A substructural solution to the Lottery and Preface Paradoxes

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Abstract The lottery [5] and the preface [7] paradoxes challenge a fundamental principle of rationality: if two propositions A and B are rationally acceptable, so is their conjunction, "A and B". The paradoxes show that there are cases in which it seems legitimate to endorse *each* proposition of a finite list, but it is not rational to endorse them *all* together. A prominent solution to the paradoxes uses substructural logics to distinguish these two senses of the universal quantifier ([12] and [8]). This paper is twofold: first, we develop this idea by giving a detailed analysis of why a logic without contraction such as linear logic is suitable for analysing the paradoxes, identifying the elements present in the paradoxes that classical logic fails to capture, and linear logic nicely formalises in its vocabulary. Second, we reinterpret Paoli and Zardini's solution pragmatically, arguing that the classical universal quantifier correctly captures the literal meaning of "all" but that linear logic captures two pragmatically enriched senses which are the default reading in some contexts, such as in the lottery of preface scenarios.

1 Introduction

The Lottery [5] and the Preface [7] paradoxes challenge a fundamental principle of rationality: if two propositions A and B are rationally acceptable, so is their conjunction, "A and B". The paradoxes show that there are cases in which it seems legitimate to endorse *each* proposition in a finite list of propositions, but it is not rational to endorse them *all* together.

Paoli [8] and Zardini [12] have argued for a distinction between two senses of the universal quantifier in a logic without contraction (such as linear logic LL or affine logic LA), which generalise two different senses of conjunction in such logics. Both

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authors argue that the two formalizations available of the universal quantifier express "any" or "each" on the one hand and "all" or "every" on the other.[1]

We will analyse this diagnosis and reinterpret it as a pragmatic enrichment of the logical vocabulary. In more detail, we will first examine why linear logic is suitable for analysing the two paradoxes, showing how a relation without contraction in which one can express a change on the premises given the conclusion is suitable for codifying the conditional probability required to solve the paradoxes. Second, we will argue that "all", "every", "any" and "each" have the same semantic meaning correctly captured by the classical universal quantifier. However, the distinction made by Paoli and Zardini is legitimate at another level: as a pragmatic enrichment of the universal quantifier in certain contexts. This interpretation gives a novel diagnostic for the two paradoxes: when formulated, their premises are strictly speaking false (that is, when formulated with their literal meaning in classical logic), although highly assertable given the particular enrichments that they exhibit, as formulated in LL, and it is this pragmatic enrichment that dissolves the paradox as we usually interpret the vocabulary in the context of the paradoxes as pragmatically enriched.

2 The paradoxes

Both the lottery and the preface paradoxes emerge from the acceptance of two principles which seem unproblematic. First, let t be some value between 0 and 1 such that 'rationally acceptable' means that the subjective probability of p (symbolised $P(p)$) is equal to or above t, $P(p) \geq t$, that is, we equate rational acceptability with attribution of high probability. We thus define the following Rational Principle:

(**R**) if $P(A) \geq t$ then it is rational to accept A,

Second, given a conjunction governed by the classical rules (such as the following presentation in the Gentzen system for LC),

$$\frac{A, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \wedge L_1 \qquad \frac{B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \wedge L_2 \qquad \frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B} \wedge R$$

we can define the Conjunction Principle (CP) of the acceptance of a conjunction of a group of propositions $\varphi_1, ..., \varphi_n$ as follows:

(**CP**) If each of the propositions $\varphi_1, ..., \varphi_n$ is rationally acceptable, so is $\varphi_1 \wedge ... \wedge \varphi_n$.

[1] Zardini endorses the distinction between "each" and "every" while Paoli codifies it as between "any" and "all". We will treat "each" as having the same sense as "any" and "every" as having the same sense as "all".

2.1 The lottery paradox

Kyburg [5] presents the lottery paradox as follows:

> Consider a fair, 1,000,000-ticket lottery, with one prize. On almost any view of probability, the probability that a given ticket (say ticket n° 1) will win is 0.000001, and the probability that it will not win is 0.999999. Surely if a sheer probability is ever sufficient to warrant the acceptance of a hypothesis, this is a case. It is hard to think of grounds on which to base a distinction between this case and the cases of thoroughly acceptable statistical hypotheses. The same argument, however, goes through for ticket n° 7, ticket n° 156, etc. In fact for any i between 1 and 1,000,000 inclusive, the argument goes through and we should rationally be entitled - indeed obligated - to accept the statement 'ticket i will not win'. A commonly accepted principle of acceptability is that if S and T are acceptable statements, then their conjunction is also acceptable. But this means that in the lottery case, since each statement of the form 'ticket i will not win' is acceptable, so is the conjunction of 1,000,000 of these statements, which is equivalent to the statement that no ticket will win in the lottery, which contradicts the statement which we initially took to be acceptable that one ticket would win the lottery. [5, p. 197]

Let $\forall$ generalize the conjunction $\wedge$. Now we can define a lottery case[2] as one which satisfies the following two facts given a set of tickets in a lottery $Lot = \{x_1, ..., x_n\}$, and being Lx defined as 'x is a losing ticket'. First, given that we can define the possibility for each Lx_i as

$$P(Lx_i) = \frac{n-1}{n}$$

assuming a value for t lower than this value, we can accept the following fact:

(i) for any ticket $x_i \in Lot$, $P(Lx_i) \geq t$

(i) together with CP, which generalises our belief about each particular ticket and conveys that Lx_i is true whatever ticket x_i one chooses, amounts to argue that it is rational to endorse the following,

$$\forall x Lx$$

We will refer to this statement as the *positive universal*. Second, it is also rational to accept the contrary of the positive universal. We know that the following is true,

(ii) $P(\neg Lx_1 \vee ... \vee \neg Lx_n) = 1$

as we know that there is a winning ticket, that is, $\exists x \neg Lx$, and hence, we also endorse the negation of the previously accepted conjunction,

$$\neg \forall x Lx$$

[2] We will define and distinguish a lottery from a preface case using the principles in [3] but diverging from them in the labels for the paradoxes.

Which we can refer to as the *negative universal*. The positive universal and the negative universal are inconsistent beliefs that we seem to endorse, so the paradox emerges.

Yet another principle constitutes a lottery paradox and distinguishes it from the preface paradox presented below. Let the conditional probability of an event x given another event y be defined as $p(x|y)$ and calculated as:

$$p(x|y) = \frac{p(x \cap y)}{p(y)}$$

In a lottery with n tickets, the probability of Lx_m given a series of already confirmed losing tickets can be calculated as follows, where $|A|$ represents the cardinality of the set A:

$$P(Lx_m|\{Lx_i, ..., Lx_j\}) = \frac{(n - |\{Lx_i, ..., Lx_j\}|) - 1}{n - |\{Lx_i, ..., Lx_j\}|}$$

Given a sufficiently high number of tickets in the set $\{Lx_i, ..., Lx_j\}$, the probability of Lx_m will be less than t, and hence:

(iii) there is some $k < n$ such that $k = |\{Lx_i, ..., Lx_j\}|$ and for all $x \in Lot$ and $x \notin \{Lx_i, ..., Lx_j\}$, $p(Lx|Lx_i, ..., Lx_j) < t$

In the lottery scenario, as we *know* that some ticket or another has the winner number, the probability of each Lx decreases as we update the number of confirmed losing tickets. And, at a limit point, it is not rational to accept, given a sufficiently high number of losing tickets, that another ticket is also a losing one.

2.2 The preface paradox

The preface paradox [7] considers the conjunction of all the sentences of a book, the author of which endorses. However, the author also believes that there has to be some false sentence in the book, that is, that conjunction of all the sentences is false, given her known fallibility. In Makinson's words,

> Suppose that in the course of his book a writer makes a great many assertions, which we shall call $s_1, \ldots, s_n$. Given each one of these, he believes that it is true. If he has already written other books and received corrections from readers and reviewers, he may also believe that not everything he has written in his latest book is true. His approach is eminently rational; he has learnt from experience. The discovery of errors among statements that previously he believed to be true gives him good grounds for believing that there are undetected errors in his latest book.
>
> However, to say that not everything I assert in this book is true is to say that at least one statement in this book is false. That is to say that at least one of $s_1, \ldots, s_n$ is false, where $s_1, \ldots, s_n$ are the statements in the book; that $(s_1 \wedge \ldots \wedge s_n)$ is false; that $\neg(s_1 \wedge \ldots \wedge s_n)$ is true. The author who writes and believes each of $s_1, \ldots, s_n$ and yet in a preface asserts and believes $\neg(s_1 \wedge \ldots \wedge s_n)$ is, it appears, behaving very rationally. Yet clearly he is holding

> logically incompatible beliefs: he believes each of $s_1, \ldots, s_n, \neg(s_1 \wedge \ldots \wedge s_n)$, which form an inconsistent set. The man is being rational though inconsistent. More than this: he is being rational even though he believes each of a certain collection of statements, which *he knows* are logically incompatible. [7, p.205]

Again, we have a positive and a negative universal, being $Book = \{x_1, \ldots, x_n\}$ the set of all the sentences written in the book and Tx defined as 'x is true',

i. for any sentence $x_i \in Book$, $p(Tx_i) \geq t$
ii. $p(\neg Tx_1 \vee \ldots \vee \neg Tx_n) \geq t$

However, there is a substantial difference between a lottery and a preface case: in the preface paradox, there is no absolute certainty of the incompatibility of the complete set of sentences, contrary to the lottery (that is, the falsity of some sentence in the book is *highly* probable, but not completely certain, as it is certain that some ticket is winning in the lottery). It seems that (iii) should be revised: while in the lottery, the confirmation that some amount of tickets are losing tickets should increase our belief that another specific ticket is a winning one, this might not be the case of the preface. Our confidence in the truth of a particular sentence might be completely unmoved given the truth of some amount of other sentences in the book (or it could even increase as the discovery that some amount of sentences is true might make us more and more confident that the rest are true too)[3]. Hence, while (i)-(ii) can be reformulated as previously (iv) should substitute (iii):

iv. for all $x_m \in Book$ and all $\{x_j, \ldots, x_k\} \subseteq Book$: $P(Tx_m | Tx_j \wedge \ldots \wedge Tx_k) \geq t$[4]

Moreover, note that a lottery can be transformed into a preface if it is a lottery in which there is *usually* but not always a winning ticket. And a preface can be transformed into a lottery if a colleague we trust tells us that she spotted one mistake in our manuscript. In other words, what distinguishes a lottery from a preface is the certainty about the truth of the negative universal, being absolute in the lottery and highly probable in the preface. In both cases, however, there is a contradiction in the positive and the negative universal being rationally acceptable.

3 Linear logic as codifying pragmatic enrichment

Our solution to the paradoxes rests on a pragmatic interpretation of the logical codification of logical connectives in substructural logics and, in particular, linear

[3] This observation does not dissolve the paradox, it makes it even more problematic: the observation affects the connection between each sentence, meaning that there is no direct mathematical explanation in terms of probabilities that explains how our confidence in some Tx_i will decrease given other confirmed true sentences. Even if there is no incompatibility in all the sentences being true, we still believe there should be some false sentence in the book, which contradicts the positive universal and needs to be explained.

[4] The formulation is in [3] for characterizing what they call *genuine paradox*, as (iv) captures the aim of any writer of a book, that is, the maximal coherence between ones claims, which nevertheless generate a paradox.

logic. This interpretation leads to a contextualist diagnostic, according to which certain pragmatic information is added to the semantic meaning of utterances. This pragmatic information is provided by the context of utterance instead of the semantic information of the words that compose the utterance.

The approach to pragmatic enrichment that we will endorse stipulates two levels of "what is said", a minimal level, in which the contribution of 'all' can be formalised with $\forall$, independently of the context of utterance, and an enriched level, that depends on the context, and in which we can distinguish between two senses of the universal quantifier codified by $\sqcap$ and $\bigwedge$. This is possible with the distinction between *completions* and *expansion* by [1]: while completions conform the step from an utterance to a complete proposition expressed by the literal meaning of the utterance components, together with the disambiguation of ambiguous terms and the interpretation of indexicals, expansion leads to a different proposition conformed by a pragmatically enriched sense of the utterance components.[5]

3.1 Linear logical consequence

The plurality of formal languages that capture different senses of the logical vocabulary emerges by the endorsement of more than one notion of *legitimacy* in the following definition of logical consequence:

Definition 1 (Logical Consequence) Δ is a logical consequence$_x$ of Γ iff there is a deduction of Δ from Γ by a finite number of legitimate$_x$ rules of inference.

The rejection/acceptance of structural rules can be interpreted as codifying different (and legitimate) notions of "follows from". Consider the following structural rules for logical consequence $\vdash$:

$$\frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\, WL \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}\, WR$$

$$\frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\, CL \qquad \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}\, CR$$

$$\frac{\Gamma, A, B, \Delta \vdash \Pi}{\Gamma, B, A, \Delta \vdash \Pi}\, EL \qquad \frac{\Gamma \vdash \Delta, A, B, \Pi}{\Gamma \vdash \Delta, B, A, \Pi}\, ER$$

We can define different notions of consequence depending on the acceptance or rejection of structural rules. We will focus on two of them for this paper: first, a logic of truth-preservation, as LC, accepts all structural rules and determines logical consequence using the following definition:

Definition 2 (Legitimacy for LC**)** A rule of inference from Γ to Δ is legitimate if it preserves truth.

[5] See also [9] and [2] for a contextualist view on the meaning of utterances in which pragmatic content contributes to determining what is said.

Second, a logic without weakening and contraction, such as linear logic, is sensitive to the repetition of the premises to derive a conclusion. An intuitive way of understanding why a consequence would be sensitive to the repetition of the premises is because obtaining the consequence modifies the premises so that they might no longer be available once the conclusion is derived. This intuition is captured in Girard's work on linear logic [4]:

> [R]eal implication is causal. A causal implication cannot be iterated since the conditions are modified after its use; this process of modification of the premises (conditions) is known in physics as reaction. [4]

We can define the reaction as follows:

Definition 3 (Reaction) A reaction is the effect that the *use* of Γ to get Δ has on Γ.

The reaction codified by $\vdash$ in LL (henceforth $\vdash_{LL}$) entails that the connection between the premises and conclusion in an inference $\Gamma \vdash_{LL} \Delta$ expresses something different to what $\Gamma \vdash_{LC} \Delta$ codifies. We will use the notion of *instability* introduced by [11] but slightly changing its definition.[6] Zardini refers to states of affairs that might change after the conclusions that follow from them are derived. These states of affairs are *unstable*:

> Stable states-of-affairs, if they obtained, would co-obtain with all of their consequences; unstable states-of-affairs would not. [11]

Two main assumptions distinguish our view from Zardini's. First, reaction and instability are interpreted here as pragmatic enrichments of the vocabulary; that is, the information that the premises might change after the conclusion obtains is pragmatically derived given the particular utterance and context. In more detail, in a sequent $\Gamma \vdash \Delta$, whether Δ is unstable with respect to Γ depends on our contextual assumptions about Δ and Γ. Moreover, and related to this first difference, while for Zardini instability is an intrinsic property of certain states of affairs, in our view, instability is a property of the consequence: in LL, the premises are unstable *with respect to* their conclusion:

Definition 4 (Instability) A logical consequence is unstable if the obtaining of the conclusion does not co-obtain with the premises.[7]

We already have the necessary ingredients to capture the sense of legitimacy of a logic that rejects both weakening and contraction:

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[6] Note that Zardini endorses the multiplicative fragment of affine logic and not linear logic, as he accepts weakening and rejects contraction. However, his observations about logical consequence (and about the universal quantifier) can also be applied to interpret linear logic, being the

[7] Leitgeb [6] uses a notion of stability to solve the paradoxes. However, although both notions could be connected, Leitgeb's stability should not be confused with the definition given here: stability is defined in Leitgeb as the property of robustness of a set of beliefs conditionalised on any other belief of an agent. In our case, the notion remains more general, as the connection according to which the premises might change given the conclusion. Although we keep it general, we will see how the lottery exhibits a precissification of this idea. Thanks to an anonymous reviewer for a comment that motivates this fn.

Definition 5 (Legitimacy for LL) A rule of inference from Γ to Δ is legitimate if it uses Γ to get Δ.

Assuming that the term *use* is accompanied by a possible change of Γ given Δ, when Δ has been used and, hence, possibly transformed.

We will see now why rejecting weakening and contraction determines a legitimate sense of "follows from" and how it is compatible with the classical truth-preserving relation.

Given Instability, Weakening and Contraction should be rejected. First, weakening would express that irrelevant information would entail the false idea that there is a reaction of an irrelevant conclusion / on an irrelevant premise. However, it is acceptable for codifying truth preservation.

$$\frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\, WL \qquad\qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}\, WR$$

Second, also contraction is rejected for the linear reading of $\vdash_{LL}$ (but not for truth-preservation). Modifying the analysis in [13], we can say that if $\vdash_{LL}$ expresses an unstable connection between A, A and B ($A, A \vdash B$), B may hold because using A twice leads to B because using A once leads to the situation in which using A again leads to B, but A does not co-obtain with this intermediate situation.

$$\frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\, CL \qquad\qquad \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}\, CR$$

Both $\vdash_{LC}$ and $\vdash_{LL}$ are legitimate codifications of consequence, being $\vdash_{LC}$ the literal meaning of "follows from", while we enrich logical consequence with some *reaction* and *instability* in some contexts. Consider, for instance, the following two utterances:

1. If I have 280 billion euros, I can buy the biggest palace ever constructed.
2. If I have 280 billion euros, I can appear on the Forbes list.

Or consider also,

3. If this piece of wood makes one chair, then it makes one bed. [11]
4. If this piece of wood makes one chair, I will buy it.

It is the context, together with the non-logical vocabulary used, which makes us interpret "if...then" differently in each case. Note that in 1 and 3, the conditional expresses an unstable connection between antecedent and consequent - buying the biggest palace ever constructed has a reaction in the amount of money I will have afterwards, and making one bed has a reaction on the capacity of also making one chair. However, in 2 and 4, such a reaction is not present, as appearing in the Forbes list and buying the piece of wood can coexist with the antecedent of the conditionals. But rather than being a semantic dimension of the vocabulary used, it is our knowledge about how money works, or how building a chair affects a piece of wood, that makes us pragmatically derive the information that the money is wasted after one gets the palace or the wood is not available anymore after building the chair. The literal meaning of the vocabulary does not require this information, as suggested by examples 2 and 4.

3.2 Linear Conjunction

The classical connectives in substructural logics split into two counterparts, as there is more than one (equivalent) way in which they can be formalised in the full-structural logic LC, but these different formulations are not equivalent when some structural rule is rejected. Hence, in linear logic, the classical conjunction $\wedge$ splits into two. We refer to $\sqcap$ as the additive conjunction and $\otimes$ as the multiplicative conjunction in LL:

$$\frac{A, \Gamma \vdash \Delta}{A \sqcap B, \Gamma \vdash \Delta}\,\sqcap L_1 \qquad \frac{B, \Gamma \vdash \Delta}{A \sqcap B, \Gamma \vdash \Delta}\,\sqcap L_2 \qquad \frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \sqcap B}\,\sqcap R$$

$$\frac{A, B, \Gamma \vdash \Delta}{A \otimes B, \Gamma \vdash \Delta}\,\otimes L \qquad \frac{\Gamma \vdash \Delta, A \qquad \Pi \vdash \Sigma, B}{\Gamma, \Pi \vdash \Delta, \Sigma, A \otimes B}\,\otimes R$$

In [10], I have argued that the division of the classical conjunction into two different connectives can also be interpreted pragmatically, and these different interpretations of "and" will explain the solution to the paradoxes. Hence, let us see how to distinguish the two senses of "and" in linear logic.

Observing first the left rules for each conjunction, we can distinguish two different reasons to assert "if (A and B) then C":

- Additive reasons (A-reasons): using A (or using B) one gets C (formalized as $\sqcap$).
- Multiplicative reasons (M-reasons): using A together with B one gets C (formalized as $\otimes$).

Second, observing the right rules now, we can distinguish two different reasons to assert "if C then (A and B)":

- A-reasons: using C one gets any among A and B (formalized as $\sqcap$).
- M-reasons: using C one gets both A and B (formalized as $\otimes$).

Finally, observe that the pragmatic enrichment of the conjunction depends on the content and the enrichment of $\vdash$. Consider the following two utterances:

5. If one pays 5 Euro, one gets a pack of camels and a pack of Marlboros (both) (formalised as $\otimes$).

6. If one pays 5 Euro, one get a pack of camels and a pack of Marlboros (any) (formalised as $\sqcap$).

Or consider also,

7. This piece of wood makes one chair and makes one bed (both) (formalized as $\otimes$).
8. This piece of wood makes one chair and makes one bed (any) (formalized as $\sqcap$).

In a context in which one states something Δ ("I can get a pack of Camels", or "I can make one chair") on the grounds of some state of affairs Γ, and the connection between them is unstable, the conjunction can be interpreted in two different ways: as an accumulation ("both") or as the possibility of a choice ("any").

In sum, at least three different legitimate codifications of "and" coexist: $\wedge$ captures the literal meaning of 'and', required by the classical codification of consequence, $\otimes$ captures a pragmatic enrichment of 'and' that is required by the linear codification of consequence, and $\sqcap$ captures the non-enriched conjunction in a linearly enriched context (and, given the pragmatic enrichment of 'and' in the contexts in which $\vdash$ is linearly enriched, $\sqcap$ is sometimes expressed by 'or' in natural language).

4 Pragmatic solution to paradoxes

We now have all the necessary elements to diagnose and solve the lottery and preface paradoxes. We will first review the solution that we find in [8] and [12], and second we will pragmatically reinterpret the diagnostic. This pragmatic variation will generate a novel Conjunction Principle.

4.1 The paradoxes in a logic without contraction

Following [8] and [12], we can distinguish two senses of the universal quantifier in LL, which correspond to the two different senses of "and": the generalisation of the extensional conjunction $\sqcap$ is formalised as Π and the generalisation of the intensional conjunction $\otimes$ is formalised as Λ. This distinction gives us a framework for analysing the lottery and preface paradoxes. But contrary to Zardini and Paoli, we argue that this distinction is pragmatic rather than semantic since we do not assume that substructural languages capture the semantic meaning of logical connectives. Substructural vocabulary captures different senses of logical connectives resulting from pragmatic enrichments, while LC captures their literal meaning.

The two senses of the universal quantifier that Π and Λ can be interpreted in natural language: Π expresses 'each' or 'any', while Λ expresses 'all' or 'every'. With a restriction to denumerable domains, we can set the following rules for the quantifiers from [11],

$$\frac{\Gamma, \phi(t) \vdash \Delta}{\Gamma, \Pi x\phi \vdash \Delta}\ \Pi L \qquad\qquad \frac{\Gamma \vdash \Delta, \phi(a)}{\Gamma \vdash \Delta, \Pi x\phi}\ \Pi R$$

$$\frac{\Gamma, \phi(t_1), \phi(t_2), \ldots \vdash \Delta}{\Gamma, \bigwedge x\phi \vdash \Delta}\ \bigwedge L \qquad\qquad \frac{\Gamma \vdash \Delta, \phi(t_1) \qquad \Gamma' \vdash \Delta, \phi(t_2) \qquad \ldots}{\Gamma, \Gamma', \ldots \vdash \Delta, \Delta', \ldots, \bigwedge x\phi}\ \bigwedge \mathrm{R}$$

Now we can formulate the paradoxes using the enriched language of LL. First, Zardini argues that the positive universal should be formalised with the extensional universal,

$$\Pi x Lx$$

And the negative universal should be formalized with the intensional universal,

$$\neg \bigwedge x Lx$$

Given that in a logic without contraction $\Pi x Px \nvdash \bigwedge x Px$, one can endorse both the positive and the negative universal. Hence, the solution to the paradoxes is the rejection that the union of each particular belief (about a ticket or a sentence) entails 'every' belief (but it entails 'any'); and given that 'any' does not entail 'every', there is no contradiction in one's beliefs.

Paoli has a similar diagnostic, in this case relying on an ambiguity exhibited by the classical universal quantifier. The author argues that there is no contradiction between 'for any x, Lx' and 'there is an x, $\neg Lx$', which is equivalent to 'not for every x, Lx'. Paoli presents the paradox as a contradiction between the two existential quantifiers (Σ and $\coprod$) that connect to the universal quantifiers as follows:

- $\Pi x Lx \Leftrightarrow \neg \coprod x \neg Lx$: expresses that there is no particular x such that $\neg Lx$,
- $\neg \bigwedge x Lx \Leftrightarrow \Sigma x Lx$: expresses that there is a connection between each ticket x such that if the first is a losing ticket then, if the second is a losing ticket then... the last one is not a losing ticket Lx.

Given that the negative universal is asserted because some ticket has to be the winning ticket ($\Sigma x \neg Lx$) and the positive universal is asserted because each particular ticket has an extremely low probability of being the winning ticket ($\coprod x \neg Lx$), and given that $\coprod x \neg Lx \nvdash \neg \Sigma x \neg Lx$, there is no contradiction between these two claims.

4.2 Pragmatic variation

The pragmatic solution to the paradoxes rests on two facts. First, $\Pi x Px \nvdash \bigwedge x Px$ only holds when $\vdash$ is unstable, which is a pragmatic enrichment of "follows from". Second, both the lottery and the preface scenarios determine a context in which the vocabulary is pragmatically enriched.[8]

[8] Leitgeb [6] defends a solution to the paradox that relies on contextualism. We also hold a contextualist view but rooted in different principles than Leitgeb's. In his view, contextualism

To see why a pragmatic interpretation of the vocabulary gives a better diagnosis of the paradoxes, we need to understand the role that *instability* and *reaction* play in their formal analysis. An important observation to be made is that there are situations in which we learn some property for *each* member of a particular domain, and we can safely conclude that the property holds for *all*, and vice-versa, as will become clear with the analysis of the lottery below.

4.2.1 The Lottery

Recall that a Lottery case is composed by the following three conditions:

(i) for any ticket $x_i \in Lot$, $P(Lx_i) \geq t$
(ii) $p(Lx_1 \wedge ... \wedge Lx_n) < t$
(iii) there is some $k < n$ such that $k = |\{Lx_i, ..., Lx_j\}|$ and for all $x_m \in Lot$ $P(Lx_m|Lx_i, ..., Lx_j) < t$

We will derive, as we did before within the classical framework, the positive and negative universal that give rise to the paradox. Let Γ be the grounds we have for deriving each Lx_i. Principle (i) justifies the following:

- $\Gamma \vdash \Pi x Lx$

Turning our attention to (iii), we know that given a hypothetical confirmation of each Lx_i, there is a reaction on Γ such that it might not be the case anymore that it is rational to derive that another ticket is a losing ticket. In particular, we know that $P(Lx_i|\{Lx_j, ..., Lx_k\}) < P(Lx_i)$, and not only this, but for whatever threshold that we put for rationality via the value t, we can calculate the number of tickets required for a lottery of n numbers such that:

$$P(Lx_i|\{Lx_j, ..., Lx_k\}) < t$$

Hence, even if it it is rational to believe $Lx_j \wedge ... \wedge Lx_k$ and it is rational to believe Lx_i, CP fails for the conjunction $Lx_j \wedge ... \wedge Lx_k \wedge Lx_i$.

Linear logic contains the appropriate vocabulary to capture these facts. In particular, the multiplicative universal that requires that each element of the domain is derived from Γ together does not hold,

- $\Gamma \vdash \neg \bigwedge x Lx$

determines the number of possible tickets/sentences being considered in the scenario in order to determine how stable it is to assert that a ticket is losing or a sentence is true. In the present proposal, contextualism plays the role of determining whether a given scenario is such that it requires the linear enrichment of the vocabulary, given the assumed reaction of conclusions on the premises that derived them. Thanks to an anonymous reviewer for a comment that motivates this fn.

The fake lottery

Contrast the lottery scenario with the following fake lottery, in which there is no reaction on Γ given some L_i, and hence the dependency expressed by $\neg\bigwedge Lx$ is not present (to see that $\Gamma \vdash \Pi Lx$ and $\Gamma \vdash \bigwedge xLx$). Consider a lottery with 1,000,000 tickets numbered with odd numbers. Imagine that we know that *each* number in the lottery drum has been substituted by its immediate successor in the series of natural numbers and that we have learned this fact for each particular ticket and not as a general change. In this case, one can legitimately accept ΠxLx, and, contrary to the classical lottery, one should also accept that *all* tickets are losing tickets, that is, $\bigwedge xLx$ because of the lack of reaction on Γ.

4.2.2 The Preface

Recall that a Preface case is a situation in which the following holds:

i. for any sentence $x_i \in Book$, $p(Tx_i) \geq t$
ii. $P(Tx_1 \wedge \ldots \wedge Tx_n) < t$
iv. for all $x \in Book$ and all $\{x_j, \ldots, x_k\} \subseteq Book$: $p(Tx|Tx_j \wedge \ldots \wedge Tx_k) \geq t$

The solution to the lottery rested on the claim that we should not accept the multiplicative universal given that there is a reaction on the grounds for asserting each conjunct which does not permit to endorse all of the conjuncts *together*. However, the preface paradox seems to escape this diagnosis because there is not necessarily a reaction on Γ given some amount of Tx, or at least it is not as immediate as in the case of the lottery, a contrast captured by (iv).

Recall that the difference between the lottery and the preface is that the negative universal of the lottery paradox has the maxim probability 1, contrary to the negative universal of the preface. But this mismatch can be easily solved if we consider that the negative universal is just another rational belief among all the beliefs about each sentence's truth value in the preface. We might be as convinced about the truth of each sentence in the book as about the truth of the negative universal. And here lies the solution: $\neg\bigwedge xTx$ can be added to the list of our rational beliefs, together with Tx_1, Tx_2, etc. In particular, our set of beliefs can be formalised as follows:

- $\Gamma \vdash \Pi xTx \sqcap \neg\bigwedge Tx$

Or, to put things more simply,

- $\Gamma \vdash T1 \sqcap T2 \sqcap \ldots \sqcap Tn \sqcap \neg\bigwedge xTx$

Given this, and contrary to the lottery, we might become as suspicious about the negative universal as about the truth value of the rest of the book's sentences after a certain number of sentences are shown to be true. That is, given a sufficiently significant amount of confirmed true sentences, there is a reaction either on the negative universal or on the set of the book's sentences. In more detail, given a sufficiently significant number of confirmed true sentences, either the following holds:

- $P(\neg \bigwedge x Tx | Tx_1, Tx_2, ...Tx_n) < t$

Or if, on the contrary, we are completely convinced that there is some false sentence in the book, we should add the negative universal to our list of confirmed sentences. Given that this belief, together with some amount of true sentences, might have some reaction over our conviction of other sentences, we should endorse:

- $P(Tx_i | \neg \bigwedge x Tx, Tx_1, Tx_2, ...Tx_n) < t$

Again, none of these considerations about the reaction that the confirmation of different sentences of the book has on Γ is part of the meaning of $\vdash$ or 'and'. Rather, it is a pragmatic enrichment of the vocabulary induced by the context.

4.3 The Conjunction Principle revised

Recall that the Lottery and Preface paradoxes emerged from the acceptance of the following two principles:

(**R**) if $P(A) > t$ then it is rational to accept A,
(**CP**) Conjunction Principle: If each of the propositions $\varphi_1, ..., \varphi_n$ are rationally acceptable, so is $\varphi_1 \wedge ... \wedge \varphi_n$.

In light of the linear formalisation of the two universal quantifiers, (CP) should be reconsidered, as one can only conjoin $\varphi_1, ..., \varphi_n$ if there is no reaction on the grounds for asserting each one of them given some amount of $\varphi_1, ..., \varphi_n$. That is:

Definition 6 (Conjunction Principle revised (CP')) If each of the propositions $\varphi_1, ..., \varphi_n$ are rationally acceptable given a background Γ, then,

i. if each φ_i is stable with respect to Γ, then $\varphi_1 \wedge ... \wedge \varphi_n$ is rationally acceptable,
ii. if some φ_i is unstable with respect to Γ, then
    - $\varphi_1 \sqcap ... \sqcap \varphi_n$ is rationally acceptable
    - $\varphi_1 \times ... \times \varphi_n$ is rationally acceptable if the reaction of φ_i does not rule out that $\varphi_1, ..., \varphi_n$ can also be deduced from Γ.

Condition (i) corresponds to the classical formalisation, required in contexts where there is no assumption of a possible reaction between a statement and the grounds that lead to it. On the other hand, (ii) is the condition to be imposed in a context where we assume a reaction between what is asserted and its grounds, so we require that this reaction does not rule out the possibility of conjoining two or more incompatible (due to instability) assertions.

Stability and reaction might not necessarily be calculated probabilistically. There can be other reasons for the dependency between conjuncts which requires the linear interpretation of logical vocabulary. But the probabilistic nature of the lottery and the preface presented in this paper is one particular instantiation of these terms. In particular, the equation that calculates the probability of some $x_m \in A$ being also B, where A is a size n set and $B \subset A$, where we know that there is one $a \in A$, but $a \notin B$ is:

$$P(Bx_m|\{Bx_i, ..., Bx_j\}) = \frac{(n - |\{Bx_i, ..., Bx_j\}|) - 1}{n - |\{Bx_i, ..., Bx_j\}|}$$

The fact that Bx_m is affected by the other elements of B confirms that the lottery exhibits a reaction, so linear logic is a good candidate to distinguish two senses of the universal quantifier in that context. The same is true for the preface, where A does not only include all the sentences of a book but also the negative universal quantifier.

In contexts in which reaction is not granted and where there is no incompatibility between several conjuncts being true together (condition (i) in definition 4.1), the classical vocabulary is suitable to formalise the universal quantifier, codifying its literal sense, where “each”, “any”, “every”, and “all” express the same content.

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