, for any $l \in (H, \pi)$, there exists no stationary solution $\tilde{h}$ to (1.5) satisfying $\tilde{h}(0) = 0$ and $\tilde{h}(1) = l$. Moreover, we have the following obvious consequence of Proposition 1.4.

Corollary 1.5. *For $l \in (H, \pi)$, the unique stationary solution h to (1.5) such that $h(1) = l$ is given by $H_l(r) = \pi - h_*(\alpha r)$, where $\alpha > 0$ is determined by $h_*(\alpha) = \pi - l$.*

On the other hand, if h is a global solution to the Dirichlet problem (1.6), then it is expected, as usual for parabolic equations, that its large-time behavior as $t \to \infty$, to be a stationary state to the equation. Thus, we deduce that if the initial datum satisfies $h(0) = 0$ and $h(1) = l \in (H, \pi)$, the solution to (1.6) with initial condition h_0 is not global. Indeed, we have

Theorem 1.6. *Let h_0 be as in Theorem 1.3 and $T > 0$ its maximum existence time. If $h_0(1) \in (H, \pi)$, then $T < \infty$ and the derivative of the solution h blows up at the origin at $t = T$.*

The proof of Theorem 1.6 is very involved and technical and requires several steps that are sketched below.

(A) We first show that indeed, if a global solution h to (1.6) exists, then its behavior as $t \to \infty$ is given by a stationary state to (1.5). In order to prove it, we need part (B) below.

(B) To prove (A), we first derive some weighted bounds in L^∞ of the derivative h_r, more precisely, uniform bounds of rh_r, that in particular imply local boundedness for h_r outside the origin. This is done in Section 4.

(C) We also need some special, explicit subsolutions that blow up in finite time. Their form proves to be quite simple, but proving that indeed they are subsolutions to (1.5) showed to be technically difficult. In fact, we have the following result proved in the Appendix:

Proposition 1.7. *Let $p \in (1, 2)$. There exists $\delta_0 > 0$, such that*

$$h(t, r) := \frac{p+4}{3} \arctan\left(\frac{r}{b(t)}\right), \quad b(t) = b_0 - t\delta_0,$$

is a subsolution to Eq. (1.5), that is

$$h_t \leq J(h)^{(p-4)/2} A(h), \quad \text{for any } (t, r) \in (0, b_0/\delta) \times (0, 1), \tag{1.11}$$

for any $b_0 > 0$ and $\delta \in (0, \delta_0)$.

With all these steps, the proof of Theorem 1.6 follows easily from a contradiction argument. The proof of claim (A) above together with this final contradiction argument may be found in Section 5.

Finally, we can complete the panorama of blow up in finite time by showing that it may also occur, although not as a general phenomenon, even when $h(1) < H$. More precisely

Theorem 1.8. *For any $l \in (0, \pi)$, there exist h_0, h and T as in Theorem 1.3 such that $h_0(1) = l$ and $T < \infty$.*

The proof of this theorem is similar to the one of [3, Proposition 2.2] or [11, Theorem 4] and it will be sketched at the end of Section 5.

2 Comparison principle

In this section we prove a comparison principle for (1.5) that will be often useful in the sequel. It is analogous, but technically more involved, to Lemma 4.2 in [12], where the comparison principle is proved for $p = 1$. To this end, we establish the notions of sub- and supersolution to (1.5)

Definition 2.1. *A function $h \in C^{1,2}([0,T] \times [0,1])$ is called a subsolution (resp. a supersolution) to Eq.* (1.5) *if*

$$h_t \leq J(h)^{(p-4)/2} A(h), \quad (\text{resp.} \quad h_t \geq J(h)^{(p-4)/2} A(h)), \text{ for all } (t,r) \in (0,T) \times (0,1).$$

We then prove

Proposition 2.2. *Assume that h, $f \in C^{1,2}([0,T] \times [0,1])$ are two functions that satisfy $h(t,0) = f(t,0) = 0$ for any $t \in [0,T]$ and that*

$$inf\{J(\theta h + (1-\theta)f) : (t,r,\theta) \in (0,T) \times (0,1) \times (0,1)\} > 0.$$

Assume that h is a supersolution to (1.5) *and f is a subsolution to* (1.5) *and that*

$$h(t,1) \geq f(t,1), \quad h(0,r) \geq f(0,r),$$

for all $t \in [0,T]$, respectively $r \in [0,1]$. Then $h \geq f$ in $[0,T] \times [0,1]$.

Proof. Set

$$F(r,x,y,z) = \left(y^2 + \frac{\sin^2 x}{r^2}\right)^{(p-4)/2} \left\{(p-1)y^2 z + (p-3)\left[\frac{y^2 \sin x \cos x}{r^2} - \frac{y \sin^2 x}{r^3}\right] + \frac{y^3}{r} + \frac{z \sin^2 x}{r^2} - \frac{\sin^3 x \cos x}{r^4}\right\}$$

and notice that

$$h_t \geq F(r,h,h_r,h_{rr}), \quad f_t \leq F(r,f,f_r,f_{rr}).$$

Let $w := h - f$. Denoting

$$v(t,r,\theta) := \theta h(t,r) + (1-\theta) f(t,r), \quad (t,r,\theta) \in (0,T) \times (0,1) \times (0,1),$$

it follows by the mean-value theorem that w satisfies the following inequality

$$w_t - A(t,r) w_{rr} - B(t,r) w_r - C(t,r) w \geq 0, \quad (t,r) \in (0,T) \times (0,1), \tag{2.1}$$

where

$$A(t,r) := \int_0^1 F_z \, d\theta, \quad B(t,r) := \int_0^1 F_y \, d\theta, \quad C(t,r) := \int_0^1 F_x \, d\theta,$$

in all of these, the integral being taken for $F(r,v,v_r,v_{rr})$. Moreover, $w(t,0) = 0$ and $w(t,1) \geq 0$ for all $t \in [0,T)$ and $w(0,r) \geq 0$ for all $r \in [0,1]$. Thus, we have to prove that (2.1) satisfies the standard comparison principle for linear parabolic equations. To this end,

we show that $A(t,r) \geq 0$ and $C(t,r)$ is bounded from above, for all $(t,r) \in (0,T) \times (0,1)$. We have

$$F_z(r,x,y,z) = \left(y^2 + \frac{\sin^2 x}{r^2}\right)^{(p-4)/2} \left((p-1)y^2 + \frac{\sin^2 x}{r^2}\right) \geq 0,$$

thus $A(t,r) \geq 0$ for all $(t,r) \in (0,T) \times (0,1)$. On the other hand, after some calculations we find

$$F_x(r,x,y,z) = \left(y^2 + \frac{\sin^2 x}{r^2}\right)^{(p-6)/2} (P+Q+R), \tag{2.2}$$

where

$$P(r,x,y,z) := \frac{\sin 2x}{r^2}\left[\frac{(p-2)(p-3)}{2}y^2 + \frac{p-2}{2}\frac{\sin^2 x}{r^2}\right] z,$$

$$Q(r,x,y) := \frac{(p-4)\sin 2x}{2r^2}\left[(p-3)\left(\frac{y^2 \sin x \cos x}{r^2} - \frac{y \sin^2 x}{r^3}\right) + \frac{y^3}{r} - \frac{\sin^3 x \cos x}{r^4}\right],$$

and

$$R(r,x,y) := \left(y^2 + \frac{\sin^2 x}{r^2}\right)\left[(p-3)\left(\frac{y^2 \cos 2x}{r^2} - \frac{y \sin 2x}{r^3}\right) - \frac{3\sin^2 x \cos^2 x - \sin^4 x}{r^4}\right].$$

Since $J(v) > 0$ up to $r > 0$ by hypothesis and $h_r > 0$ near $r = 0$, we have

$$v_r(t_0, r, \theta_0) > 0, \quad \cos v(t_0, 0, \theta_0) = \cos 2v(t_0, 0, \theta_0) = 1$$

for any $(t_0, \theta_0) \in [0,T] \times [0,1]$ and r sufficiently close to 0. Thus, fixing $a \in (0,1)$ and any $(t_0, \theta_0) \in [0,T] \times [0,1]$, there exists some $r_a > 0$ sufficiently small, that can be chosen independent of (t_0, θ_0), such that

$$y = v_r(t,r,\theta) > 0, \quad a < \cos v(t,r,\theta) \leq 1, \quad a < \cos 2v(t,r,\theta) \leq 1, \tag{2.3}$$

for all $(t,r,\theta) \in W_a(t_0,\theta_0)$, where $W_a(t_0,\theta_0)$ is the following neighborhood of (t_0,θ_0):

$$W_a(t_0,\theta_0) := \{(t,r,\theta) \in [0,T] \times (0,1) \times [0,1] : r \in (0,r_a),\ |t - t_0| < r_a,\ |\theta - \theta_0| < r_a\}.$$

Similarly as in [12], we denote $m := \min\limits_{W_a} v_r > 0$, $M := \max\limits_{W_a} v_r$ and choose r_a small enough such that $m \geq aM$. Then, by the mean-value theorem, we have

$$ma \leq \frac{\sin x}{r} \leq M, \quad 2ma^2 \leq \frac{\sin 2x}{r} \leq 2M \quad \text{in } W_a. \tag{2.4}$$

We can thus estimate the three terms P, Q, R in W_a. First of all, we get the following estimate for P:

$$P(r,x,y,z) \leq \frac{1}{r} M_1(a), \quad M_1(a) > 0. \tag{2.5}$$

Indeed, by simply bounding from above all the terms and taking modulus, using (2.4) we can take

$$M_1(a) = 2M\left(\frac{(p-2)(p-3)}{2}M^2 + \frac{2-p}{2}M^2\right) \max_{W_a} |v_{rr}(t,r,\theta)| > 0$$

and derive (2.5). In order to estimate Q and R, we state and prove the following

Claim: There exists $M_2(a)$, $M_3(a)$ such that

$$Q(r,x,y) \le \frac{1}{r^2} M_2(a), \quad R(r,x,y) \le \frac{1}{r^2} M_3(a) \tag{2.6}$$

in W_a, with $M_2(a) + M_3(a) < 0$ for $a \in (0,1)$ sufficiently close to 1.

End of the proof. Assume for a moment that the claim is true. Then, from (2.2), (2.5) and (2.6) we deduce that $F_x(r,x,y,z) < 0$ in W_a provided $r \in [0, -(M_2(a)+M_3(a))/M_1(a))$. Since the only singularity is at $r = 0$, by continuity we get that F_x is bounded from above for any $r \in (0,1)$ and (t,θ) such that $|t - t_0| < r_a$ and $|\theta - \theta_0| < r_a$. Recalling that r_a was taken independent of the starting point (t_0, θ_0), we conclude that F_x, whence also $C(t,r)$, is bounded from above in $(0,T) \times (0,1)$. Then the maximum principle applies to (2.1) and we reach the conclusion.

Proof of the claim. To estimate $Q(r,x,y)$, using (2.4), we notice that

$$\frac{y^2 \sin x \cos x}{r^2} - \frac{y \sin^2 x}{r^3} = \frac{1}{r}\left[y^2 \frac{\sin 2x}{2r} - y \left(\frac{\sin x}{r} \right)^2 \right] \le \frac{1}{r}(M^3 - a^2 m^3),$$

hence we find

$$Q(r,x,y) \frac{2r^2}{(p-4)\sin 2x} \ge \frac{1}{r}\left[(p-3)(M^3 - a^2 m^3) + m^3 - M^3 \right]. \tag{2.7}$$

Since $a < 1$, $p \in (1,2)$ and obviously $m < M$, the right-hand side of (2.7) is negative. Thus, using once more (2.4), we finally get

$$Q(r,x,y) \le \frac{1}{r^2} M_2(a), \quad M_2(a) = \frac{M(p-4)}{2}\left[(p-3)(M^3 - a^2 m^3) + m^3 - M^3 \right] > 0. \tag{2.8}$$

In order to estimate R, we first notice that (2.4) and (2.3) imply

$$\frac{y^2 \cos 2x}{r^2} - \frac{y \sin 2x}{r^3} \ge \frac{am^2 - 2M^2}{r^2},$$

hence

$$\left(y^2 + \frac{\sin^2 x}{r^2} \right)^{-1} R(r,x,y) \le \frac{1}{r^2}\left[(p-3)(am^2 - 2M^2) - a^3 m^2 - 2a^4 m^2 \right].$$

It follows that

$$R(r,x,y) \le \frac{1}{r^2} M_3(a), \quad M_3(a) = (1+a^2)m^2\left[(p-3)(am^2 - 2M^2) - a^3 m^2 - 2a^4 m^2 \right]. \tag{2.9}$$

We notice that, for a sufficiently close to 1, $M_3(a) < 0$. Moreover, from (2.8) and (2.9), and taking into account that $aM \le m \le M$, we have

$$\begin{aligned} M_2(a) + M_3(a) &= \frac{(p-4)M}{2}\left[(p-3)(M^3 - a^2 m^3) + m^3 - M^3 \right] \\ &\quad + (1+a^2)m^2\left[(p-3)(am^2 - 2M^2) - a^3 m^2 - 2a^4 m^2 \right] \\ &\le \frac{(p-4)M}{2}\left[(p-4)M^3 + (a^2(3-p)+1)a^3 M^3 \right] \\ &\quad + a^2(1+a^2)M^2\left[2(3-p)M^2 - a^2(a(3-p) + a^3 + 2a^2)M^2 \right] \\ &= M^4 G(a,p), \end{aligned}$$

where it is immediate to see that

$$\lim_{a\to 1} G(a,p) = -2p < 0.$$

Thus, for a sufficiently close to 1 we get that $M_2(a) + M_3(a) < 0$ and the claim is proved. □

3 Local well-posedness

In this section we show that the Dirichlet problem (1.6) is well-posed at least locally when the initial condition h_0 satisfies (1.7).

3.1 Uniqueness. Proof of Theorem 1.2

We begin with the following easy fact.

Lemma 3.1. *For all $\lambda > 0$, the function $\Phi_\lambda(r) := 2\arctan(\lambda r)$ is a subsolution to (1.5) and the function $\Psi_\lambda(r) := \pi - \Phi_\lambda(r)$ is a supersolution to (1.5).*

Proof. We note that

$$\Phi'_\lambda(r) = \frac{2\lambda}{1+\lambda^2 r^2}, \quad \Phi''_\lambda(r) = -\frac{4\lambda^3 r}{(1+\lambda^2 r^2)^2},$$

and

$$\sin\Phi_\lambda(r) = \frac{2\lambda r}{1+\lambda^2 r^2}, \quad \cos\Phi_\lambda(r) = \frac{1-\lambda^2 r^2}{1+\lambda^2 r^2}.$$

After straightforward calculations, we find that

$$A(\Phi_\lambda(r)) = \frac{32(2-p)\lambda^5 r}{(1+\lambda^2 r^2)^2} > 0,$$

since $1 < p < 2$. In a similar way Ψ_λ is a supersolution, we omit the details. □

Proof of Theorem 1.2. The uniqueness follows directly from Proposition 2.2. Let then h be a classical solution with initial condition h_0 satisfying (1.7).

Step 1. We show first that $h(t,0) = 0$ for all $t \in (0,T)$. We argue by contradiction and assume that this is not true; hence, by continuity there exists $t_0 \in (0,T)$ such that $h(t_0,0) \in (-\pi/2, \pi/2) \setminus \{0\}$. We then deduce from the equation (1.5) that, as $t = t_0$ and $r \to 0^+$,

$$\begin{aligned} h_t &= \left(h_r^2 + \frac{\sin^2 h}{r^2}\right)^{(p-4)/2} \left[h_{rr}\left((p-1)h_r^2 + \frac{\sin^2 h}{r^2}\right)\right. \\ &\quad \left. + \frac{h_r}{r}\left(h_r^2 + (3-p)\frac{\sin^2 h}{r^2}\right) - \frac{\sin h\cos h}{r^2}\left((3-p)h_r^2 + \frac{\sin^2 h}{r^2}\right)\right] \\ &= -\frac{\cos h}{r^p}(1+o(1)) \quad \text{as } r \to 0^+, \end{aligned}$$

which contradicts the regularity of h as a classical solution. Hence $h(t,0) = 0$ for any $t \in (0,T)$.

Step 2. We prove the rest of part (a) in Theorem 1.2. By assumptions in (1.7), there exists $T_1 > 0$ such that $h_r(t,0) > 0$ and $h(t,r) \in (0,\pi)$ for all $(t,r) \in (0,T_1) \times (0,1]$. Thus, we apply Proposition 2.2 in $(0,T_1) \times (0,1]$ coupled with Lemma 3.1 with λ sufficiently small, to get

$$\Phi_\lambda(r) \leq h(t,r) \leq \Psi_\lambda(r), \quad (t,r) \in (0,T_1) \times (0,1].$$

But since $h(T_1,0) = 0$, we notice that

$$h_r(T_1,0) = \lim_{r\to 0^+} \frac{h(T_1,r)}{r} \geq \lim_{r\to 0^+} \frac{2\arctan(\lambda r)}{r} = 2\lambda > 0,$$

thus we can restart the argument from $t = T_1$ and extend it up to T by a standard maximality argument. Finally, since

$$\frac{\arctan(x)}{x} \geq \frac{\pi}{4}, \quad \text{for all } x \in (0,1],$$

we can replace $\Phi_\lambda(r)$ by δr and $\Psi_\lambda(r)$ by $\pi - \delta r$ in the above inequality, provided $2\delta \leq \pi\lambda$. In particular, we also get that $h_r(t,0) \geq \delta$, for all $t \in [0,T)$.

Step 3. We prove part (b) in Theorem 1.2. Since $h(t,0) = 0$ for all $t > 0$, we have $h_t(t,0) = 0$, hence $A(h) = 0$ at $(t,0)$. We evaluate $A(h)$ at (t,r) as $r \to 0^+$, taking into account that,

$$\frac{h_r^2 \sin h \cos h}{r^2} \sim \frac{h_r^3}{r}, \quad \frac{h_r \sin^2 h}{r^3} \sim \frac{h_r^3}{r}, \quad \text{as } r \to 0^+$$

It follows that the singular terms in the expression of $A(h)$ cancel (in a first order approximation) as $r \to 0^+$, whence $p h_{rr}(t,0) h_r^2(t,0) = 0$ which implies the first part of the conclusion. The second one is trivial, since

$$0 = h_t(t,1) = F(h(t,\cdot))|_{r=1}.$$

□

3.2 Local existence and continuous dependence

As explained in the Introduction, Eq. (1.5) has singular coefficients, thus the standard existence theory for ODEs cannot be applied directly. In order to simplify its writing and, in a first step, transform its trigonometric nonlinearities into algebraic ones, we introduce the following change of variables:

$$h(t,r) = 2\arctan(ru(t,r)).$$

The equation satisfied by u reads as follows:

$$\begin{aligned}\frac{2j^{(4-p)/2}}{1+r^2u^2} u_t = \frac{8}{(r^2u^2+1)^4} \big[&(ru_{rr}(r^2(p-1)u_r^2 + 2r(p-1)uu_r + pu^2)(r^2u^2+1) \\ &+ 2r^5(1-p)uu_r^4 - r^2u_r^3(r^2(6p-7)u^2 - 2p + 1) \\ &- ruu_r^2(3r^2(3p-4)u^2 - 5p + 4) + u_r u^2(3p - r^2(9p-16)u_r^2) + 4r(2-p)u^5)\big],\end{aligned}$$

with $j(r,u,w) = \dfrac{4(u+rw)^2+4u^2}{(1+r^2u^2)^2}$. The above equation can be written in a simplified form as

$$u_t = \left[a(r,u,u_r)u_{rr} + \frac{b(r,u,u_r)}{r}u_r + f(r,u,u_r)\right] =: G(u), \tag{3.1}$$

with

$$a(r,u,w) = j(r,u,w)^{(p-4)/2}\frac{4(p-1)(u+rw)^2+4u^2}{(1+r^2u^2)^2}, \tag{3.2}$$

$$b(r,u,w) = j(r,u,w)^{(p-4)/2}\frac{12pu^2}{(1+r^2u^2)^3}, \tag{3.3}$$

and $f(r,u,w)$ gathers the remaining terms in the equation above, we omit its precise expression. The initial and Dirichlet conditions become

$$u_r(t,0) = 0,\ u(t,1) = u_0(1), \quad \text{for } t>0, \quad u(0,r) = u_0(r), \quad \text{for } r\in[0,1]. \tag{3.4}$$

We then follow the ideas in [11, Section 4] and consider the linearization of the right-hand side above around compatible initial data u_0 such that $u_0 > 0$ and $u_{0,r}(0) = 0$. To this end, we consider the Fréchet derivative of the right-hand side of (3.1) around a generic point $(u_0, u_{0,r})$ and obtain that the linearized operator has the form:

$$L(v) := a(r,u_0,u_{0,r})v'' + \frac{b(r,u_0,u_{0,r})}{r}v' + c(r,u_0,u_{0,r})v' + d(r,u_0,u_{0,r})v, \tag{3.5}$$

where a, b are given respectively in (3.2) and (3.3) and

$$c = \frac{\partial a}{\partial w}u_{0,rr} + \frac{\partial b}{\partial w}\frac{u_{0,r}}{r} + \frac{\partial f}{\partial w}$$

and

$$d = \frac{\partial a}{\partial u}u_{0,rr} + \frac{\partial b}{\partial u}\frac{u_{0,r}}{r} + \frac{\partial f}{\partial u}.$$

We note that no singular terms near $r=0$ appear in c and d since $\frac{u_{0,r}}{r} \overset{r\to 0^+}{\sim} u_{0,rr}$. Without loss of generality, we will assume that $a(r,u_0,u_{0,r})|_{r=0} = 1$. We define next the following operators:

$$L_0 v := v'' + \frac{3}{r}v', \tag{3.6}$$

and

$$L_\varepsilon v := a(r)v'' + \frac{b(r)}{r+\varepsilon}v' + c(r)v' + d(r)v, \tag{3.7}$$

where we have omitted from the expressions of a, b, c and d the dependence on u_0 for simplicity. From the previous considerations, we formally expect that

$$Lv \sim L_0 v, \text{ as } r\to 0^+, \quad Lv \sim L_\varepsilon v, \text{ for } r > r_0 > 0. \tag{3.8}$$

Following [11, Section 4], we will employ the theory of sectorial operators. We do not recall the definition of a sectorial operator, that can be found in [17, Definition 2.0.1] or [11, Definition 4.1], but we state the following sufficient condition that we use in the sequel.

Proposition 3.2. *Let X be a complex Banach space and $A : D(A) \subseteq X \mapsto X$ be a linear operator. If there exist $\omega \in \mathbb{R}$ and $M > 0$ such that the resolvent $\varrho(A) = \{\lambda \in \mathbb{C} : \lambda - A$ is invertible$\}$ contains the set $S := \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \geq \omega\}$ and*

$$\|\lambda(\lambda - A)^{-1}x\| \leq M\|x\| \tag{3.9}$$

holds true for all $\lambda \in S$ and $x \in X$, then A is sectorial from $D(A)$ to X.

Our strategy is to show that the linearized operator L in (3.5) is sectorial from $D(L) := \{v \in C^2([0,1]) : v'(0) = v(1) = 0\}$ to $X = C([0,1])$. In order to do it, we take as starting point the formal approximations in (3.8) and the fact that the operators L_0 and L_ε, defined in (3.6) and (3.7) respectively, are both sectorial. For L_0, this is proved by Angenent in [1, Lemma 4.3] in a more general context, see also [11, Theorem 4.3], while for L_ε, this follows from a more general theory in [17, Corollary 3.1.21.(ii) and Theorem 3.1.19]. We can combine these two facts into the following result:

Proposition 3.3. *The operator L defined in (3.5) is sectorial from $D(L) := \{v \in C^2([0,1]) : v'(0) = v(1) = 0\}$ to $X = C([0,1])$. In addition, there exists $C = C(M) > 1$ depending on*

$$M := (\inf a)^{-1} + (\inf b)^{-1} + \|a\|_\infty + \|b\|_\infty + \|c\|_\infty + \|d\|_\infty,$$

where the infima and norms are taking over $r \in [0,1]$, such that, for all $v \in D(L)$ and $\lambda \in \mathbb{C}$ such that $\mathrm{Re}\lambda \geq C(M)$, we have

$$|\lambda|\|v\|_\infty + |\lambda|^{1/2}\|v'\|_\infty + \|v''\|_\infty \leq C(M)\|(\lambda - L)v\|_\infty. \tag{3.10}$$

Proof. The proof follows ideas in [11, Proposition 4.5] and uses Proposition 3.2. We only prove the injectivity of the operator $\lambda - L$ for $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > 0$ sufficiently large. The rest of the steps required for an application of Proposition 3.2, that is, the surjectivity of $\lambda - L$ and the estimates (3.9) and (3.10), are a straightforward adaptation of those in the proof of [11, Proposition 4.5].

Let $v \in D(L)$ be such that $(\lambda - L)v = 0$. We want to show that for $\mathrm{Re}\lambda$ sufficiently large, this implies $v \equiv 0$. We begin by the following calculation:

$$\begin{aligned}
0 &= \frac{1}{a}\left[r\overline{v}(\lambda - L)v + rv\overline{(\lambda - L)v}\right] \\
&= \frac{1}{a}\left[2r\mathrm{Re}\lambda|v|^2 - ar(\overline{v}v'' + v\overline{v}'') - b(\overline{v}v' + v\overline{v}') - cr(\overline{v}v' + v\overline{v}') - 2rd|v|^2\right] \\
&= -r(\overline{v}v'' + v\overline{v}'') - \left(\frac{b}{a} + \frac{rc}{a}\right)(|v|^2)' - \frac{2r(d - \mathrm{Re}\lambda)}{a}|v|^2.
\end{aligned}$$

Let $a_0 = \inf a > 0$, $b_0 = \inf b > 0$, $A_0 = \|a\|_\infty$, $B_0 = \|b\|_\infty$, $C_0 = \|c\|_\infty$, $D_0 = \|d\|_\infty$. We integrate over $(0,1)$ the previous equality and estimate it term by term as follows.

(i) After integrating by parts and taking into account that $v(1) = 0$ and that $\lim_{r\to 0}(b(r)/a(r)) = 3$, we have:

$$\begin{aligned}
-\int_0^1 r(\overline{v}v'' + v\overline{v}'')\,dr - \int_0^1 \frac{b}{a}(|v|^2)'\,dr &= 2\int_0^1 r|v'|^2\,dr - \int_0^1 \left(\frac{b}{a} - 1\right)(|v|^2)'\,dr \\
&= 2\int_0^1 r|v'|^2\,dr - \left(\frac{b}{a} - 1\right)|v|^2\Big|_0^1 + \int_0^1 \left(\frac{b}{a}\right)'|v|^2\,dr \\
&= 2\int_0^1 r|v'|^2\,dr + 2|v(0)|^2 + \int_0^1 \left(\frac{b}{a}\right)'|v|^2\,dr.
\end{aligned}$$

(ii) We easily find that

$$2\mathrm{Re}\lambda \int_0^1 \frac{r}{a}|v|^2\,dr \geq \frac{2\mathrm{Re}\lambda}{A_0}\int_0^1 r|v|^2\,dr.$$

(iii) After using Hölder and Young's inequalities, we get

$$\begin{aligned}\left|\int_0^1 \frac{rc}{a}(|v|^2)'\,dr\right| &\leq \frac{2C_0}{a_0}\left(\int_0^1 r|v|^2\,dr\right)^{1/2}\left(\int_0^1 r|v'|^2\,dr\right)^{1/2}\\ &\leq \int_0^1 r|v'|^2\,dr + \frac{C_0^2}{a_0^2}\int_0^1 r|v|^2\,dr.\end{aligned}$$

(iv) Obviously,

$$\left|\int_0^1 \frac{2rd}{a}|v|^2\,dr\right| \leq \frac{2D_0}{a_0}\int_0^1 r|v|^2\,dr.$$

Gathering the estimates (i)-(iv) above, we obtain

$$2\int_0^1 r|v'|^2\,dr + 2|v(0)|^2 + \left(\frac{2\mathrm{Re}\lambda}{A_0} - \frac{C_0^2}{a_0^2} - \frac{2D_0}{a_0}\right)\int_0^1 r|v|^2\,dr \leq -\int_0^1 \left(\frac{b}{a}\right)'|v|^2\,dr. \quad (3.11)$$

On the other hand,

$$\begin{aligned}\left(\frac{b}{a}\right)'(r) = &-\frac{6rpu_0}{\left[r^2(p-1)u_{0,r}^2 + 2r(p-1)u_0u_{0,r} + pu_0^2\right]^2(r^2u_0^2+1)^2}\times\\ &\Big\{(p-1)u_0u_{0,rr}(ru_{0,r}+u_0)(r^2u_0^2+1) + r(1-p)u_{0,r}^3\\ &+u_0^2\left[3r^2(p-1)u_0u_r^2 + u_{0,r}(r(4p-3)u_0^2) + pu_0^3\right] + \frac{(p-1)u_0^2u_{0,r}}{r}\Big\}\end{aligned}$$

and taking into account that $u_0(0) > 0$, $u_0'(0) = 0$ and $u_0 \in C^2([0,1])$, we find that there exists some $\overline{C} > 0$ sufficiently large such that

$$\left|\left(\frac{b}{a}\right)'(r)\right| \leq \overline{C}r. \quad (3.12)$$

Replacing this estimate into (3.11), we obtain that

$$2\int_0^1 r|v'|^2\,dr + 2|v(0)|^2 + \left(\frac{2\mathrm{Re}\lambda}{A_0} - \frac{C_0^2}{a_0^2} - \frac{2D_0}{a_0} - \overline{C}\right)\int_0^1 r|v|^2\,dr \leq 0,$$

whence $v \equiv 0$ if $\lambda \in \mathbb{C}$ is taken with $\mathrm{Re}\lambda$ sufficiently large. □

In order to prove the existence of solutions to our nonlinear PDE, we follow [11] and use results about sectorial operators introduced by Lunardi, more precisely [11, Theorem 4.6] or [17, Theorem 8.1.1 and Corollary 8.3.3] to which we refer. We recall the definition of the following spaces, for $\alpha \in (0,1)$

$$C_\alpha^\alpha((0,T];C^2([0,1])) = \{v \in L^\infty((0,T);C^2([0,1])) : (t \mapsto t^\alpha v(t)) \in C^\alpha((0,t];C^2([0,1]))\},$$

and

$$B_\alpha((0,T];C^{2\alpha}([0,1])) = \{g : (0,T] \mapsto C^{2\alpha}([0,1]) : \sup_{t\in(0,T)} t^\alpha \|g(t)\|_{C^{2\alpha}} < \infty\}.$$

With this notation, we state the following result:

Proposition 3.4. *Let $u_0 \in C^2([0,1])$ such that $\inf u_0 > 0$, $u_{0r}(0) = 0$ and $G(u_0)|_{r=1} = 0$, where $G(u)$ is defined in (3.1). Then there exists a time $T > 0$ and a unique $u \in C^{1,2}([0,T]\times[0,1]) \cap C^\alpha_\alpha((0,T];C^2([0,1]))$ for all $\alpha \in (0,1)$, which solve the problem (3.1)-(3.4). Moreover, u depends continuously on u_0 and $u_t \in B_\alpha((0,T];C^{2\alpha}([0,1]))$ for any $\alpha \in (0,1)\setminus\{1/2\}$.*

The proof is an immediate adaptation of that of [11, Proposition 4.7] and we omit it. The previous existence result allows us to prove Theorem 1.3.

Proof of Theorem 1.3. Let $h_0 \in C^2([0,1])$ satisfy conditions (1.7) and (1.10). Then the function

$$u_0(r) := \frac{1}{r}\tan\left(\frac{h_0(r)}{2}\right)$$

is well defined and belongs to $C^2([0,1])$, and it is easy to check that $\lim_{r\to 0^+} = h_0'(0)/2 > 0$, hence $\inf u_0 > 0$. Moreover, from (1.10) we deduce that $u_{0r}(0) = 0$ and that $G(u_0)|_{r=1} = F(h_0)|_{r=1} = 0$. By Proposition 3.4, there exist $T > 0$ and a unique $u \in C^{1,2}([0,T]\times[0,1]) \cap C^\alpha_\alpha((0,T];C^2([0,1]))$ with the properties described in Proposition 3.4. Thus, undoing the change of variables, the function

$$h(t,r) = 2\arctan(ru(t,r))$$

is the claimed solution. Then, noticing that

$$h_r(t,r) = \frac{2(u(t,r)+ru_r(t,r))}{1+r^2u^2(t,r)}, \quad (t,r)\in[0,T]\times[0,1],$$

in order to prove part (b) in Theorem 1.3, it is enough to show that

$$u_r \in C^{1,2}([\varepsilon,T]\times[\varepsilon,1]) \quad \text{for any } \varepsilon > 0. \tag{3.13}$$

Let $\varepsilon > 0$ and $\alpha \in (1/2,1)$. It follows from Proposition 3.4 that

$$\sup_{t\in(\varepsilon,T)} \|u_t(t,\cdot)\|_{C^{2\alpha}([0,1])} < \infty. \tag{3.14}$$

Since $u \in C^{1,2}([0,T]\times[0,1])$ and $\inf u > 0$, we obtain that the mapping

$$(t,r) \mapsto (\overline{a}(t,r), \overline{f}(t,r)) = (a(r,u(t,r),u_r(t,r)), f(r,u(t,r),u_r(t,r)),$$

is of class $C([0,T];C^1([0,1]))$, while

$$(t,r) \mapsto \overline{b}(t,r) = b(r,u(t,r)) \in C([0,T];C^2([0,1])).$$

Recalling that $\inf \overline{a} > 0$, we define

$$l(t,r) := \frac{1}{\overline{a}(t,r)}(u_t(t,r) - \overline{f}(t,r)) = u_{rr}(t,r) + \frac{\overline{b}(t,r)}{\overline{a}(t,r)}\frac{u_r(t,r)}{r},$$

then $l \in L^\infty([\varepsilon, T]; C^1([0,1]))$.

In order to simplify the notations, in the following technical estimates we omit the dependence on time variable. We first define

$$w(r) = \left(\frac{u(0)}{u(r)}\right)^{6(p-1)/p} r^3 \exp\left[\int_0^r \left(\frac{\overline{b}}{s\overline{a}}(s) - \frac{3}{s} + \frac{6(p-1)u_r}{pu}\right) ds\right] \tag{3.15}$$

and we note that

$$\frac{w_r}{w} = \frac{\overline{b}}{\overline{a}r}$$

and

$$w(r) = r^3(1 + o(r)),$$

where the $o(r)$ symbol is taken as $r \to 0^+$. By standard integration by parts, we obtain

$$u_r = \frac{1}{w(r)}\int_0^r w(\varrho) l(\varrho)\, d\varrho = \frac{1}{w(r)}\left[l(r)\int_0^r w(s)\, ds - \int_0^r \left(\int_0^\varrho w(s)\, ds\right) l_r(\varrho)\, d\varrho\right], \tag{3.16}$$

We also have

$$u_{rr} = l - \frac{\overline{b}}{\overline{a}r}u_r,$$

whence

$$\begin{aligned} u_{rrr} &= l_r - \left(\frac{\overline{b}}{\overline{a}}\right)_r \frac{u_r}{r} + \frac{\overline{b}}{\overline{a}}\left[\left(1 + \frac{\overline{b}}{\overline{a}}\right)\frac{u_r}{r^2} - \frac{l}{r}\right] \\ &= l_r - \left(\frac{\overline{b}}{\overline{a}}\right)_r \frac{u_r}{r} - \frac{\overline{b}}{\overline{a}r^2}\left(1 + \frac{\overline{b}}{\overline{a}}\right)\frac{1}{w}\int_0^r \left(\int_0^\varrho w(s)\, ds\right) l_r(\varrho)\, d\varrho \\ &+ \frac{\overline{b}l}{\overline{a}}\left[\left(1 + \frac{\overline{b}}{\overline{a}}\right)\frac{1}{wr^2}\int_0^r w(s)\, ds - \frac{1}{r}\right] = L_1 + L_2 + L_3, \end{aligned} \tag{3.17}$$

where $L_1 = l_r - \left(\frac{\overline{b}}{\overline{a}}\right)_r \frac{u_r}{r}$. Our next aim is to show that u_{rrr} is bounded for $r \in [0,1]$. Recall that u_r/r is bounded in $[0,1]$ and notice that u_{rrr} is uniformly bounded for $r \in [\delta, 1]$ for any $\delta > 0$, thus we will only have to prove that the uniform boundedness can be extended up to $r = 0$. In order to estimate L_1, we take into account (3.12) and the fact that $u_r(0) = 0$, and we get

$$\lim_{r \to 0^+} I_1 = l_r(0). \tag{3.18}$$

On the other hand, since

$$1 + \frac{\overline{b}}{\overline{a}} = 4 - \frac{6(p-1)ru_r}{pu} + o(r) = 4 + o(r), \quad \text{as } r \to 0^+,$$

we can write

$$\begin{aligned} L_3 &= \frac{\overline{b}l}{\overline{a}}\left[\frac{4+o(r)}{r^5(1+o(r))}\left(\frac{r^4}{4}+\int_0^r s^3 o(s)\,ds\right)-\frac{1}{r}\right] \\ &= \frac{\overline{b}l}{\overline{a}}\left[\frac{1}{r(1+o(r))}-\frac{1}{r}+\frac{4}{r^5(1+o(r))}\int_0^r s^3 o(s)\,ds - o(1)\right] \to 0 \quad \text{as } r\to 0^+. \end{aligned} \tag{3.19}$$

In a similar way, we can estimate

$$L_2 = -\frac{\overline{b}}{\overline{a}}\left(1+\frac{\overline{b}}{\overline{a}}\right)\frac{1}{r^5(1+o(r))}\int_0^r \frac{s^4}{4}(1+o(s))l_r(s)\,ds \to -\frac{3}{5}l_r(0), \quad \text{as } r\to 0^+, \tag{3.20}$$

Gathering (3.19), (3.20) and (3.18) and taking limits as $r\to 0^+$ in (3.17), we obtain that $u_{rrr}\in L^\infty([\varepsilon,T];C([0,1]))$, for any $\varepsilon>0$, and moreover $u_{rrr}(t,0)=2l_r(t,0)/5$. This implies that $\overline{a},\overline{b}$ and $\overline{f}$ belong to $L^\infty([\varepsilon,T];C^2([0,1]))$. Then, we obtain from (3.14) that $l\in L^\infty([\varepsilon,T];C^{2\alpha}([0,1]))$, whence

$$\sup_{t\in(\varepsilon,T)}\|u_{rr}(t,\cdot)\|_{C^{2\alpha}([\varepsilon,1])}<\infty. \tag{3.21}$$

Using classical interpolation results (see e.g. [17, Lemma 5.1.1]), it follows from (3.14) and (3.21) that

$$\sup_{r\in(\varepsilon,1)}\|u_{rr}(\cdot,r)\|_{C^{\alpha}([\varepsilon,T])}<\infty.$$

This readily shows that $u_{rr}\in C^{\alpha,2\alpha}([\varepsilon,T]\times[\varepsilon,1])$, and using the equation we obtain that

$$u\in C^{1+\alpha,2+2\alpha}([\varepsilon,T]\times[\varepsilon,1]). \tag{3.22}$$

Now, to obtain the regularity stated in (3.13), one may follow the end of the proof of [11, Theorem 2]. □

4 Bounds on the derivative

In this section, we establish some uniform interior bounds for h_r, where h is a generic solution to (1.5). This will be done through a maximum and minimum principle applied to some suitable PDEs.

Proposition 4.1. *Let h_0 satisfy* (1.7) *and* (1.10). *Then, there exists $K=K(h_0(1))$ such that the classical solution h to* (1.6) *with initial condition h_0, given by Theorems 1.2 and 1.3, satisfies*

$$\sup_{(t,r)\in[0,T)\times(0,1)}|rh_r(t,r)|\le \max\left\{\sup_{r\in[0,1]}\left(\frac{\pi}{h_0(r)}\right)^{2/3}|rh_{0r}(r)|,K\right\}. \tag{4.1}$$

Notice that the right hand side of (4.1) is finite due to assumptions in (1.7).

Proof. We prove the lower and the upper bound in two different steps.

Step 1. The upper bound. We first take

$$d(t,r) := \frac{(rh_r(t,r))^{1/\gamma}}{h(t,r)^{(1-\gamma)/\gamma}},$$

with $\gamma > 0$ to be chosen later. Taking into account that

$$d_r = \frac{d}{r\gamma} + \frac{\gamma-1}{\gamma}\frac{d^{\gamma+1}}{rh^\gamma} + \frac{rd^{1-\gamma}}{\gamma h^{1-\gamma}}h_{rr}, \tag{4.2}$$

after some straightforward computations, we then notice that the partial differential equation satisfied by d has the form:

$$z(r,h,d)d_t = a(r,h,d)d_{rr} + b(r,h,d,d_r)d_r + c(h,d), \tag{4.3}$$

where

$$z(r,h,d) = r^p((d^\gamma h^{1-\gamma})^2 + \sin^2 h)^{\frac{6-p}{2}} > 0$$
$$a(r,h,d) = r^2 \sin^2 h(p(d^\gamma h^{1-\gamma})^4 + \sin^2 h) > 0,$$

$b(r,h,d,d_r) \in C([0,1]\times[0,\pi]\times\mathbb{R}^2)$ and

$$c(h,d) = \sum_{j=0}^{7} c_j d^{1+(j-1)\gamma},$$

where c_j are polynomials in $(h, \sin h, \cos h)$; we omit the details of the calculation. Notice that c does not depend on r, due to the homogeneity of Eq. (1.5). We thus only want to evaluate the sign of c_7, that is, the dominating term with respect to the powers of d, in order to verify that a maximum principle holds true. We notice that

$$c_7(h) = \frac{\gamma-1}{\gamma}(p-1)h^{4-6\gamma}((p-1)\gamma + 2 - p)$$

If we choose now $\gamma = 1/3$ so that $4-6\gamma = 2$, then $c_7(h) < 0$, which implies that there exists a universal constant $d_0 > 0$ such that $c(h,d) < 0$ for all $d > d_0$ and $h \in [0,\pi]$. Let us define

$$K_0 := \max\left\{d_0,\ \sup_{r\in[0,1]}\frac{(rh_{0r}(r))^3}{h_0^2(r)}, K_1\right\}, \quad K_1 := \left(\frac{3(3-p)}{4(p-1)h_0^{1/3}(1)}\right)^{3/2}.$$

Assume by contradiction that there exists a point (t_0', r_0') such that $d(t_0', r_0') > K_0$, and let (t_0, r_0) be the point where d attains its maximum in the compact set $[0, t_0']\times[0,1]$. Then $d(t_0,r_0) > K_0$ and obviously $t_0 > 0$, $r_0 > 0$. We also want to avoid $r = 1$. At $r = 1$, we have $h_t = 0$ from the Dirichlet condition, hence (everything being evaluated at $(t,1)$)

$$h_{rr} = \frac{\left[h_r^2(3-p) + \sin^2 h\right]\sin h\cos h - \left[h_r^2 + (3-p)\sin^2 h\right]h_r}{(p-1)h_r^2 + \sin^2 h}. \tag{4.4}$$

We also notice that, for $p \in (1,2)$, we have

$$1 < \frac{(3-p)h_r^2 + \sin^2 h}{(p-1)h_r^2 + \sin^2 h} < \frac{3-p}{p-1}, \quad 1 < \frac{h_r^2 + (3-p)\sin^2 h}{(p-1)h_r^2 + \sin^2 h} < \frac{3-p}{p-1}, \tag{4.5}$$

whence

$$h_{rr}(t,1) < \frac{(3-p)}{2(p-1)} - h_r(t,1) = \frac{(3-p)}{2(p-1)} - d(t,1)^{1/3} h_0(1)^{2/3}. \tag{4.6}$$

Then, replacing (4.6) into (4.2), we obtain

$$d_r(t,1) < \frac{3d(t,1)^{2/3}}{h_0(1)^{2/3}} \left[\frac{3-p}{2(p-1)} - \frac{2}{3} d(t,1)^{2/3} h_0(1)^{1/3} \right],$$

whence $d_r(t,1) < 0$ if $d(t,1) > K_0 > K_1$. It follows that $r_0 < 1$, hence $d_t(t_0, r_0) \geq 0$, $d_r(t_0, r_0) = 0$ and $d_{rr}(t_0, r_0) \leq 0$. Evaluating (4.3) at (t_0, r_0), we find $c(h(t_0, r_0), d(t_0, r_0)) \geq 0$, which contradicts the choice of $d(t_0, r_0) > K_0 > d_0$. Thus, $d(t,r) \leq K_0$ in $[0,T) \times [0,1]$, hence

$$r h_r(t,r) = d(t,r)^{1/3} h(t,r)^{2/3} \leq K_0^{1/3} \pi^{2/3} =: K,$$

which is the claimed upper bound.

Step 2. The lower bound. For the lower bound, things are technically simpler. We let

$$d(t,r) = r h_r(t,r)(\pi + h(t,r))$$

and we follow the same ideas as in Step 1. In this case, the equation satisfied by d has the same form as in (4.3), where

$$z(r,h,d) = r^p (\pi + h)^{p+2} ((\pi + h)^2 \sin^2 h + p d^2) > 0,$$

$$a(r,h,d) = r^2 (\pi + h)^6 \sin^2 h ((\pi + h)^2 \sin^2 h + p d^2) > 0,$$

$b(r,h,d,d_r) \in C([0,1] \times [0,\pi] \times \mathbb{R}^2)$ and

$$c(h,d) = \sum_{j=0}^{7} c_j d^j,$$

where c_j are polynomials in $(h, \sin h, \cos h)$. In this case,

$$c_7 = p(p-1) > 0.$$

Thus there exists a universal constant $d_1 > 0$ such that $c(h,d) > 0$ for all $d \in (-\infty, -d_1]$ and $h \in [0,\pi]$. Let us define, as in Step 1,

$$\overline{K}_0 := \max \left\{ d_1, \sup_{r \in (0,1)} |r(\pi + h_0(r)) h_{0r}(r)|, \overline{K}_1 \right\},$$

$$\overline{K}_1 := \frac{2\pi^2}{p-1} \left[\sqrt{\frac{4(p-2)^2 \pi + (p-1)^2}{\pi}} - 2(2-p) \right].$$

Assume by contradiction that there exists a point $(t_0', r_0') \in [0,T)\times[0,1]$ such that $d(t_0', r_0') < -\overline{K}_0$, and let (t_0, r_0) be the point where $d(t,r)$ attains its minimum in the compact set $[0, t_0'] \times [0,1]$; obviously, $d(t_0, r_0) < -\overline{K}_0$ and, by the conditions on h_0, we have $t_0 > 0$, $r_0 > 0$. We want to avoid the case $r = 1$; thus, evaluating (1.5) at $r = 1$, we obtain (4.4) and we use again the estimates (4.5), arriving to the following inequality:

$$h_{rr}(t,1) > -\frac{1}{2} - \frac{(3-p)d(t,1)}{(p-1)(\pi + h_0(1))},$$

hence (everything being taken at $(t,1)$, that we omit from the writing)

$$\begin{aligned} d_r &= (\pi+h)h_r + h_r^2 + (\pi+h)h_{rr} \geq d + \frac{d^2}{(\pi+h)^2} - \frac{1}{2}(\pi+h) - \frac{3-p}{p-1}d \\ &= \frac{d^2}{(\pi+h)^2} - \frac{2(2-p)}{p-1}d - \frac{1}{2}(\pi+h) \geq \frac{d^2}{4\pi^2} - \frac{2(2-p)}{p-1}d - \pi > 0, \end{aligned}$$

when $d(t,1) < -\overline{K}_0$, which implies that $r_0 < 1$. Thus, $d_t(t_0, r_0) \leq 0$, $d_r(t_0, r_0) = 0$ and $d_{rr}(t_0, r_0) \geq 0$. Evaluating (4.3) at (t_0, r_0) and taking into account the previous inequalities, we obtain that $c(h(t_0, r_0), d(t_0, r_0)) \leq 0$, which contradicts the assumption that $d(t_0, r_0) < -\overline{K}_0 < -d_1$. Therefore $d(t,r) > -\overline{K}_0$ in $[0,T)\times[0,1]$, yielding

$$rh_r(t,r) > -\frac{\overline{K}_0}{\pi + h} > -\frac{\overline{K}_0}{\pi},$$

which is the desired lower bound. □

5 Blow up in finite time

As described in the Introduction, in this section we prove Theorem 1.6. It relies on three steps marked as (A), (B) and (C) in the Introduction, among which it remains to prove only part (A), that is made precise in the following statement

Proposition 5.1. *Let h_0, T and h be as in Theorem 1.3. If h is a global solution (that is, $T = \infty$), then for any sequence $t_n \to \infty$, there exists a subsequence (not relabeled) and a stationary state $\overline{g} \in C^\infty((0,1])$ of (1.5) such that*

$$\lim_{n\to\infty} \|h(t_n) - \overline{g}\|_{L^\infty([\varepsilon,1])} = 0, \quad \text{for any } \varepsilon > 0$$

.

The proof is very similar to that of [11, Lemma 7.1], thus we only sketch it. Before starting, let us define the following

$$B(h) := \left((p-1)h_r^2 + \frac{\sin^2 h}{r^2}\right)^{-1} A(h) = h_{rr} + R(h)\frac{h_r}{r} - S(h)\frac{\sin h \cos h}{r^2}, \tag{5.1}$$

with

$$R(h) := \frac{(rh_r)^2 + (3-p)\sin^2 h}{(p-1)(rh_r)^2 + \sin^2 h}, \quad S(h) := \frac{(3-p)(rh_r)^2 + \sin^2 h}{(p-1)(rh_r)^2 + \sin^2 h}.$$

Proof of Proposition 5.1. Assume that $T = \infty$ and take any sequence $t_n \to \infty$ and any $\varepsilon \in (0,1)$. In this proof, by C_ε we will always understand a generic positive constant depending on ε, which may change without relabeling. From Proposition 4.1, there exists a constant $C_\varepsilon > 0$, depending on ε, such that

$$|h_r| \leq C_\varepsilon, \quad \frac{1}{C_\varepsilon} \leq \left|\frac{\sin h}{r}\right| \leq C_\varepsilon, \quad J(h) \leq C_\varepsilon, \tag{5.2}$$

for any $(t,r) \in [0,\infty) \times [\varepsilon, 1]$. Using then the Arzelà-Ascoli theorem, there exists a subsequence (not relabeled) such that $h(t_n) \to \overline{h}$ locally uniformly in $(0,1]$, as $n \to \infty$, the argument being made completely rigorous in the proof of [11, Lemma 7.1]. Following as there, we let

$$g_n(s,r) := h(t_n + s, r), \quad (s,r) \in [0,\infty) \times [0,1],$$

whence g_n satisfies the following bounds uniformly in n, which are a consequence of (5.2):

$$|g_{nr}| \leq C_\varepsilon, \quad \frac{1}{C_\varepsilon} \leq \left|\frac{\sin g_n}{r}\right| \leq C_\varepsilon, \quad J(g_n) \leq C_\varepsilon, \tag{5.3}$$

for any $(s,r) \in [0,\infty) \times [\varepsilon, 1]$. Then

$$B(g_n) = \left((p-1)h_r^2 + \frac{\sin^2 h}{r^2}\right)^{-1} J(g_n)^{(4-p)/2} g_{ns} \leq \frac{r^2}{\sin^2 g_n} C_\varepsilon g_{ns},$$

whence

$$B(g_n)^2 \leq C_\varepsilon r g_{ns}^2, \quad (s,r) \in [0,\infty) \times [\varepsilon, 1].$$

On the other hand, it is straightforward to check that

$$1 \leq R(h), S(h) \leq \frac{3-p}{p-1},$$

for any $r \in [0,1]$, h and $p \in (1,2)$, thus, from the definition of B and the bounds (5.3), we get

$$\begin{aligned} g_{nrr} &= B(g_n) - R(g_n)\frac{g_{nr}}{r} + S(g_n)\frac{\sin g_n \cos g_n}{r^2} \\ &\leq B(g_n) - \frac{g_{nr}}{r} + \frac{3-p}{p-1}\frac{C_\varepsilon}{r} \leq C_\varepsilon(1 + |B(g_n)|), \end{aligned}$$

by modifying conveniently the constant C_ε in each step. We thus have the same bounds as in the proof of [11, Lemma 7.1], allowing us to continue as there and deduce that there exists a limit $\overline{g}$ as $n \to \infty$ and it coincides with our previous $\overline{h}$, giving the desired locally uniform convergence.

It only remains to show that $\overline{g}$ is a classical stationary solution to (1.5), that is, $B(\overline{g}) = 0$. Let $\varepsilon > 0$ and some test function $\varphi \in C_0^\infty((\varepsilon, 1])$. We know that $B(g_n) \to 0$ in $L^2([0,1]; L^2_{\mathrm{loc}}((0,1])))$, thus

$$0 = \lim_{n\to\infty} \int_0^1 \int_0^1 B(g_n)\varphi dr dt = \lim_{n\to\infty} \int_0^1 \int_0^1 \left(g_{nrr} + \frac{R(g_n) g_{nr}}{r} - \frac{S(g_n)\sin g_n \cos g_n}{r^2}\right)\varphi dr dt.$$

Following the same argument as in part (II) of the proof of [11, Lemma 7.1] which involves the Aubin-Simon compactness criterium, we obtain that, for a subsequence (not relabeled), $g_n \to \overline{g}$ in $L^2([0,1]; H^1([\varepsilon,1]))$ and in particular $g_{nr} \to \overline{g}_r$ a. e. Thus, we can easily pass to the limit in the right-hand side above to obtain that $B(\overline{g}) = 0$ and that $\overline{g}$ has the required smoothness. We omit the details. □

Joining Proposition 5.1 and Corollary 1.5, we have the following immediate consequence whose proof we skip.

Corollary 5.2. *Let h_0, h, T be as in Theorem 1.3 and such that $h_0(1) = l \in (H, \pi)$ and $T = \infty$. Then we have*

$$\lim_{t\to\infty} \|h(t) - H_l(t)\|_{L^\infty([\varepsilon,1])} = 0,$$

for all $\varepsilon > 0$, where H_l is the stationary solution to (1.5) *introduced in Corollary 1.5.*

We finally have all the needed ingredients for the proof of Theorem 1.6

Proof of Theorem 1.6. Assume by contradiction that h is a global solution such that $h(0) = 0$, $h(1) = l \in (H, \pi)$. Then, Corollary 5.2 holds true for h. Since $H_l(0) = \pi > (p+4)\pi/6$, we can take some $\theta > 0$ such that $H_l(\theta) > (p+4)\pi/6$. By Corollary 5.2, there exists $t_0 > 0$ such that

$$h(t,\theta) > \frac{(p+4)\pi}{6}, \quad \text{for all } t \geq t_0.$$

Define now the rescaled function $\tilde{h}(t,r) := h(t_0 + \theta t, \theta r)$. Then $\tilde{h}$ is a solution to (1.5) in $[0,\infty) \times [0,1]$, such that $\tilde{h}(0,r) = h(t_0, \theta r)$ for $r \in [0,1]$, and the Dirichlet conditions

$$\tilde{h}(t,0) = 0, \quad \tilde{h}(t,1) > \frac{(p+4)\pi}{6}, \quad \text{for } t > 0.$$

Consider now f a subsolution from the family defined in Proposition 1.7, choosing b_0 so large such that

$$f(0,r) = \frac{p+4}{3} \arctan\left(\frac{r}{b_0}\right) \leq \tilde{h}(0,r) = h(t_0, \theta r).$$

Since $f(t,1) \leq (p+4)\pi/6 \leq \tilde{h}(t,1)$, for any $t > 0$, by the comparison principle in Section 2, we obtain $f \leq \tilde{h}$ in $[0, b_0/\delta_0) \times [0,1]$. But this is a contradiction with the regularity of $\tilde{h}$ when taking limits as $t \to b_0/\delta_0$, hence $T < \infty$. □

We end with the proof of the nongeneric blow up in finite time.

Proof of Theorem 1.8. We only sketch the proof, as it follows the lines of [11, Theorem 4]. Let g_0 be as in Theorem 1.3 with $g_0(1) = l \in (0,\pi)$ and $g_0(1/2) > (p+4)\pi/6$, and let g be the unique solution to the Dirichlet problem (1.6) with initial datum g_0. Then by continuity, there exists a time $t_0' > 0$ such that $g(t,1/2) > (p+4)\pi/6$, for $t \in [0, t_0']$. Let now f be a subsolution from the family defined in Proposition 1.7, with $b_0 > 0$ such that $b_0/\delta_0 < t_0'$. Thus, the desired initial datum is a function h_0 satisfying the regularity hypothesis in Theorem 1.3, $h_0(1) = l$ and

$$h_0(r) \geq \begin{cases} \max\{f(0,r), g_0(r)\}, & \text{if } r \in (0,1/2), \\ g_0(r), & \text{if } r \in (1/2,1). \end{cases}$$

Let h be the solution with initial datum h_0 and T its maximal existence time; let also $t_0 = \min\{b_0/\delta_0, T\} < t_0'$. By standard comparison, $h \geq g$, thus in particular $h(t, 1/2) \geq g(t, 1/2) \geq (p+4)\pi/6 > f(t, 1/2)$ for all $t \in [0, t_0]$. We can thus apply once more the comparison principle to compare h and f to get that $f \leq h$ in the parabolic cylinder $[0, t_0] \times [0, 1/2]$. Since $h(t, 0) = 0$ for all $t < T$ and

$$\lim_{t \to (b_0/\delta_0)} h(t, r) \geq \lim_{t \to (b_0/\delta_0)} f(t, r) = \frac{(p+4)\pi}{6}, \quad \text{for all } r \in (0, 1],$$

it follows that $T < b_0/\delta_0 < \infty$, as claimed. □

6 Appendix

This section is devoted to prove Proposition 1.7. We first note that the statement is somehow analogous to [11, Lemma 5.1], although by letting $p \to 1$ we do not recover the same subsolutions as there, but their form is similar. However, the proof in our case is much more involved and highly technical, due to the complicated form of (1.5). We will first state and prove an inequality that will be useful at several points in the proof.

Lemma 6.1. *For $k \in (2, 4)$, the following inequality holds true:*

$$\sin\left(k \arctan\left(\frac{1}{x}\right)\right) < \frac{kx}{1+x^2} \tag{6.1}$$

Proof. Let $\theta := k \arctan(1/x)$. We first note that if $\theta \in [\pi, 2\pi]$, then the inequality is trivial. Let now

$$g(k, x) := \frac{kx}{1+x^2} - \sin\left(k \arctan\left(\frac{1}{x}\right)\right).$$

We differentiate it with respect to k

$$\frac{\partial g}{\partial k}(k, x) = \frac{x}{1+x^2} - \cos\left(k \arctan\left(\frac{1}{x}\right)\right) \arctan\left(\frac{1}{x}\right),$$

and

$$\frac{\partial^2 g}{\partial k^2}(k, x) = \sin\left(k \arctan\left(\frac{1}{x}\right)\right) \arctan^2\left(\frac{1}{x}\right).$$

Since $\theta \in (0, \pi)$, this second derivative is strictly positive, whence the first derivative increases with k. It follows that

$$\frac{\partial g}{\partial k}(k, x) \geq \frac{\partial g}{\partial k}(2, x) = \frac{x}{1+x^2} - \frac{x^2-1}{x^2+1} \arctan\left(\frac{1}{x}\right) > 0,$$

the last inequality being trivial for $x \leq 1$ and following from the fact that

$$\arctan\left(\frac{1}{x}\right) < \frac{1}{x} = \frac{x}{x^2} < \frac{x}{x^2-1},$$

when $x > 1$. Thus, g increases with k for all $x \geq 0$, hence $g(k, x) \geq g(2, x) = 0$. □

Proof of Proposition 1.7. By differentiating it with respect to time and letting $s := b/r$, (1.11) writes

$$\frac{p+4}{3}\delta \le r(1+s^2)J(h)^{(p-4)/2}A(h) = \frac{J(h)^{(p-4)/2}}{r^3(1+s^2)^{3/2}}\left(r^4(1+s^2)^{5/2}A(h)\right) \tag{6.2}$$

Thinking at $s \in (0,\infty)$ as an independent variable (not depending on time), it is enough to prove that the right-hand side of (6.2) is uniformly positive for any $p \in (1,2)$. To this end, we first notice, by straightforward calculations, that, as $s \to \infty$

$$\frac{J(h)^{(p-4)/2}}{r^3(1+s^2)^{3/2}} \sim 2^{(p-4)/2}\left(\frac{p+4}{3}\right)^{p-4}\frac{1}{(rs)^{p-1}} = C_1(p)b^{1-p} > 0,$$

since $b < b_0 < \infty$ and $p > 1$. On the other hand, letting

$$A^*(h) := r^4(1+s^2)^{5/2}A(h), \quad \theta = \theta(s,p) := \frac{p+4}{3}\arctan\left(\frac{1}{s}\right),$$

and considering (s,p) as independent variables from now on, we have

$$\begin{aligned} A^*(s,p) &= -\frac{3(1+s^2)^2\sin(2\theta)\left[9(1+s^2)^2\sin^2\theta + (3-p)(p+4)^2s^2\right]}{54(1+s^2)^{3/2}} \\ &\quad - \frac{2s(p+4)\left[9(1+s^2)^2((p-3)s^2+p-1)\sin^2\theta + (p+4)^2s^2(2p-3-s^2)\right]}{54(1+s^2)^{3/2}} \\ &= -\frac{1}{54(1+s^2)^{3/2}}(I_1+I_2+I_3+I_4), \end{aligned}$$

where

$$I_1 = 27(1+s^2)^4\sin(2\theta)\sin^2\theta, \quad I_2 = 3(3-p)(p+4)^2s^2(1+s^2)^2\sin(2\theta),$$

and

$$I_3 = 18(p+4)s(1+s^2)^2\left[(p-3)s^2+p-1\right]\sin^2\theta, \quad I_4 = 2(p+4)^3s^3(2p-3-s^2).$$

By a rather long but easy calculation that takes into account that $\sin\theta \sim (p+4)/3s$, as $s \to \infty$, we find

$$A^*(s,p) \to -\frac{(p+4)(p^5+10p^4+72p^3+256p^2-128p-1536)}{729} = C_2(p) > 0 \quad \text{as } s \to \infty,$$

which holds true for $p < 2$. Thus, in order to get the desired uniform positivity, it only remains to prove that $I_1+I_2+I_3+I_4 < 0$ for any $s \in (0,\infty)$. This will be done in a series of technical lemmas.

Lemma 6.2. *With the notation and conditions above, $I_1+I_3 < 0$ for $p \in (1,2)$, $s \in (0,\infty)$.*

Proof. It suffices to show that

$$I_5 = I_5(s,p) := 3(1+s^2)^2\sin(2\theta) + 2s(p+4)\left[(p-3)s^2+p-1\right] \le 0.$$

This will be done by showing that I_5 is strictly increasing with $p \in (1,2]$ and $I_5(s,2) = 0$ for any $s > 0$. Differentiating it once with respect to p, we get

$$\frac{\partial I_5}{\partial p} = 6(1+s^2)^2 \cos(2\theta)\theta_p + 2s\left[(2p+1)s^2 + 2p + 3\right].$$

If $\theta \in [0, \pi/4] \cup [3\pi/4, \pi]$, then obviously $\frac{\partial I_5}{\partial p} > 0$. If $\theta \in (\pi/4, 3\pi/4)$, we differentiate two times more with respect to p:

$$\frac{\partial^2 I_5}{\partial p^2} = 4(1+s^2)\left[s - \frac{1}{3}(1+s^2)\arctan^2\left(\frac{1}{s}\right)\sin(2\theta)\right],$$

and

$$\frac{\partial^3 I_5}{\partial p^3} = -\frac{4}{9}(1+s^2)^2 \arctan^3\left(\frac{1}{s}\right)\cos(2\theta) > 0,$$

taking into account the range of θ. It follows that $\frac{\partial^2 I_5}{\partial p^2}$ is increasing with p and in particular attains its minimum either for $p \to 1$ or for the value of p for which $\theta = \pi/4$, in case this value of p is greater than 1. In the latter case, we can restrict the range of s by noticing that $p = 3\pi/(4\arctan(1/s)) - 4 > 1$, and at the same time $\theta = (p+4)\arctan(1/s)/3 = \pi/4$, whence this second case holds only if

$$\frac{1}{\tan(3\pi/20)} < s < \frac{1}{\tan \pi/8}.$$

In this range we have

$$\frac{\partial^2 I_5}{\partial p^2} \geq 4(1+s^2)\left[s - \frac{1+s^2}{3}\arctan^2\left(\frac{1}{s}\right)\right]. \tag{6.3}$$

But it is easy to check that in the indicated range for s (which in particular implies $s > 1$), we have

$$\arctan^2\left(\frac{1}{s}\right) \leq \frac{1}{s^2} < \frac{3s}{1+s^2},$$

whence the right-hand side in (6.3) is positive. In the former case (when the minimum is attained as $p \to 1$), we get

$$\frac{\partial^2 I_5}{\partial p^2} \geq 4(1+s^2)\left[s - \frac{1+s^2}{3}\arctan^2\left(\frac{1}{s}\right)\sin\left(\frac{10}{3}\arctan\left(\frac{1}{s}\right)\right)\right]. \tag{6.4}$$

To prove that the right-hand side in (6.4) is positive, we first notice that if $\theta \in [\pi/2, 3\pi/4)$ we are done, since $\sin(2\theta) \leq 0$. It remains to study the case $\theta \in [\pi/4, \pi/2)$, which implies again a restriction for the range of s to

$$\frac{1}{\tan(3\pi/10)} < s < \frac{1}{\tan(3\pi/20)}.$$

We use then Lemma 6.1 for $k = 10/3$, obtaining that

$$\sin\left(\frac{10}{3}\arctan\left(\frac{1}{s}\right)\right) < \frac{10s}{3(1+s^2)}, \tag{6.5}$$

and, due to the previous restriction on s we also get that $\arctan^2(1/s) < 9\pi^2/100$. Joining this with (6.5), we obtain that the right-hand side of (6.4) is positive in our range, which shows furthermore that $\frac{\partial I_5}{\partial p}$ increases with p.

In particular, $\frac{\partial I_5}{\partial p}$ attains its minimum either for $p \to 1$ or for the value of p for which $\theta = \pi/4$, in case this value of p is greater than 1. In the latter, we trivially get

$$\frac{\partial I_5}{\partial p} \geq 2s\left[(2p+1)s^2 + 2p + 3\right] > 0.$$

In the former, we replace $p = 1$ to obtain

$$\frac{\partial I_5}{\partial p} \geq 2\left[(1+s^2)^2 \cos\left(\frac{10}{3}\arctan\left(\frac{1}{s}\right)\right)\arctan\left(\frac{1}{s}\right) + s(3s^2+5)\right]$$

In order to prove that the right-hand side above is positive for $\theta \in (\pi/4, 3\pi/4)$, we restrict the range of s to $s \in (1/\tan(9\pi/20), 1/\tan(3\pi/20))$. Since $10/3 \arctan(1/s) \in [\frac{\pi}{2}, \frac{5\pi}{3}]$, then $\cos\left(\frac{10}{3}\arctan\left(\frac{1}{s}\right)\right) > -\frac{1}{2}$. Hence, we only have to check that

$$l(s) := s(3s^2+5) - \frac{1}{2}(1+s^2)^2 \arctan\left(\frac{1}{s}\right) > 0, \quad \text{for any } s \in \left(\frac{1}{\tan(9\pi/20)}, \frac{1}{\tan(3\pi/20)}\right).$$

It is easily seen that l is increasing in s and there exists a unique $s_1 > 0$ such that $l(s_1) = 0$. By approximation using for example Newton's method, we find that $s_1 < 0.15 < 1/\tan(9\pi/20)$, which shows that we have covered all the admissible interval for s.

Thus, I_5 is increasing with respect to p and it follows that $I_5 < 0$ for $p \in (1,2)$, since $I_5|_{p=2} \equiv 0$. □

Lemma 6.3. *Under the notation and conditions above, $I_2 + I_3 < 0$ for any $p \in (1,2)$ and $s > \sqrt{3/7}$.*

Proof. The inequality to prove is equivalent to

$$(3-p)(p+4)s\cos\theta + 3\left[(p-3)s^2 + p - 1\right]\sin\theta < 0,$$

or furthermore

$$\varphi(s) := \theta(s,p) - \operatorname{arccot}\left(-\frac{3(p-3)s^2 + 3(p-1)}{(3-p)(p+4)s}\right) > 0.$$

We differentiate once with respect to s and find

$$\varphi'(s) = \frac{(2-p)(p+4)\left[s^2(p-3)(p^2+7p+24) + 18(p-1)\right]}{3(1+s^2)\left[9s^4(p-3)^2 + s^2(p-3)(p^3+5p^2+10p-66) + 9(p-1)^2\right]} = \frac{A(s,p)}{B(s,p)}.$$

It is obvious that $B(s,p) > 0$. On the other hand, as it easy to check, for $1 < p < 2$,

$$\frac{18(p-1)}{(3-p)(p^2+7p+24)} \leq \frac{3}{7} < s^2,$$

whence $A(s,p) < 0$ in our range for s and p. Thus, $\varphi(s)$ is decreasing with s for $s > \sqrt{3/7}$, hence $\varphi(s) > \lim_{s\to\infty} \varphi(s) = 0$, which ends the proof. □

Lemma 6.4. *Under the notation and conditions above, $I_1 + I_4 < 0$ for any $p \in (1,2)$.*

Proof. Let $\theta = \theta(s,p) = \frac{p+4}{3}\arctan(\frac{1}{s}) \in [0,\pi]$ and let

$$I_6 := 27(1+s^2)^4 \sin^3(\theta)\cos(\theta) + s^3(p+4)^3(2p-s^2-3).$$

We differentiate I_6 with respect to p and we obtain

$$\frac{\partial I_6}{\partial p} = 27(1+s^2)^4\theta_p(4\sin^2(\theta)\cos^2(\theta) - \sin^4(\theta)) - s^3(p+4)^2(3s^2-8p+1).$$

In case that $\theta \in [\frac{2\pi}{3}, \pi]$ (which implies that $s \leq 1/\sqrt{3}$), both summands are non-negative and therefore $\frac{\partial I_6}{\partial p} \geq 0$. This implies that $I_6 \leq I_6|_{p=2} = 0$.

For the case in which $\theta \in [0, \frac{2\pi}{3})$, we use Lemma 6.1 and we obtain that

$$\begin{aligned} I_1 + I_4 &= 27(1+s^2)^4 \sin(2\theta)\sin^2(\theta) + 2s^3(p+4)^3(2p-s^2-3) \\ &< 18(p+4)s(1+s^2)^3\sin^2\theta + 2s^3(p+4)^3(2p-s^2-3), \end{aligned}$$

hence we will study the sign of

$$\overline{I}_6 := 9(1+s^2)^3\sin^2\theta + s^2(p+4)^2(2p-s^2-3).$$

As in the previous Lemma 6.2, we differentiate several times with respect to p, and obtain

$$\frac{\partial \overline{I}_6}{\partial p} = 9(1+s^2)^3\sin(2\theta)\theta_p + 2s^2(p+4)(3p+1-s^2),$$

$$\frac{\partial^2 \overline{I}_6}{\partial p^2} = 18(1+s^2)^3\cos(2\theta)\theta_p^2 + 2s^2(6p+13-s^2),$$

$$\frac{\partial^3 \overline{I}_6}{\partial p^3} = 12\left[s^2 - 3(1+s^2)^3\sin(2\theta)\theta_p^3\right],$$

and finally,

$$\frac{\partial^4 \overline{I}_6}{\partial p^4} = -\frac{8}{9}(1+s^2)\cos(2\theta)\arctan^4\left(\frac{1}{s}\right).$$

Therefore

$$\frac{\partial^3 \overline{I}_6}{\partial p^3} \geq \begin{cases} \frac{\partial^3 \overline{I}_6}{\partial p^3}|_{p=2}, & \text{if } \theta(s,2) \leq \frac{\pi}{4}, \\ \frac{\partial^3 \overline{I}_6}{\partial p^3}|_{\theta=\frac{\pi}{4}}, & \text{if } \theta(s,1) \leq \frac{\pi}{4} \leq \theta(s,2), \\ \frac{\partial^3 \overline{I}_6}{\partial p^3}|_{p=1}, & \text{otherwise.} \end{cases}$$

In the first case,

$$\frac{\partial^3 \overline{I}_6}{\partial p^3} \geq \frac{16}{3}s(1-s^4)\arctan^3\left(\frac{1}{s}\right) + 12s^2 \geq \frac{4}{3s^2}(5s^4+4) > 0,$$

the intermediate inequality following after some algebraic manipulations from the standard fact that $\arctan(1/s) \leq 1/s$ for any $s > 0$.

In the second and third case, we can estimate

$$\frac{\partial^3 \overline{I}_6}{\partial p^3} \geq 12s^2 - \frac{4}{3}(1+s^2)^3 \arctan^3\left(\frac{1}{s}\right). \tag{6.6}$$

It remains to show that the right hand side in (6.6) is positive within the range

$$\frac{1}{\sqrt{3}} < s < \frac{1}{\tan(\frac{\pi}{8})}.$$

In order to prove it, we introduce

$$l(s) := 12s^2 - \frac{4}{3}(1+s^2)^3 \arctan^3\left(\frac{1}{s}\right).$$

We differentiate it and, taking into account again that $\arctan(1/s) \leq 1/s$ we obtain

$$\begin{aligned} l'(s) &= 4\left[(1+s^2)^2 \arctan^2\left(\frac{1}{s}\right)\left(1 - 2s\arctan\left(\frac{1}{s}\right)\right) + 6s\right] \\ &\geq 4\left[6s - (1+s^2)^2 \arctan^2\left(\frac{1}{s}\right)\right] =: \tilde{l}(s). \end{aligned}$$

and furthermore

$$\begin{aligned} \tilde{l}'(s) &= 8\left[(1+s^2)\arctan\left(\frac{1}{s}\right)\left(1 - 2s\arctan\left(\frac{1}{s}\right)\right) + 3\right] \\ &> 8\left[3 - (1+s^2)\arctan\left(\frac{1}{s}\right)\right] > 0, \end{aligned}$$

as it is easy to check in the range $1/\sqrt{3} < s < 1/\tan(\pi/8)$. Then, $\tilde{l}$ increases with s which implies that $l'(s) > \tilde{l}(1/\sqrt{3}) > 0$; hence, l increases with s yielding $l(s) > l(1/\sqrt{3}) = 4 - \frac{256\pi^3}{2187} > 0$.

Hence $\frac{\partial^2 \overline{I}_6}{\partial p^2}$ is increasing with respect to p. Thus, letting $p \to 1$, we get

$$\frac{\partial^2 \overline{I}_6}{\partial p^2} \geq 2\cos(2\theta)\arctan^2\left(\frac{1}{s}\right)(1+s^2)^3 + 2s^2(19 - s^2).$$

We notice that the right-hand side above is clearly positive when $\cos(2\theta) > 0$, whence it remains to prove it when $\theta \in (\pi/4, 2\pi/3)$, that is (taking into account that $p = 1$)

$$\frac{1}{\sqrt{3}} < s < \frac{1}{\tan(3\pi/20)}.$$

In this range,

$$\frac{\partial^2 \overline{I}_6}{\partial p^2} \geq 2\left[s^2(19 - s^2) - (1+s^2)^3 \arctan^2\left(\frac{1}{s}\right)\right].$$

Now, if $\frac{1}{\sqrt{3}} < s < 1$, we use the inequality $\arctan(1/s) \leq \frac{3s}{1+s^2}$ and we obtain that

$$\frac{\partial^2 \overline{I}_6}{\partial p^2} \geq 20s^2(1 - s^2) > 0.$$

If instead $s > 1$ we use $\arctan(1/s) \leq 1/s$ and we obtain that

$$\frac{\partial^2 \overline{I}_6}{\partial p^2} \geq \frac{2}{s^2}[s^4(19 - s^2) - (1 + s^2)^3] > 0.$$

for $1 \leq s \leq 1/\tan(3\pi/20)$.

It follows that $\frac{\partial^2 \overline{I}_6}{\partial p^2} > 0$ and the first derivative is increasing with respect to p. We again let $p \to 1$ to get

$$\frac{\partial \overline{I}_6}{\partial p} \geq 3(1 + s^2)^3 \sin\left(\frac{10}{3}\arctan\left(\frac{1}{s}\right)\right)\arctan\left(\frac{1}{s}\right) + 10s^2(4 - s^2) \tag{6.7}$$

To prove that the right-hand side above is positive, we divide the interval for s as follows

$$\left(\frac{1}{\sqrt{3}}, \infty\right) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\tan(3\pi/10)}\right) \cup \left[\frac{1}{\tan(3\pi/10)}, \frac{1}{\tan(\frac{3\pi}{20})}\right] \cup \left(\frac{1}{\tan(\frac{3\pi}{20})}, \infty\right) = J_1 \cup J_2 \cup J_3,$$

and notice that for $s \in J_2$, both terms in the right-hand side of (6.7) are positive. Also for $s \in J_1$, we easily get

$$\begin{aligned}\frac{\partial \overline{I}_6}{\partial p} &\geq 10s^2(4 - s^2) - 3(1 + s^2)^3\frac{3\pi}{10} \\ &\geq \frac{10}{3}\left(4 - \frac{1}{\tan^2(3\pi/10)}\right) - \frac{9\pi}{10}\left(1 + \frac{1}{\tan^2(3\pi/10)}\right)^3 > 0\end{aligned}$$

Finally, for $s > 1/\tan(3\pi/20)$, we can invert the sine function and the inequality to prove is equivalent to

$$\psi(s) := \frac{10}{3}\arctan\left(\frac{1}{s}\right) - \arcsin\left(\frac{10s^2(s^2 - 4)}{3(1 + s^2)^3\arctan(1/s)}\right) > 0.$$

We show that ψ is decreasing with respect to s, or equivalently $\psi'(s) < 0$. This is, after straightforward calculations, equivalent to

$$3(s^4 - 16s^2 + 8) - \arctan\left(\frac{1}{s}\right)\sqrt{9(s^2 + 1)^6\arctan^2\left(\frac{1}{s}\right) - 100s^4(s^2 - 4)^2} < 0,$$

which can be proved by estimating $\arctan(1/s)$ in the convenient way at each point, using the well-known inequalities

$$\frac{1}{s} - \frac{1}{3s^3} < \arctan\left(\frac{1}{s}\right) < \frac{1}{s},$$

we omit the details. It follows that ψ is decreasing, whence it is positive for $s \in (2, \infty)$, since $\lim_{s\to\infty} \psi(s) = 0$.

We finally obtain that $\frac{\partial \overline{I}_6}{\partial p} > 0$ for any $s > \frac{1}{\sqrt{3}}$ and $p \in (1, 2)$, hence $\overline{I}_6$ is increasing with $p \in (1, 2)$. The conclusion follows then from the fact that $\overline{I}_6|_{p=2} \equiv 0$. $\square$

Lemma 6.5. *Under the notation and conditions above, $I_2 + I_4 < 0$ for any $p \in (1,2)$ and $s \in (0, \sqrt{3/7})$.*

Proof. The conclusion is equivalent to

$$I_7 := 3(3-p)(1+s^2)^2 \sin(2\theta) + 2s(p+4)(2p-3-s^2) < 0.$$

We differentiate it with respect to p:

$$\frac{\partial I_7}{\partial p} = -3(1+s^2)^2 \sin(2\theta) + (3-p)(1+s^2)^2 \cos(2\theta)\theta_p + 2s(4p+5-s^2).$$

Since $s \in (0, \sqrt{3/7})$, we have

$$\theta = \frac{p+4}{3} \arctan\left(\frac{1}{s}\right) > \frac{5}{3} \arctan\left(\sqrt{\frac{7}{3}}\right) =: \theta_0 > \pi/2,$$

and if $\theta \in [3\pi/4, \pi]$, the first derivative is obviously positive since $\cos(2\theta) > 0$, $\sin(2\theta) < 0$ and $4p + 5 - s^2 > 9 - 3/7 > 0$. In the complementary case $\theta \in (\theta_0, 3\pi/4)$, we differentiate again with respect to p to find

$$\frac{\partial^2 I_7}{\partial p^2} = -12(1+s^2)^2 \left[\cos(2\theta) + (3-p)\sin(2\theta)\theta_p\right]\theta_p + 8s > 0,$$

since $\sin(2\theta) < 0$ and $\cos(2\theta) < 0$ in the considered range of θ. Hence $\frac{\partial I_7}{\partial p}$ increases with p for all angles $\theta \in (\theta_0, \pi)$, in particular attaining its minimum as $p \to 1$. This implies that

$$\begin{aligned} \frac{\partial I_7}{\partial p} \geq\ & -3(1+s^2)^2 \sin\left(\frac{10}{3}\arctan\left(\frac{1}{s}\right)\right) \\ & + 4(1+s^2)^2 \cos\left(\frac{10}{3}\arctan\left(\frac{1}{s}\right)\right)\arctan\left(\frac{1}{s}\right) + 2s(9-s^2). \end{aligned} \tag{6.8}$$

In order to prove that the right-hand side in (6.8) is positive, we first notice that this is trivial for $\theta_s := \frac{5}{3}\arctan(1/s) \in [3\pi/4, \pi]$, since $\sin(2\theta_s) < 0$ and $\cos(2\theta_s) > 0$. For $\theta_s \in (\theta_0, 3\pi/4)$, we drop the first term and prove the stronger fact that

$$4(1+s^2)^2 \cos\left(\frac{10}{3}\arctan\left(\frac{1}{s}\right)\right)\arctan\left(\frac{1}{s}\right) + 2s(9-s^2) > 0,$$

or equivalently

$$\frac{s(9-s^2)}{2(s^2+1)^2 \arctan(1/s)} + \cos\left(\frac{10}{3}\arctan\left(\frac{1}{s}\right)\right) > 0.$$

It is easy to check that there exists a unique $s_0 \in (0, \sqrt{3/7})$ such that

$$\frac{s(9-s^2)}{2(s^2+1)^2 \arctan(1/s)} \geq 1 \text{ for } s \geq s_0, \quad \frac{s(9-s^2)}{2(s^2+1)^2 \arctan(1/s)} < 1 \text{ for } s < s_0.$$

In the interval $[s_0, \sqrt{3/7})$, we are done. In the remaining part $s \in (0, s_0)$, we can invert the cosine function, in a similar manner as in proof of Lemma 6.4, and then prove that the resulting function is decreasing by differentiation and estimates. We skip the technical details.

Thus, I_7 is increasing with $p \in (1,2)$ and thus it is negative, since $I_7|_{p=2} \equiv 0$. □

From Lemmas 6.2, 6.3, 6.4 and 6.5 we deduce that $I_1 + I_2 + I_3 + I_4 < C_0(p) < 0$ for any $p \in (1, 2)$ fixed, $s \in (0, \infty)$, which, together with the good behavior as $s \to \infty$, end the proof. $\square$

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