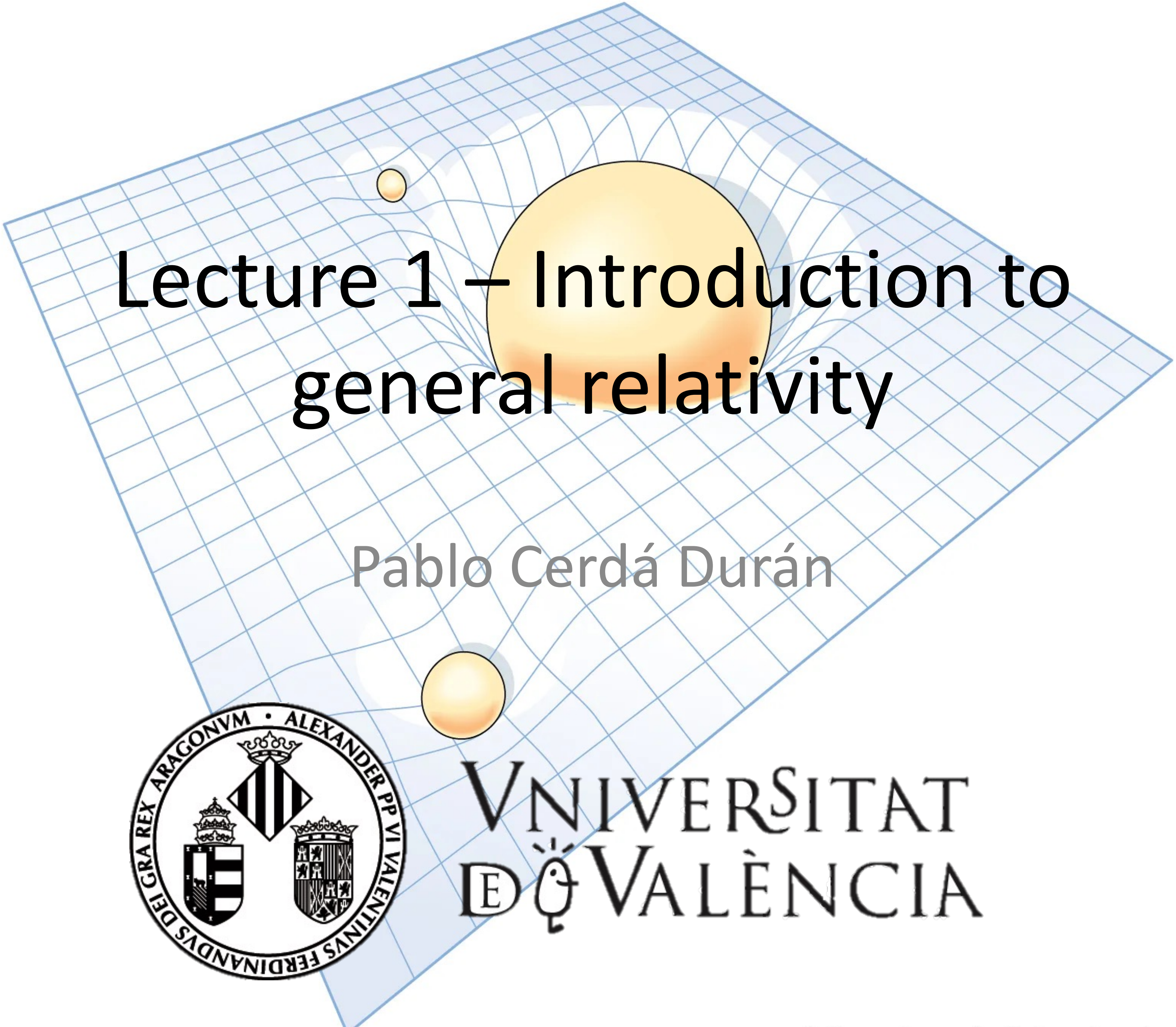

Lecture 1 – Introduction to general relativity

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Lecture 1 - content

1.1 Introduction

1.2 Special relativity

1.3 Equivalence principle

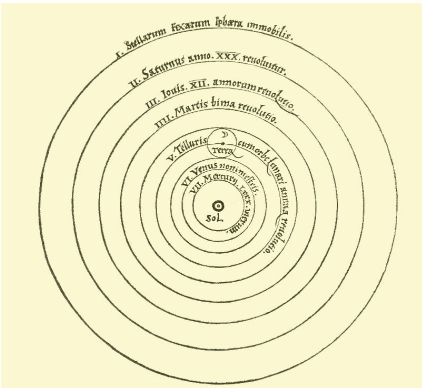
1.4 Curved manifolds

1.5 Einstein equations

1.6 Observers in a curved space-time

1.1 INTRODUCTION

1.1.1 Relativity in classical physics



Heliocentric model: Copernicus (1473-1543)

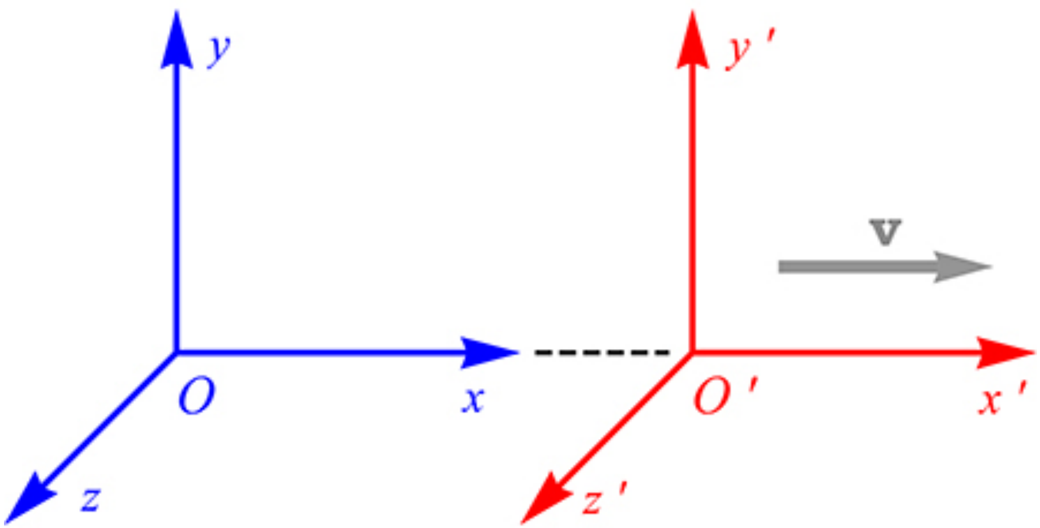
Galilean principle of relativity: “It is not possible to measure, through an experiment, the absolute speed of an observer." (Galileo, 1564-1642)

Law of inertia: “Every body continues in its state of rest or uniform motion in the absence of external forces” (Newton, 1643-1727)

1.1.1 Relativity in classical physics

Invariance under velocity transformation

$$\vec{v}(t) \rightarrow \vec{v'}(t) = \vec{v}(t) - \vec{v}_{\mathcal{O}\mathcal{O}'}$$ (Galilean law of velocity addition)



- Newton's laws of dynamics and of gravitation are invariant under such transformations
- Inertial observer (classical definition): not subject to any acceleration

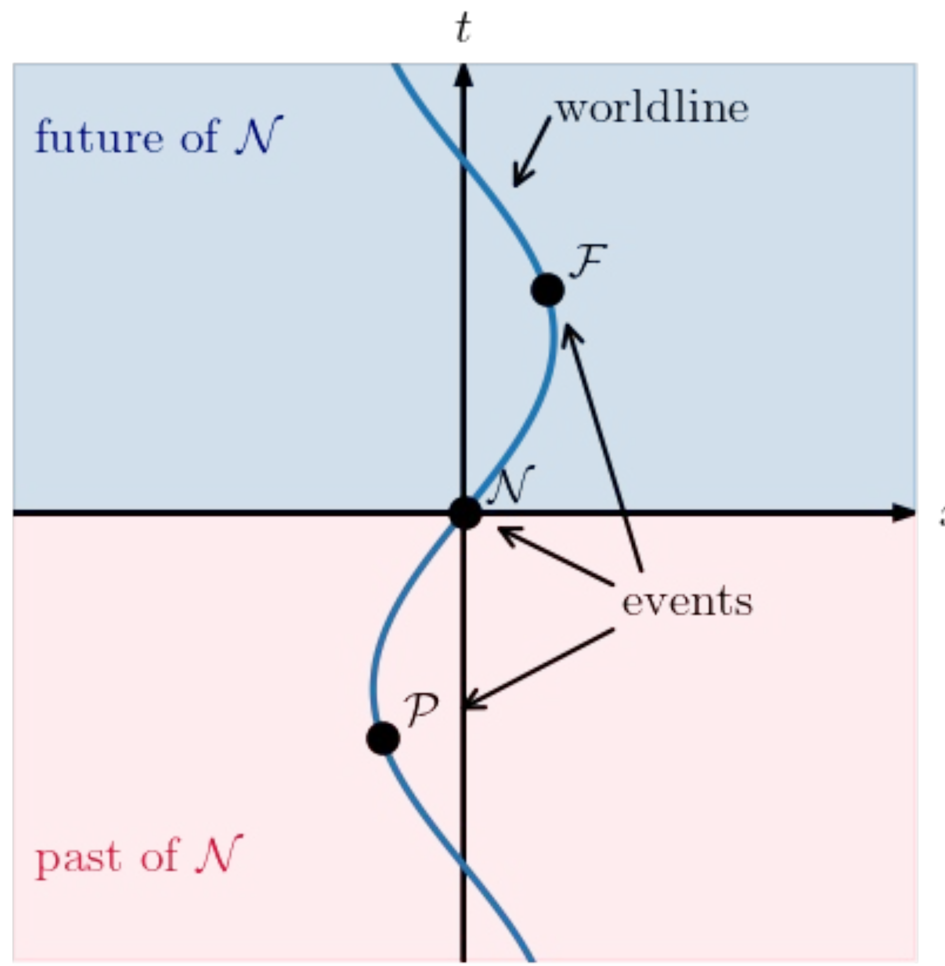
1.1.2 Pre-relativistic structure of spacetime: definitions

Observer / reference frame / frame capable of measuring time (t) and space location (x,y,z) at any point.

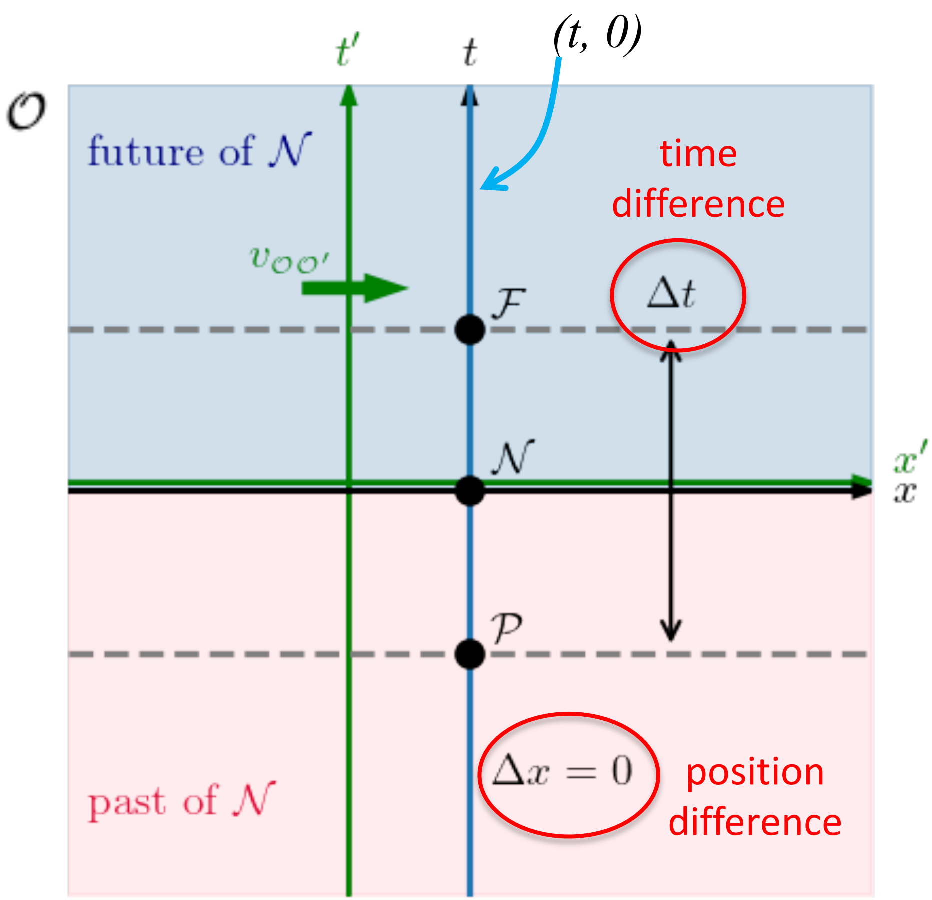
Event: point in the spacetime.

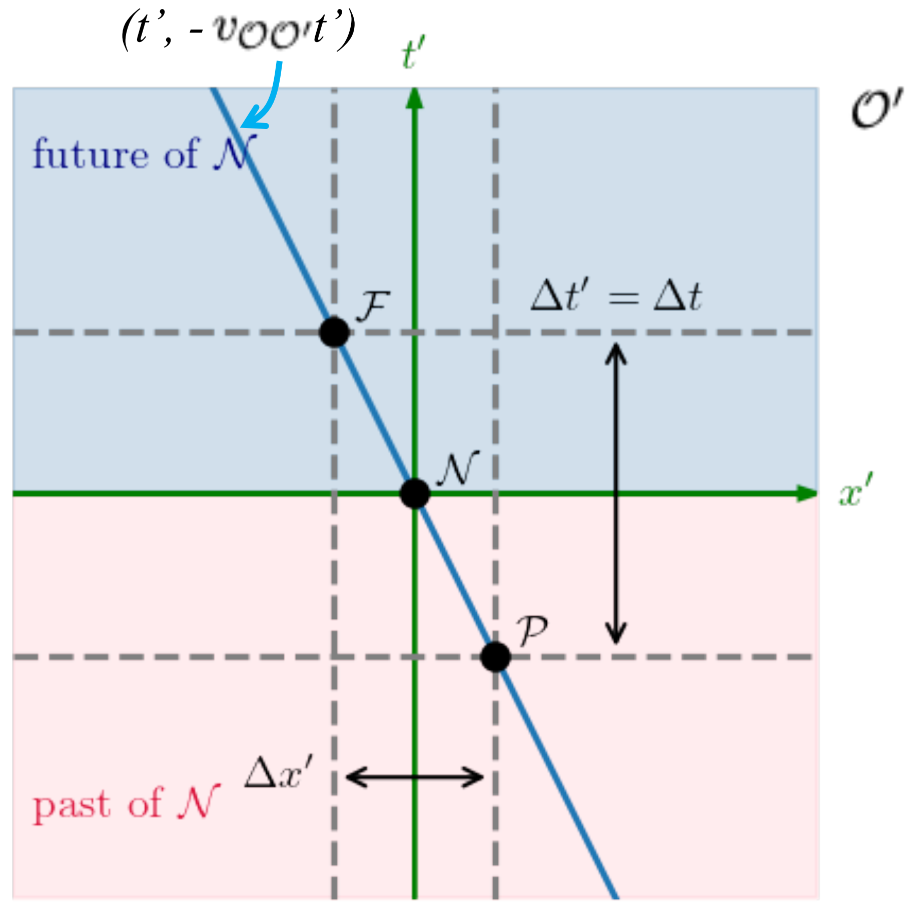
Worldline: collection of events representing a particle.

Curve: Any collection of events parametrized by a single parameter.



1.1.2 Pre-relativistic structure of spacetime: particle in absence of forces

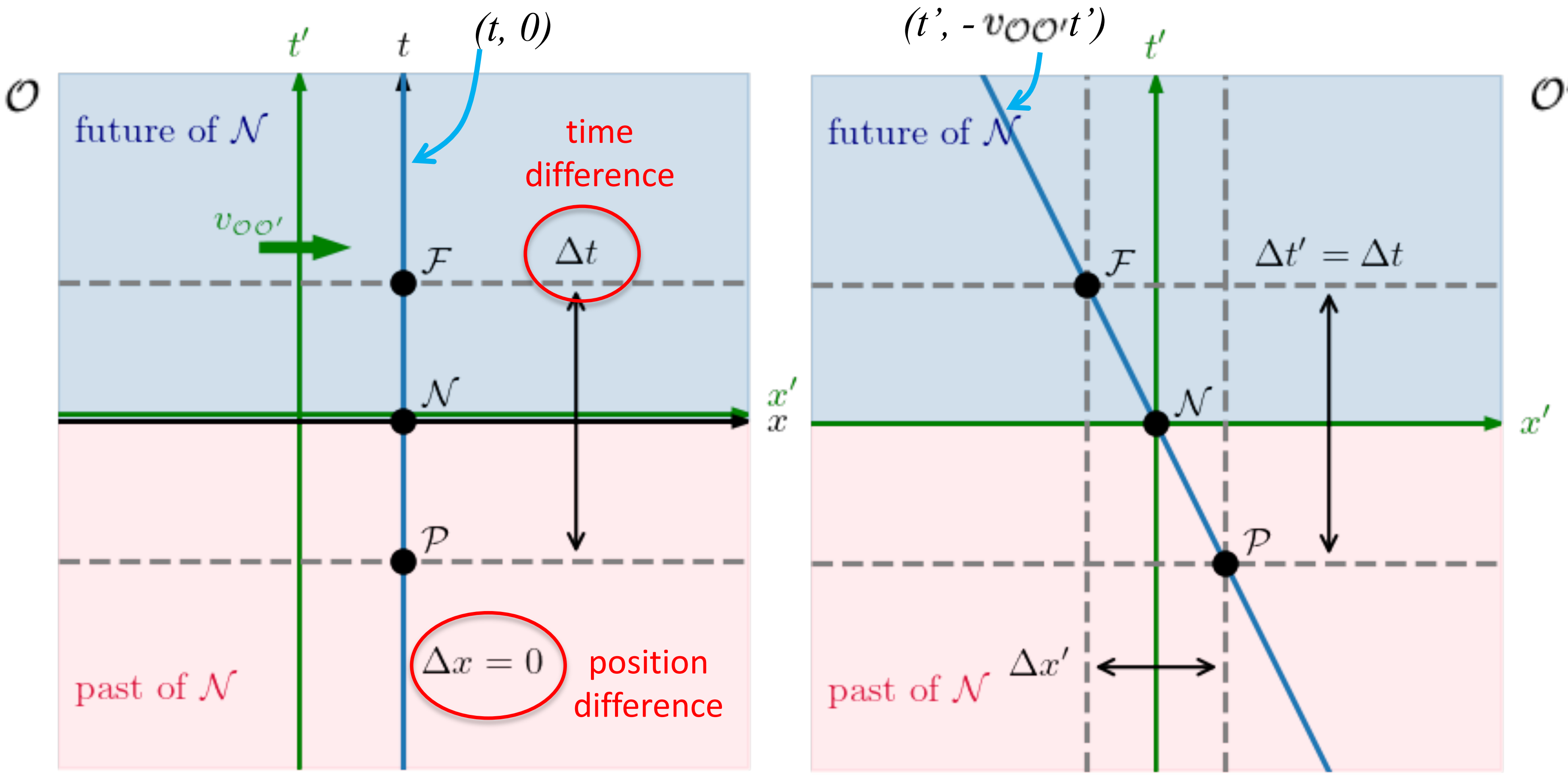




$\mathcal{O}$: Inertial observer for which the particle is at rest

$\mathcal{O}'$: Inertial observer for which the particle moves at constant velocity $v_{\mathcal{O}\mathcal{O}'}$

1.1.2 Pre-relativistic structure of spacetime: particle in absence of forces



$$\boxed{\begin{aligned} t' &= t, \\ x' &= x - v_{\mathcal{O}\mathcal{O}'}\, t. \end{aligned}}$$

Coordinate transformation

$\mathcal{O} \rightarrow \mathcal{O}'$

Equivalent to the Galilean relativity principle

- Same time interval (all clocks tick at the same time)

$$\Delta t' = \Delta t.$$

- Different spatial interval (but same ruler)

$$\Delta x = 0 \neq \Delta x' = -v_{\mathcal{O}\mathcal{O}'}\Delta t'$$

1.1.3 Electromagnetism and relativity

Maxwell equations (1861-62)

$$
\begin{aligned}
\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} - \vec{\nabla} \times \vec{B} &= -4\pi \vec{j}, \\
\frac{\partial \vec{B}}{\partial t} - \vec{\nabla} \times \vec{E} &= 0, \\
\vec{\nabla} \cdot \vec{E} &= 4\pi \rho_q, \\
\vec{\nabla} \cdot \vec{B} &= 0,
\end{aligned}
$$

Equations in vacuum $(\rho = 0 \text{ and } \vec{j} = 0)$

$$
\boxed{\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} - \vec{\nabla}^2 \vec{E} = 0 \quad , \quad \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} - \vec{\nabla}^2 \vec{B} = 0,}
$$

- Electromagnetic waves predicted (propagating at c)
- Experimental demostration by Hertz (1887)

1.1.3 Electromagnetism and relativity

Maxwell equations (1861-62)

$$\begin{aligned} \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} - \vec{\nabla}\times\vec{B} &= -4\pi\vec{j}, \\ \frac{\partial \vec{B}}{\partial t} - \vec{\nabla}\times\vec{E} &= 0, \\ \vec{\nabla}\cdot\vec{E} &= 4\pi\rho_q, \\ \vec{\nabla}\cdot\vec{B} &= 0, \end{aligned}$$

→ Not invariant under Galilean transformations

$$\begin{aligned} t' &= t, \\ x' &= x - v_{OO'}\,t. \end{aligned}$$

- Are the equations not universal? Only valid on a specific frame?
- What is this special frame (absolute reference)?
- Luminiferous ether?

Experimental search for ether:

- Effect in stellar aberration (Fresnel)
- Michelson and Morley experiment (1887)

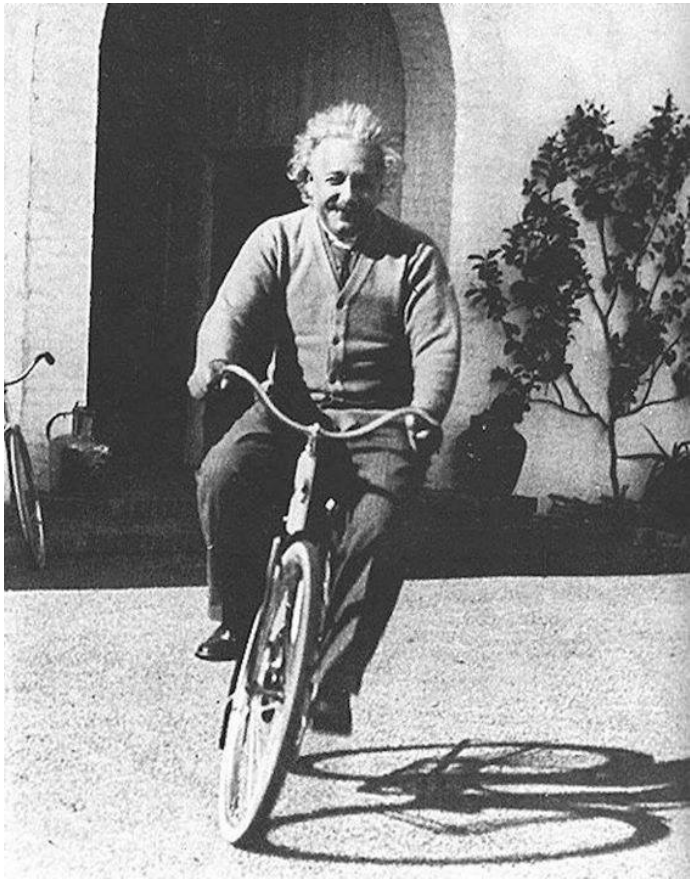
Explanation in terms of ether-matter interaction:

- Lorentz transformations (1892)
- Poincaré transformations (1900-1905)

Equations in vacuum $(\rho = 0 \text{ and } \vec{j} = 0)$

$$\boxed{\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} - \vec{\nabla}^2\vec{E} = 0 \quad , \quad \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} - \vec{\nabla}^2\vec{B} = 0,}$$

- Electromagnetic waves predicted (propagating at c)
- Experimental demostration by Hertz (1887)



1.2 SPECIAL RELATIVITY

Special relativity

A. Einstein (1879-1955)

What would be riding a beam of light? (~1895)

Special relativity

A. Einstein (1879-1955)

What would be riding a beam of light? (~1895)

Galilean relativity
+
E. Mach ideas on
absolute space

Universality of
Maxwell equations
+
No absolute space

Special relativity

A. Einstein (1879-1955)

What would be riding a beam of light? (~1895)

Galilean relativity
+
E. Mach ideas on
absolute space

Universality of
Maxwell equations
+
No absolute space

Principles of special relativity

1. No experiment can measure an observer;s absolute velocity (principle of relativity)

2. The speed of light in vacuum is constant.

(for now we consider the case without gravity)

1.2.1 Observers

Observer (reference frame): coordinate system (t,x,y,z) equipped with a set of rulers and clocks to measure the position and time at each point (evant) of spacetime.

Inertial observer (special relativity, i.e. absence of gravity):

1. The distance between two points is independent of time.
2. Clocks at different points in spacetime are syncronized and run at the same rate.
3. The geometry of space at any time is Euclidean.

$\Downarrow$

Inertial observer = non-accelerated observer (special relativity)

Notes:

- If not indicated explicitely, all observers will be inertial observers, for the moment.
- Observer is not just the person/particle observing, but the whole coordinate system built around it.

1.2.2 Worldlines

Note: we use units of c=1

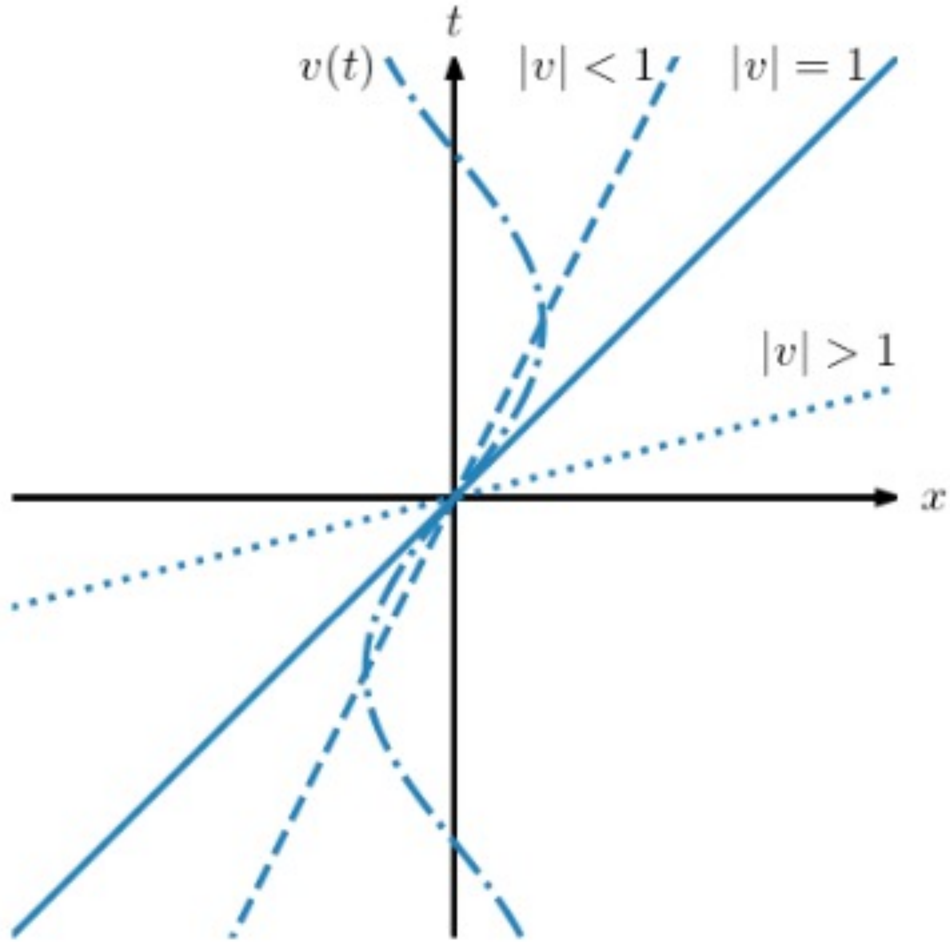
Worldline ($\mathcal{P}(\lambda)$) : series of causally connected points parametrized by a parameter λ.

Consequences of the principles of special relativity:

- The slope of the worldline is related to the speed of the particle

$$dt/dx = 1/v.$$

- The worldline of a light ray is a straight line of slope 1, for every observer.

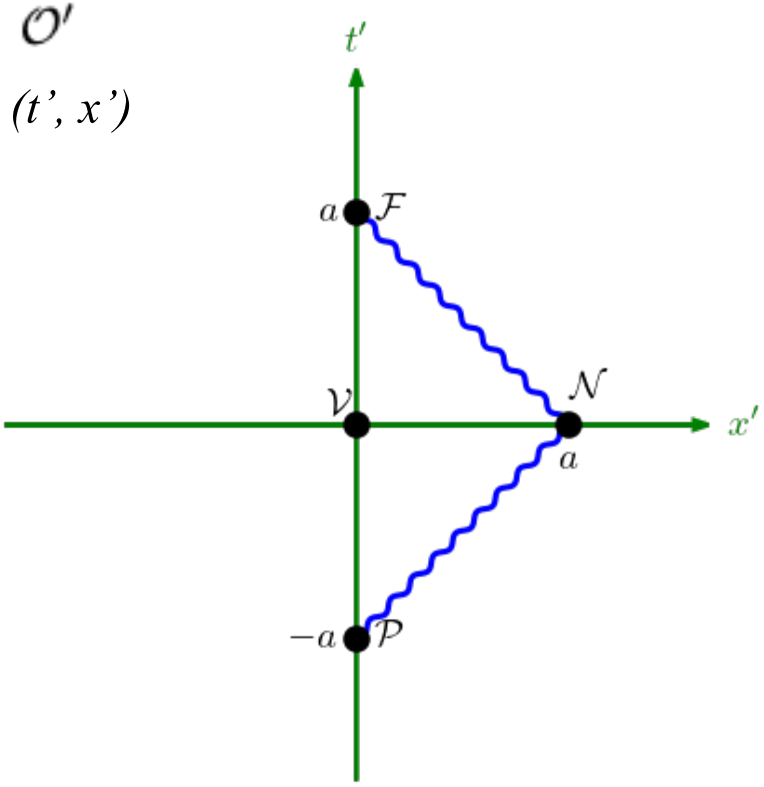


1.2.3 Coordinates observed by a different inertial observer

$\mathcal{O}'$ moves with a speed v (to the right) with respect to $\mathcal{O}$

$\mathcal{O}'$

(t', x')



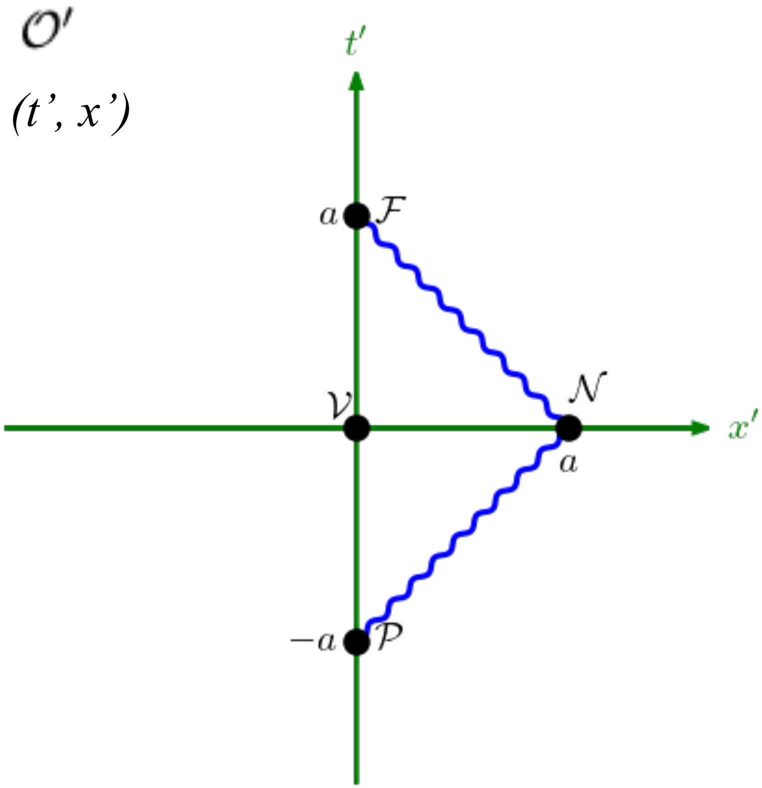
t' axis: events with $x' = y' = z' = 0$

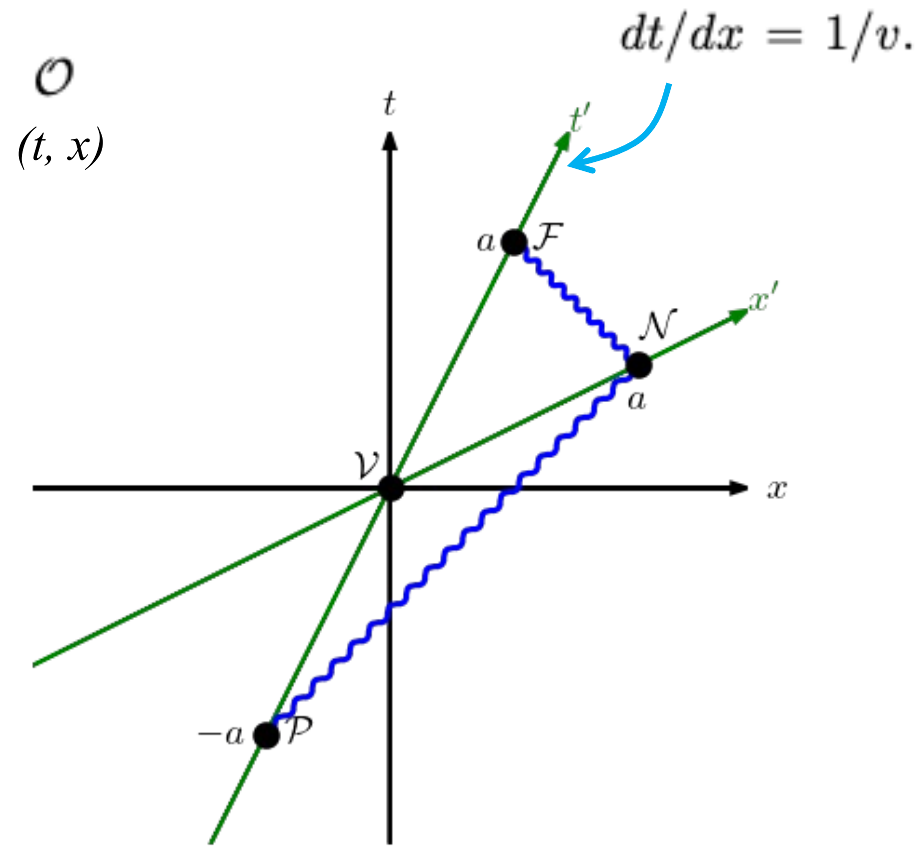
(worldline of a particle at rest for $\mathcal{O}'$)

x' axis: events with $t' = y' = z' = 0$

1.2.3 Coordinates observed by a different inertial observer

$\mathcal{O}'$ moves with a speed v (to the right) with respect to $\mathcal{O}$





t' axis: events with $x' = y' = z' = 0$
(worldline of a particle at rest for $\mathcal{O}'$)

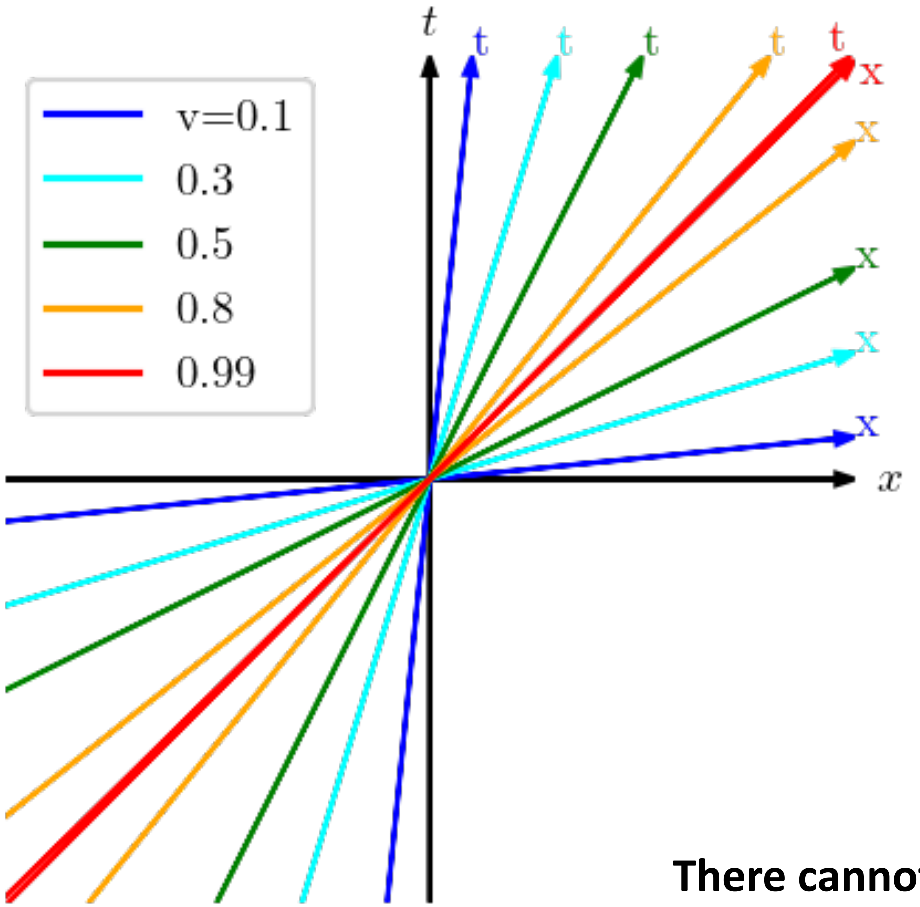
x' axis: events with $t' = y' = z' = 0$

t' axis: wordline of a particle moving at a speed v with respect to $\mathcal{O}$

x' axis: line connecting $\mathcal{V}$ and $\mathcal{N}$

y' , z' axis = y, z axis

1.2.3 Coordinates observed by a different inertial observer



Series of observers (colored axes) at different velocities with respect to $\mathcal{O}$

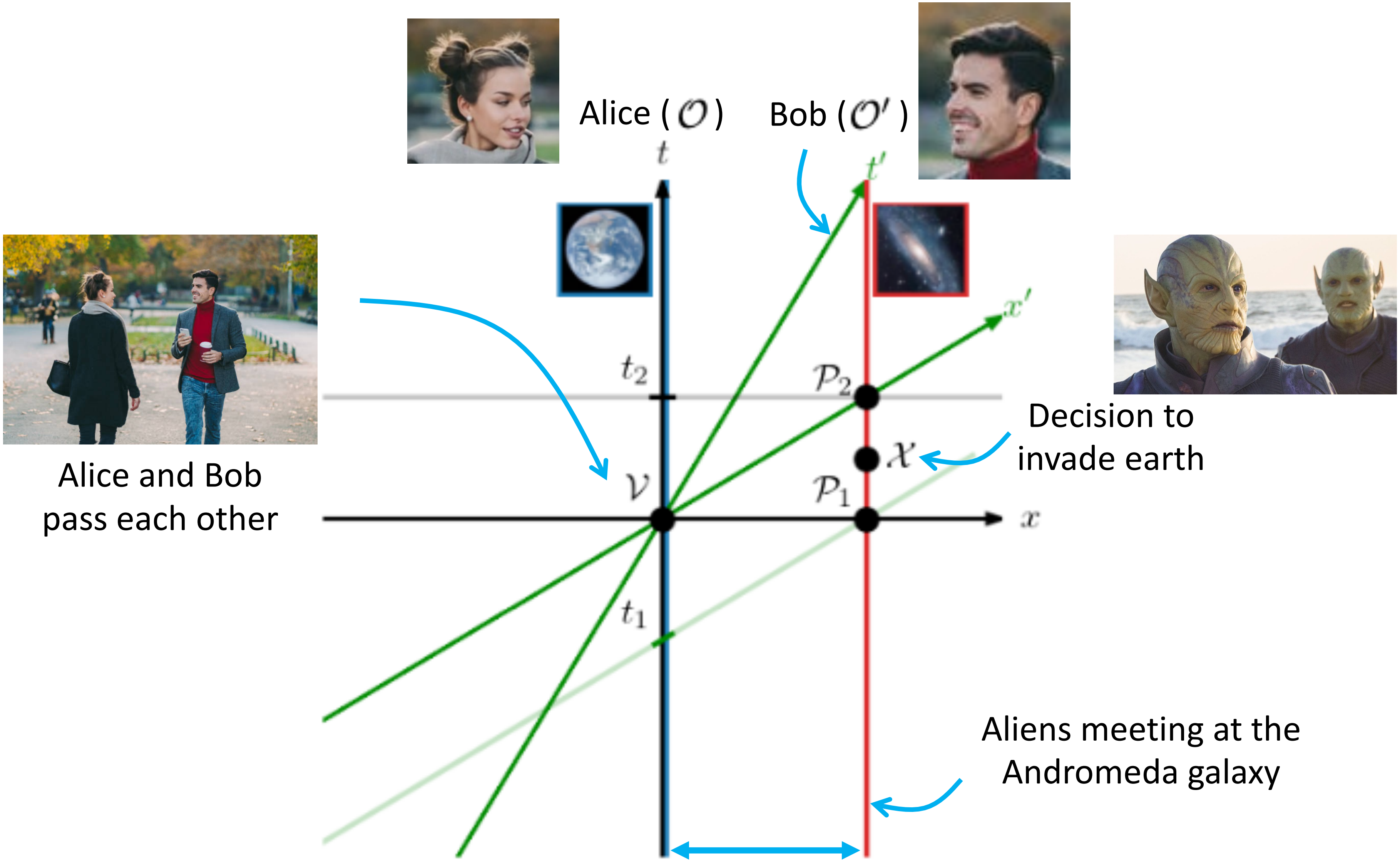
$v \rightarrow 1$: x' and t' axes converge

$v=1$: x' and t' axes coincide: coordinate system not well defined

$v>1$: Not possible to contruct the x' axis (no possible photon trajectories between $\mathcal{N}$ and $\mathcal{F}$.

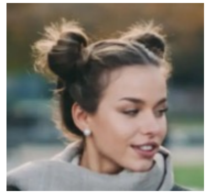
There cannot exist an inertial observer moving at a speed $v \geq 1$ relative to any other given one.

1.2.4 Simultaneity



Spacetime diagram for the observer $\mathcal{O}$ (Alice).

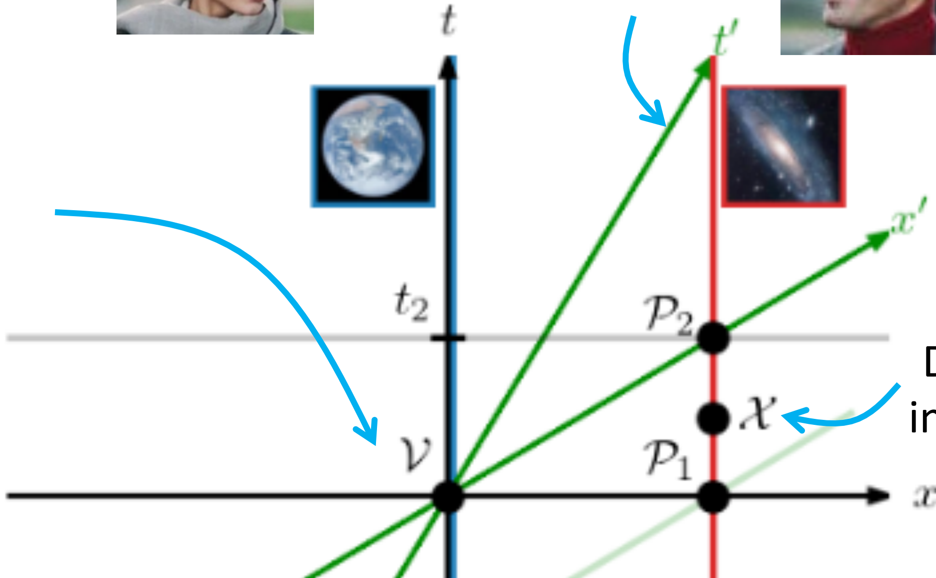
1.2.4 Simultaneity



Alice ($\mathcal{O}$)

Bob ($\mathcal{O}'$)

Alice and Bob pass each other



Decision to invade earth

What if Bob warns Alice about the invasion?

Can Alice avoid the war sending a message to the Andromeda galaxy before thay take any decision to invade earth?

the

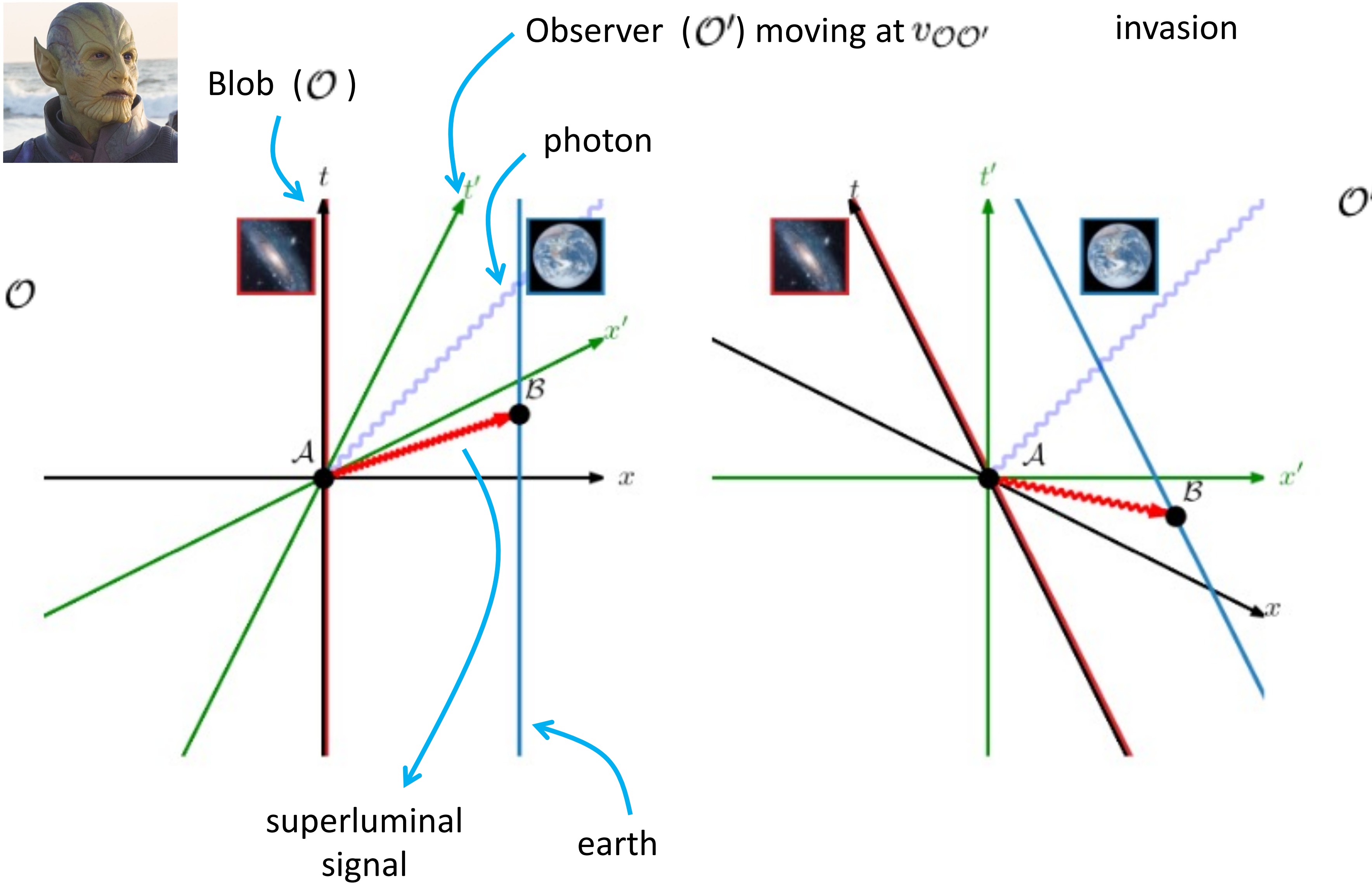
xy

(2.5x10^6 ly)

Spacetime diagram for the observer $\mathcal{O}$ (Alice).

1.2.5 Causality

Blob tries to warn earth about the invasion

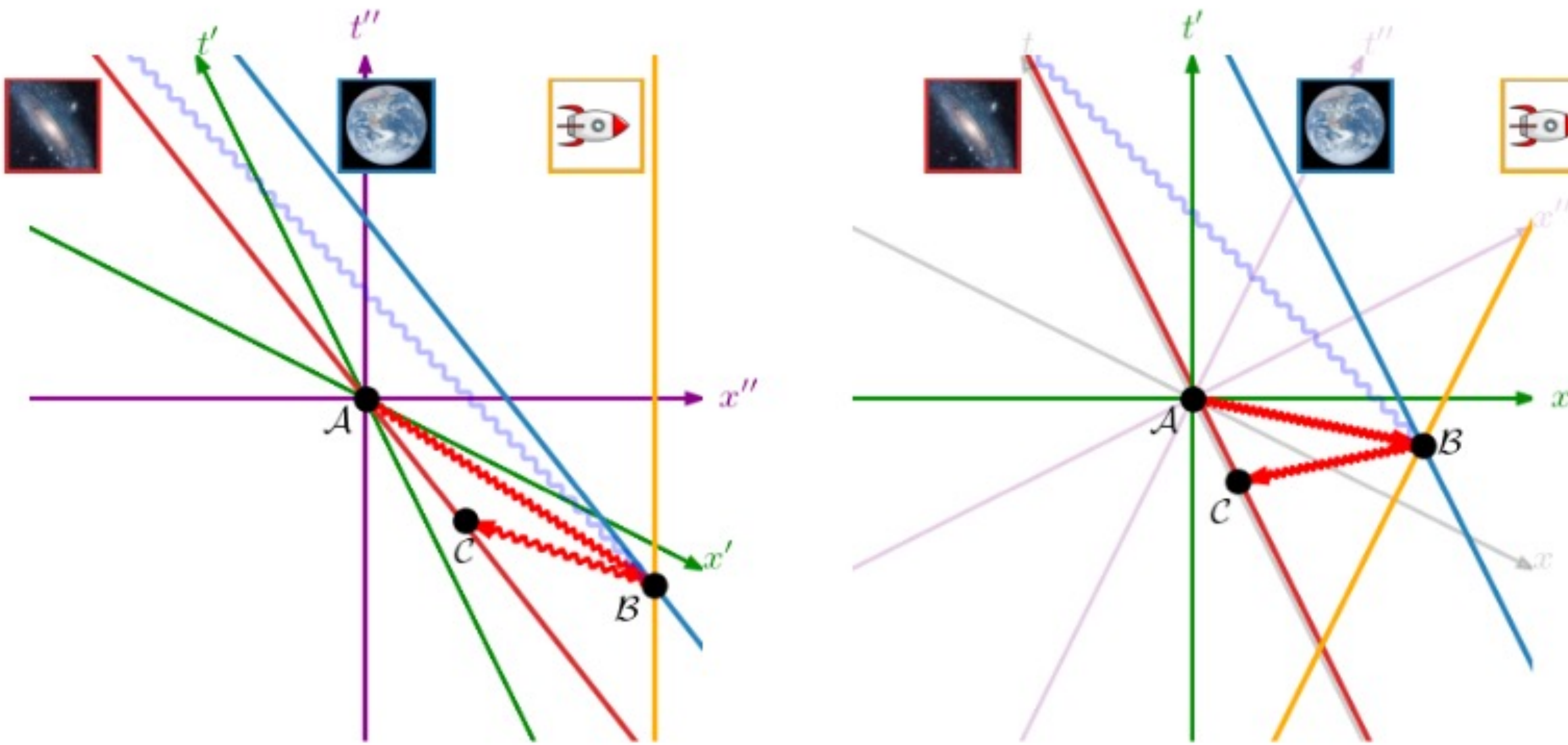


There is always an observer ($\mathcal{O}'$) for which the superluminal signal travels back in time.

1.2.5 Causality

Alice gets the message and tries to send a message back to Andromeda using a spaceship.

$\mathcal{O}''$ (spaceship) is moving even faster to the right.



The message arrives at C, before the event A happened in Blob's timeline (Andromeda)

What happened to causality?

1.2.5 Causality

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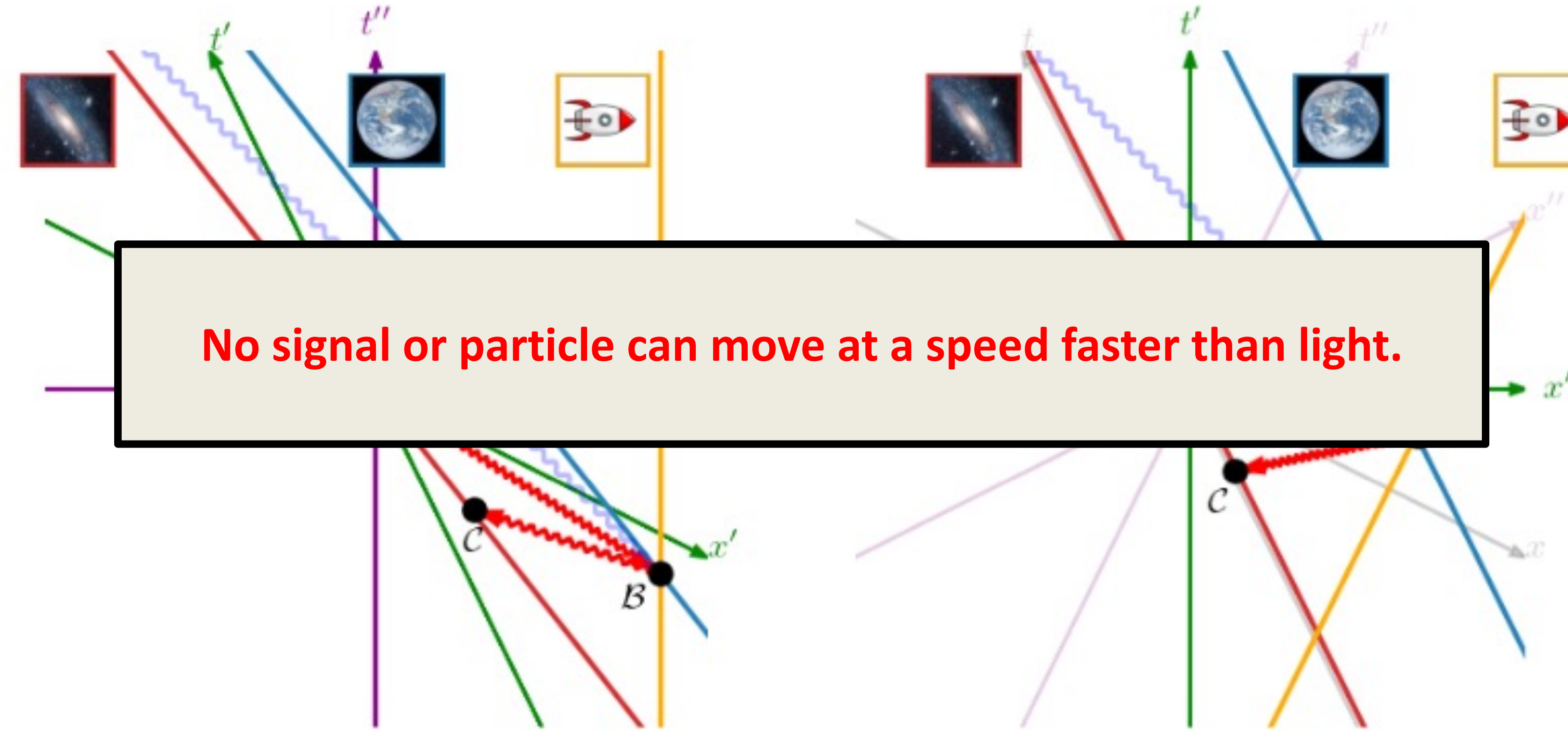

No signal or particle can move at a speed faster than light.

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1.2.5 Causality

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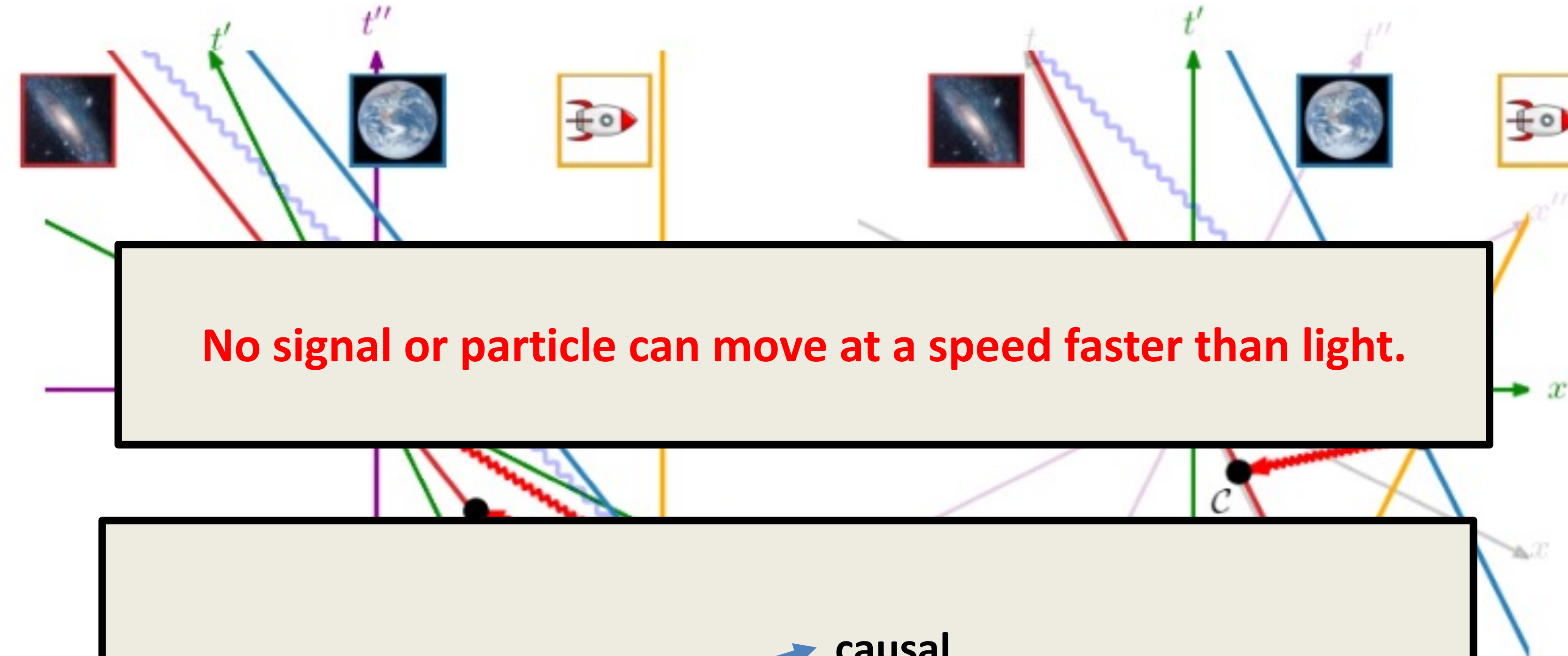


No signal or particle can move at a speed faster than light.

Case v=1? → **causal**

→ **No inertial observer**

The r ... meda)

What happened to causality?

1.2.6 Space-time intervals

1.2.6 Space-time intervals

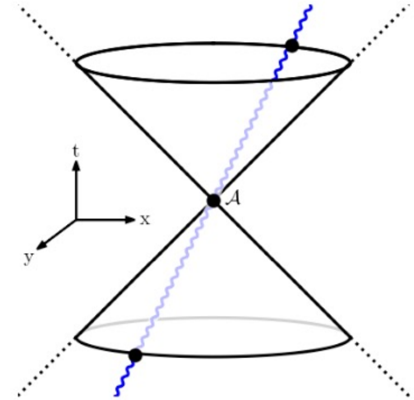
Space-time interval between two events

$$\Delta s^2 = -(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

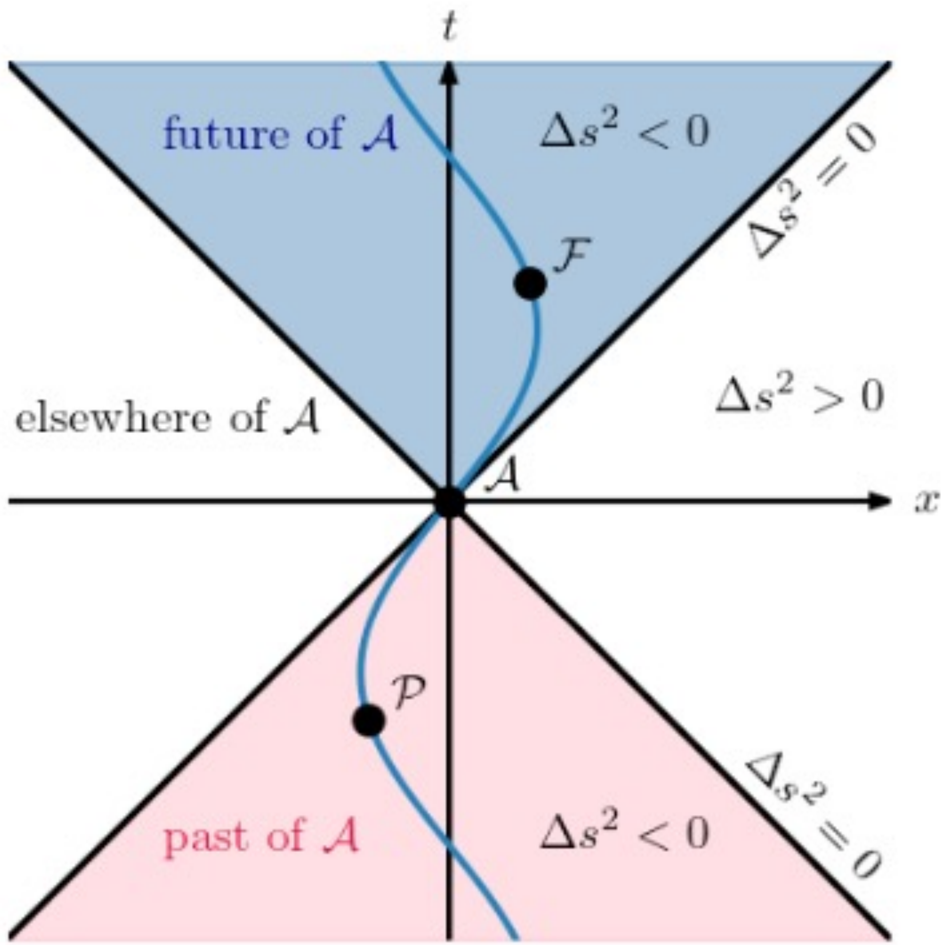
Space-time interval between an events and differential displacement around it:

$$ds^2 = -(dt)^2 + (dx)^2 + (dy)^2 + (dz)^2.$$

1.2.7 Light cones



Ginev an event A, we can build its **light cone**



Classification of an event depending on the interval to A:

$\Delta s^2 > 0$: spacelike

$\Delta s^2 < 0$: timelike

$\Delta s^2 = 0$: null or isotropic

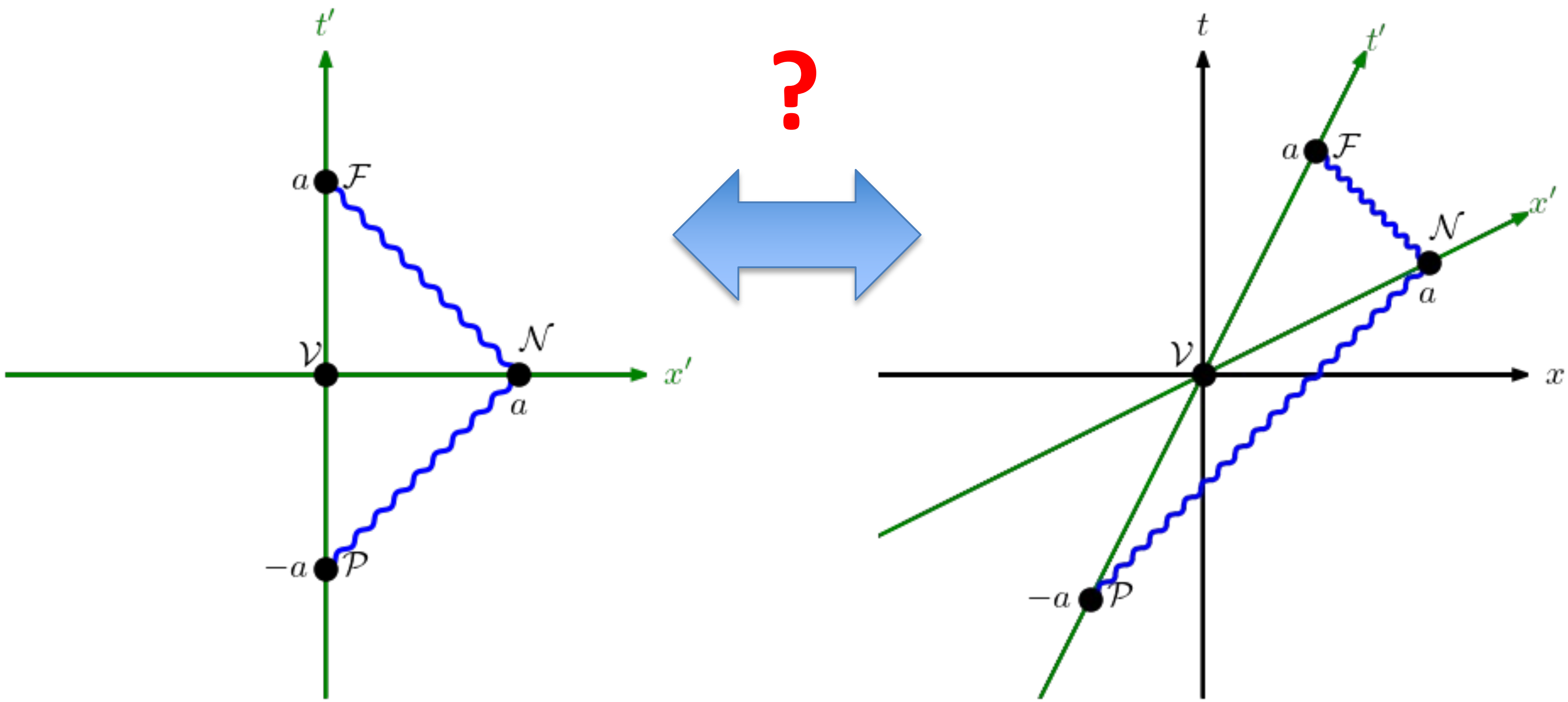
Division of the space-time in regions **independent of the observer**

Worldlines crossing A remain inside the light cylinder

Past, future, elsewhere. Simultaneity? Common past and future of two events. Interval classification

1.2.8 Lorentz transformations

What is the relation between
(t, x, y, z) and (t', x', y', z') ?



1.2.8 Lorentz transformations

What is the relation between (t, x, y, z) and (t', x', y', z') ?

Invariance of the interval Δs^2

Lorentz transformations (geometric demostration using invariant hyperbolae, see Schutz'85)

$\mathcal{O}'$ moves with velocity v along the x axis with respect to $\mathcal{O}$

$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} W & -Wv & 0 & 0 \\ -Wv & W & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}$$

Lorentz transformation or **boost** (matrix form)

$$W \equiv \frac{1}{\sqrt{1-v^2}}$$ **: Lorentz factor**

1.2.8 Lorentz transformations

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Lorentz transformation or **boost** (matrix form)

$$W \equiv \frac{1}{\sqrt{1 - v^2}}$$ **: Lorentz factor**

$|v| < 1$ → $W \geq 1$

$|v| \geq 1$ → No transformation possible

⇕

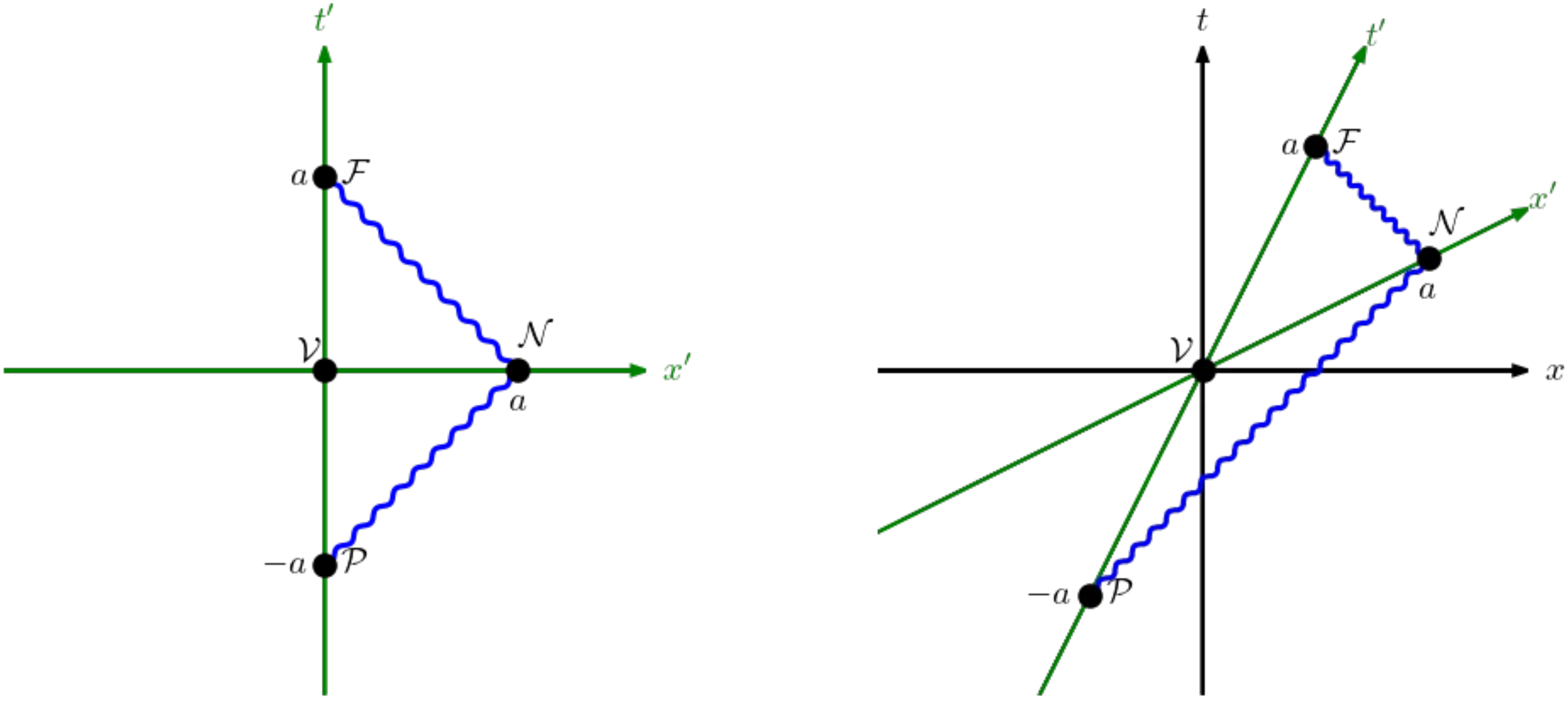
No inertial observers with speed $|v| \geq 1$

Inverse matrix

⇕

Substitute v by $-v$

1.2.8 Lorentz transformations



$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} W & -Wv & 0 & 0 \\ -Wv & W & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}$$

1.2.8 Lorentz transformations

$\mathcal{O}'$ moves with velocity $\vec{v}$ with respect to $\mathcal{O}$

$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} W & -Wv^x & -Wv^y & -Wv^z \\ -Wv^x & 1+(W-1)(v^x)^2/v^2 & (W-1)v^x v^y/v^2 & (W-1)v^x v^z/v^2 \\ -Wv^y & (W-1)v^y v^x/v^2 & 1+(W-1)(v^y)^2/v^2 & (W-1)v^y v^z/v^2 \\ -Wv^z & (W-1)v^z v^x/v^2 & (W-1)v^z v^y/v^2 & 1+(W-1)(v^z)^2/v^2 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}$$

$$\mathbf{x}' = \Lambda^{\text{boost}}(\vec{v})\,\mathbf{x}$$

Inverse transform: $$(\Lambda^{\text{boost}})^{-1}(\vec{v}) = \Lambda^{\text{boost}}(-\vec{v})$$

1.2.8 Lorentz transformations

Lorentz group

$$\mathbf{x}' = R\,\Lambda^{\text{boost}}(\vec{v})\,\mathbf{x} = \Lambda\,\mathbf{x}$$

(*boost* + rotation) ⟷ **Lorentz group**
$SO^{+}(1,3)$
("rotations" in 4D preserving the interval, orientation and direction of time)

rotation ⟷ Group of spatial rotations $SO(3)$
(3D transformations preserving Euclidean norm and orientation)

Poncaré group

$$\mathbf{x}' = \Lambda\,\mathbf{x} + \mathbf{a}$$

(Lorentz T. + translation) ⟷ **Poincaré group**

Constant vector (space-time translation)

The properties of invariance under Lorentz transformation (Lorentz invariance) also apply to the general case of Poincaré transformations.

1.2.9 Time dilation, length contraction, and other paradoxes

1.2.10 Velocity addition

$\mathcal{O}$ Measures the velocity of a particle, v, along the x axis.

$\mathcal{O}'$ moves at a speed $v_{\mathcal{O}\mathcal{O}'}$ along the x axis with respect to $\mathcal{O}$

Speed of the particle as measured by $\mathcal{O}'$

$$v = \frac{dx}{dt} \qquad ; \qquad v' = \frac{dx'}{dt'}$$

$$v' = \frac{dx'}{dt'} = \frac{W(dx - v_{\mathcal{O}\mathcal{O}'}dt)}{W(dt - v_{\mathcal{O}\mathcal{O}'}dx)} = \frac{dx/dt - v_{\mathcal{O}\mathcal{O}'}}{1 - v_{\mathcal{O}\mathcal{O}'}dx/dt} = \frac{v - v_{\mathcal{O}\mathcal{O}'}}{1 - v_{\mathcal{O}\mathcal{O}'}v}$$

Note that if $|v|<1$ then $|v'| < 1$. If $|v| = 1$ then $|v'| = 1$.

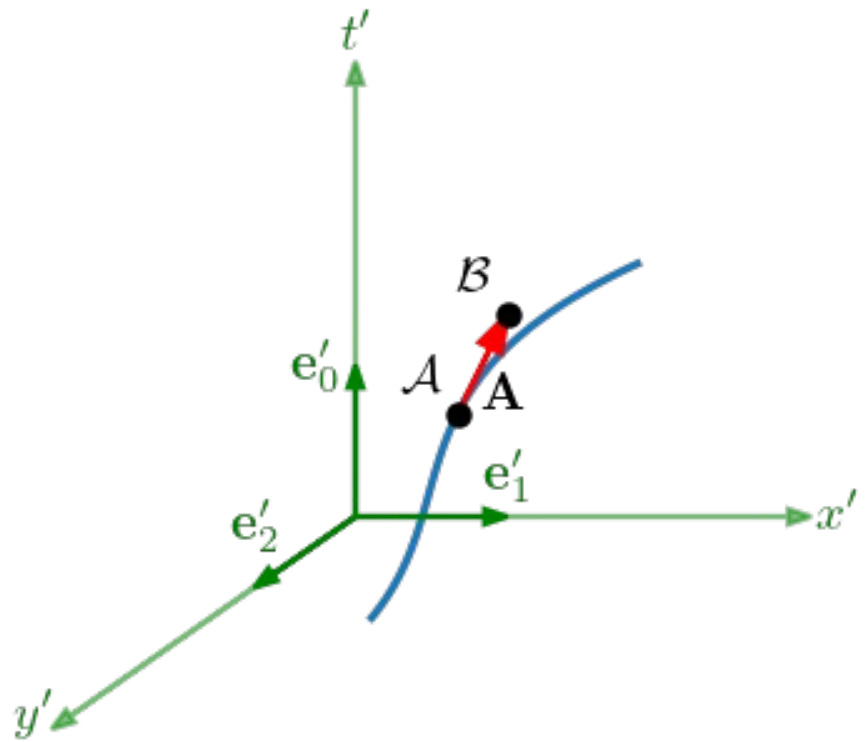
Low velocity limit ($|v| << 1$ y $|v_{\mathcal{O}\mathcal{O}'}| << 1$):

$v' \approx v - v_{\mathcal{O}\mathcal{O}'}$ (Galilean law of velocity addition)

1.2.10 Vectors and tensors

Vector

(four-vector or covariant vector)



In special relativity we could define all vectors in this way, but extension to curved spacetimes requires local definitions.

* We will use superscripts for vector components

Displacement vector : $\mathbf{\Delta x}$

(bi-local definition)

$$\mathcal{O} \rightarrow (\Delta t, \Delta x, \Delta y, \Delta z) = \Delta x^{\mu} \ *$$

$$\mathcal{O}' \rightarrow \Delta x^{\mu'} = \sum_{\nu=0}^{3} \Lambda^{\mu'}{}_{\nu}(v) \Delta x^{\nu}$$

(Einstein summation convention)

$$\Delta x^{\mu'} = \Lambda^{\mu'}{}_{\nu}(v) \Delta x^{\nu}$$

Infinitesimal displacement vector: $\mathbf{dx}$

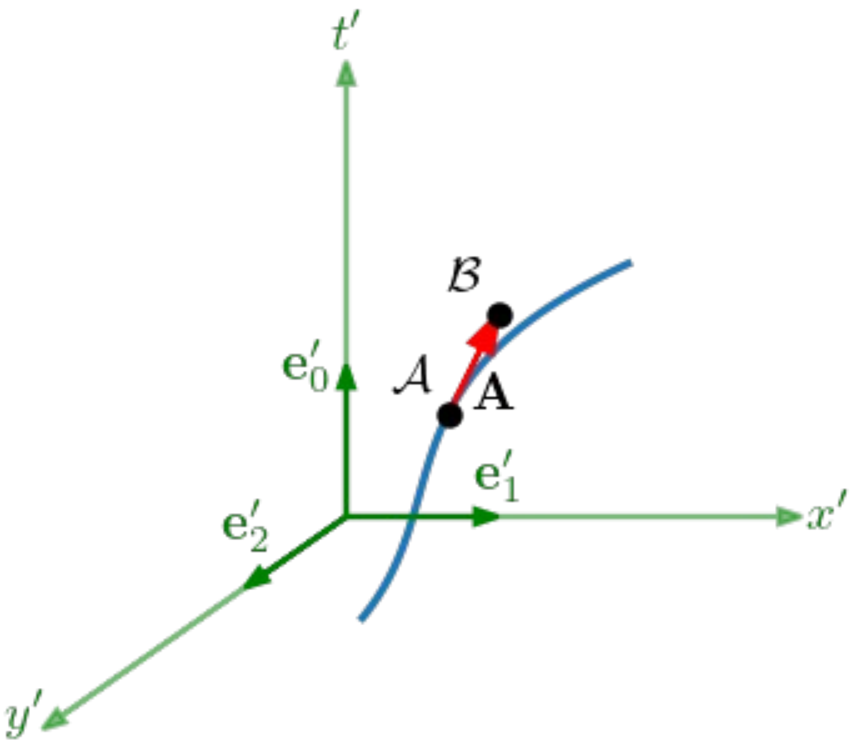
$$\mathcal{O} \rightarrow dx^{\mu}$$

Lorentz transformation

$$\begin{aligned}
\Lambda^0{}_0(v) = \Lambda^1{}_1(v) &= W, \\
\Lambda^0{}_1(v) = \Lambda^1{}_0(v) &= -Wv, \\
\Lambda^2{}_2(v) = \Lambda^3{}_3(v) &= 1.
\end{aligned}$$

Vector

(four-vector or covariant vector)



Tangent vector: $\mathbf{A}$

(local definition)

Curve: $\mathcal{P}(\lambda)$

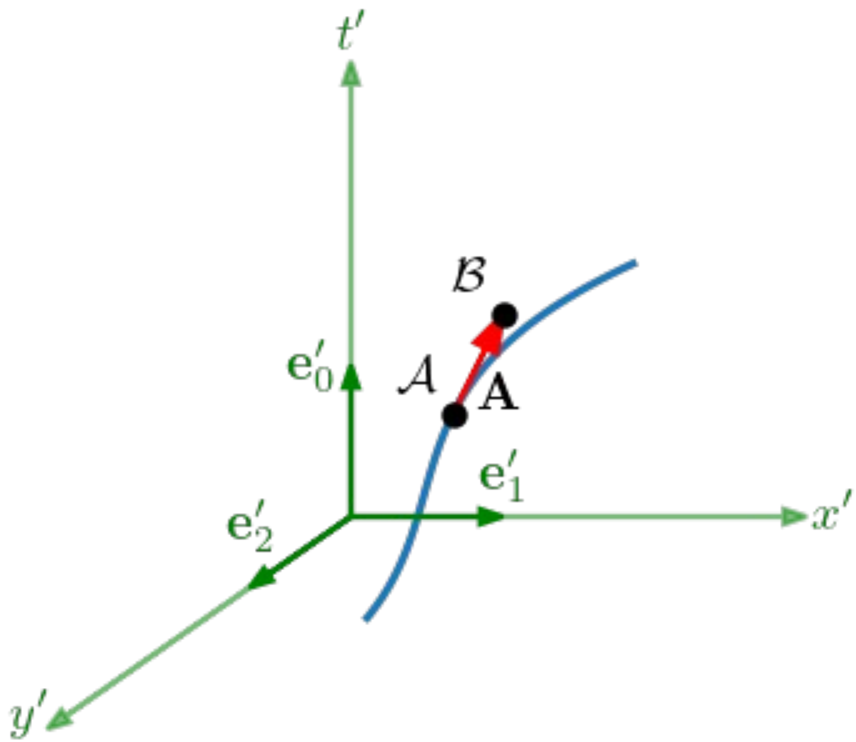
Event: $\mathcal{P}(0) = \mathcal{A}$

Geometric definition

$$\mathbf{A} = \left.\frac{d\mathcal{P}}{d\lambda}\right|_{\lambda=0}$$

Vector

(four-vector or covariant vector)



Definition:

We define the vector so that it transforms in the same way as the displacement vector (and the coordinates).

$$\mathcal{O} \rightarrow A^{\mu}$$

$$\mathcal{O}' \rightarrow \quad A^{\mu'} = \Lambda^{\mu'}{}_{\nu}(v) A^{\nu}$$

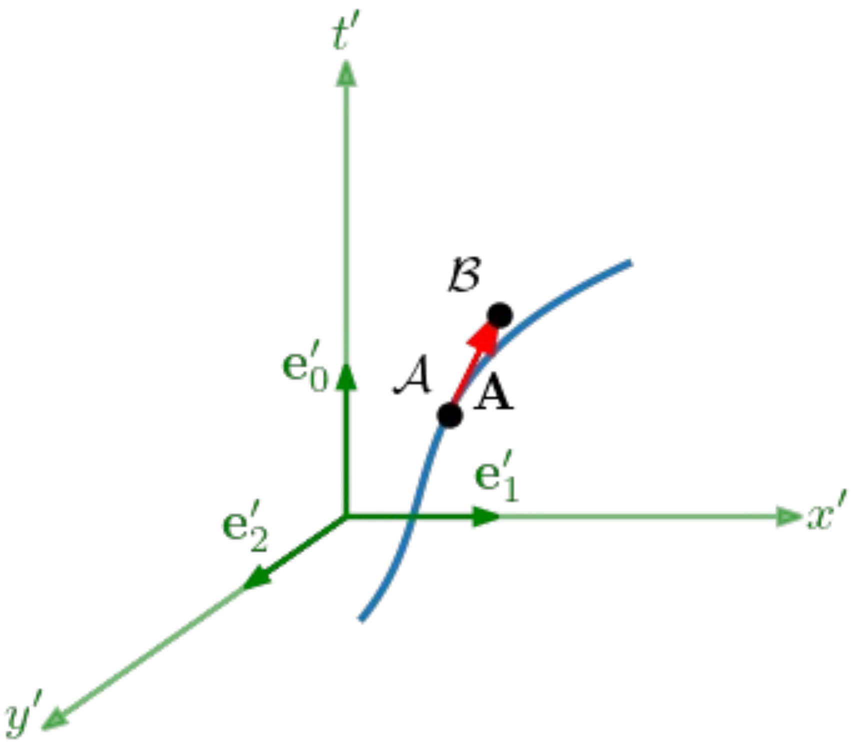
Vector space (V):

Vector sum (internal operation): $\mathbf{C} = \mathbf{A} + \mathbf{B} \quad : \quad C^{\mu} = A^{\mu} + B^{\mu}.$

Product by a scalar (external operation): $\mathbf{D} = c\,\mathbf{A} \quad : \quad D^{\mu} = cB^{\mu},$

Vector

(four-vector or covariant vector)



For the moment we will consider a Cartesian basis.

Basis: $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} = \{\mathbf{e}_\mu\}$

(μ is a label)

$$\mathcal{O}: \left.\begin{aligned} (\mathbf{e}_0)^\nu &= (1,0,0,0) \\ (\mathbf{e}_1)^\nu &= (0,1,0,0) \\ (\mathbf{e}_2)^\nu &= (0,0,1,0) \\ (\mathbf{e}_3)^\nu &= (0,0,0,1) \end{aligned}\right\} (\mathbf{e}_\mu)^\nu = \delta_\mu{}^\nu$$

(ν indicates the vector component)

(Kronecker delta)

Vector expressed in a basis

Basis of $\mathcal{O}$

$$\mathbf{A} = A^\mu \mathbf{e}_\mu$$

Vector (Geometric object independent of the observer)

Components (numbers that depend on the observer)

Vector

(four-vector or covariant vector)

Inverse Lorentz transformation

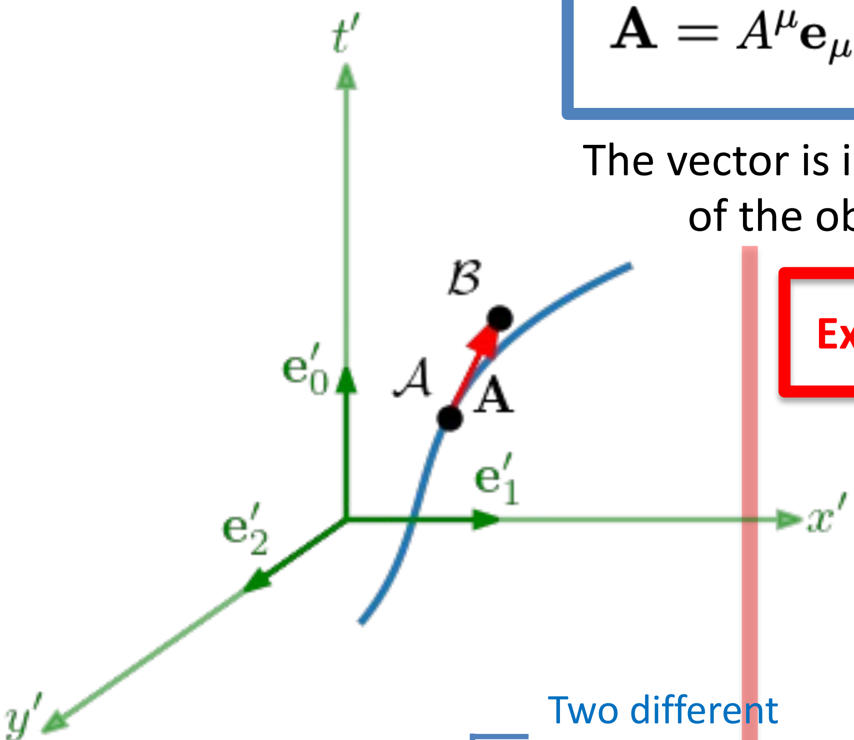
Lorentz transformation

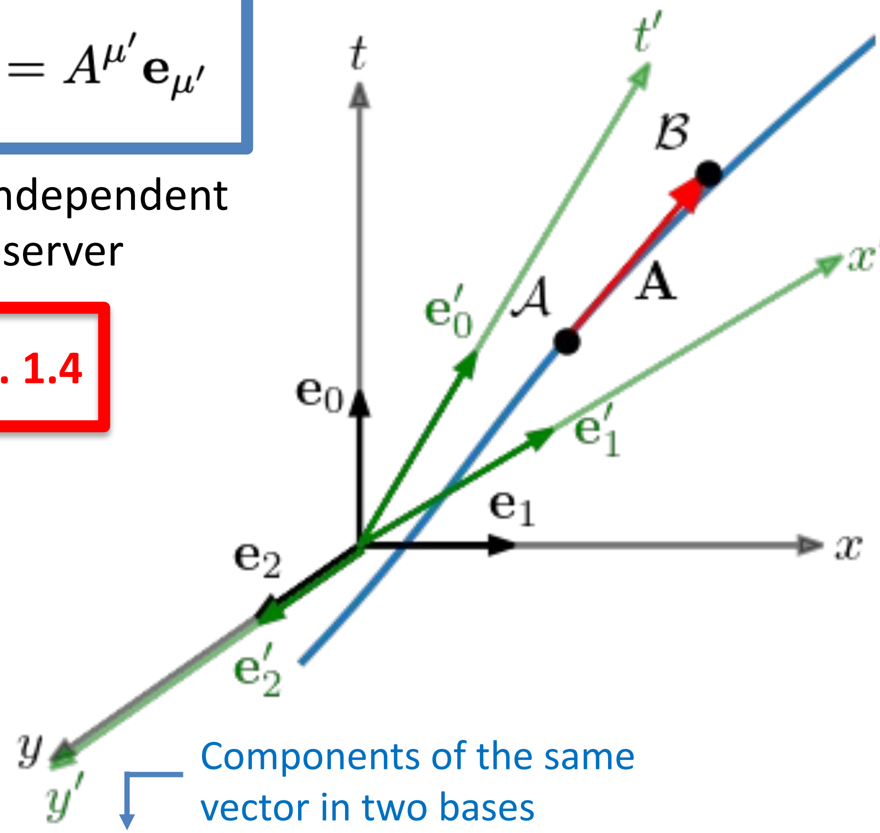
$$\Lambda^{\mu}{}_{\alpha'}(-v)\Lambda^{\alpha'}{}_{\nu}(v) = \delta^{\mu}{}_{\nu}$$

$$\boxed{\mathbf{A} = A^{\mu}\mathbf{e}_{\mu} = A^{\mu'}\mathbf{e}_{\mu'}}$$

The vector is independent of the observer

Ex. 1.4





Two different bases

Components of the same vector in two bases

$\mathcal{O}'$

$$\mathbf{e}_{\mu'} = \Lambda^{\nu}{}_{\mu'}(-v)\mathbf{e}_{\nu} \qquad A^{\mu'} = \Lambda^{\mu'}{}_{\nu}(v)A^{\nu}$$

$$\mathbf{e}_{\mu} = \Lambda^{\nu'}{}_{\mu}(v)\mathbf{e}_{\nu'} \qquad A^{\mu} = \Lambda^{\mu}{}_{\nu'}(-v)A^{\nu'}$$

$\mathcal{O}$

Tensor N times covariant

(tensor of type $\binom{0}{N}$)

Definition: Linear function of N vectors over the reals.* $\mathbf{T} : \underbrace{V \times V \times \cdots \times V}_{N} \to \mathbb{R}$

Example: tensor of type $\binom{0}{2}$

Tensor: $\mathbf{T}(\underbrace{\qquad}_{1^{\text{st}}\text{argument}}, \underbrace{\qquad}_{2^{\text{nd}}\text{argument}})$

Tensor applied to two of vectors $\mathbf{A}$ and $\mathbf{B}$: $\mathbf{T}(\mathbf{A}, \mathbf{B})$

In general: $\mathbf{T}(\mathbf{A}, \mathbf{B}) \neq \mathbf{T}(\mathbf{A}, \mathbf{B})$

Linearity: $\mathbf{T}(\alpha\,\mathbf{A} + \beta\,\mathbf{B}, \mathbf{C}) = \alpha\,\mathbf{T}(\mathbf{A}, \mathbf{C}) + \beta\,\mathbf{T}(\mathbf{B}, \mathbf{C})$

This definition is purely geometric, no observers involved..

* Nothing prevents generalizing these results to the body of complex numbers (or any other).

Metric tensor

Definition of $\mathbf{g}$: Tensor of type. $\binom{0}{2}$ Such that $\mathbf{g}(\mathbf{dx}, \mathbf{dx}) \equiv ds^2$

Scalar product of $\mathbf{A}$ **and** $\mathbf{B}$ **:** $\mathbf{g}(\mathbf{A}, \mathbf{B}) = \mathbf{A} \cdot \mathbf{B}$

Modulus squared of a vector: $\mathbf{g}(\mathbf{A}, \mathbf{A}) = \mathbf{A}^2$

Symmetric tensor: $\mathbf{g}(\mathbf{A}, \mathbf{B}) = \mathbf{g}(\mathbf{B}, \mathbf{A}) \rightarrow \mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$

(Commutative property)

Vector space + Scalar product → Hilbert space

Metric tensor

Components of the metric for the observer: $\mathcal{O}$ $\quad \mathbf{g}(\mathbf{e}_\mu, \mathbf{e}_\nu) \equiv g_{\mu\nu}$

Calculating metric components:

By definition:

$$\mathbf{g}(\mathbf{dx}, \mathbf{dx}) = ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

Using the linearity of tensors:

$$\mathbf{g}(\mathbf{dx}, \mathbf{dx}) = \mathbf{g}(dx^\mu \mathbf{e}_\mu, dx^\nu \mathbf{e}_\nu) = dx^\mu dx^\nu \mathbf{g}(\mathbf{e}_\mu, \mathbf{e}_\nu) = dx^\mu dx^\nu g_{\mu\nu}$$

Combining both:

$[g_{\mu\nu}] = \mathrm{diag}(-1, 1, 1, 1)$ **: Métrica de Minkowsky**

Transforming metric components between observers:

$$g_{\alpha'\beta'} = \Lambda^\mu{}_{\alpha'} \Lambda^\nu{}_{\beta'}\, g_{\mu\nu} = g_{\alpha\beta}$$

(Can be proved using the transformation properties of the basis)

The components of the Minkowsky metric do not change under Lorentz transformations

One-forms

(covector, covariant vector or differential form)

Definition of $\tilde{\mathbf{p}}$: Tensor of type $\binom{0}{1}$

We use the tilde to distinguish them from vectors

Components of a one-form for the observer $\mathcal{O}$: $p_\alpha \equiv \tilde{\mathbf{p}}(\mathbf{e}_\alpha)$

We use sub-indices for components (but no tilde)

Contraction of a vector and a one-form:

$$\underbrace{\tilde{\mathbf{p}}(\mathbf{A})}_{\text{Purely geometric expression}} = \tilde{\mathbf{p}}(A^\mu \mathbf{e}_\mu) = A^\mu \, \tilde{\mathbf{p}}(\mathbf{e}_\mu) = \underbrace{A^\mu p_\mu}_{\text{Expression in components}}$$

The result is a number

Components for another observer $\mathcal{O}'$

$$p_{\beta'} \equiv \tilde{\mathbf{p}}(\mathbf{e}'_\beta) = \Lambda^\alpha{}_{\beta'} p_\alpha$$

(It can be proved using the transformation properties of the basis)

The contraction of a vector and a one-form is independent of the observer

$$A^{\beta'} p_{\beta'} = A^\alpha p_\alpha$$

One-forms

(covector, covariant vector or differential form)

One forms → Vector space = Dual space (V')

Dual basis: $\{\tilde{\boldsymbol{\omega}}^{\alpha}\}$

α is a label

ν indicates the vector component

$$\mathcal{O}: \quad \underbrace{\begin{aligned} (\tilde{\boldsymbol{\omega}}^0)_\nu &= (1,0,0,0) \\ (\tilde{\boldsymbol{\omega}}^1)_\nu &= (0,1,0,0) \\ (\tilde{\boldsymbol{\omega}}^2)_\nu &= (0,0,1,0) \\ (\tilde{\boldsymbol{\omega}}^3)_\nu &= (0,0,0,1) \end{aligned}}$$

$$(\tilde{\boldsymbol{\omega}}^{\alpha})_{\beta} \equiv \tilde{\boldsymbol{\omega}}^{\alpha}(\mathbf{e}_{\beta}) = \delta^{\alpha}{}_{\beta}$$

One-form expressed in the dual basis

$$\tilde{\mathbf{p}} = p_{\alpha}\tilde{\boldsymbol{\omega}}^{\alpha}$$

Dual basis of $\mathcal{O}$

One-form (Observer-independent geometric object)

Components (numbers that depend on the observer)

Relation between the bases of two observers

$$\tilde{\boldsymbol{\omega}}^{\alpha'} = \Lambda^{\alpha'}{}_{\beta}\tilde{\boldsymbol{\omega}}^{\beta}$$

(Relation between one-forms, not between components)

Example of one-form: the gradient

Definition: $$\tilde{\mathbf{d}} f \equiv \left.\frac{df}{d\mathcal{P}}\right|_{\mathcal{P}_0}$$

Notation:

$$f_{,\alpha} \equiv \frac{\partial f}{\partial x^\alpha}$$

$$\mathcal{O}: \quad (\tilde{\mathbf{d}} f)_\mu = \left.\frac{\partial f}{\partial x^\mu}\right|_{x_0^\mu} = f_{,\alpha}$$

¿Why it is a one-form?

Directional derivative:

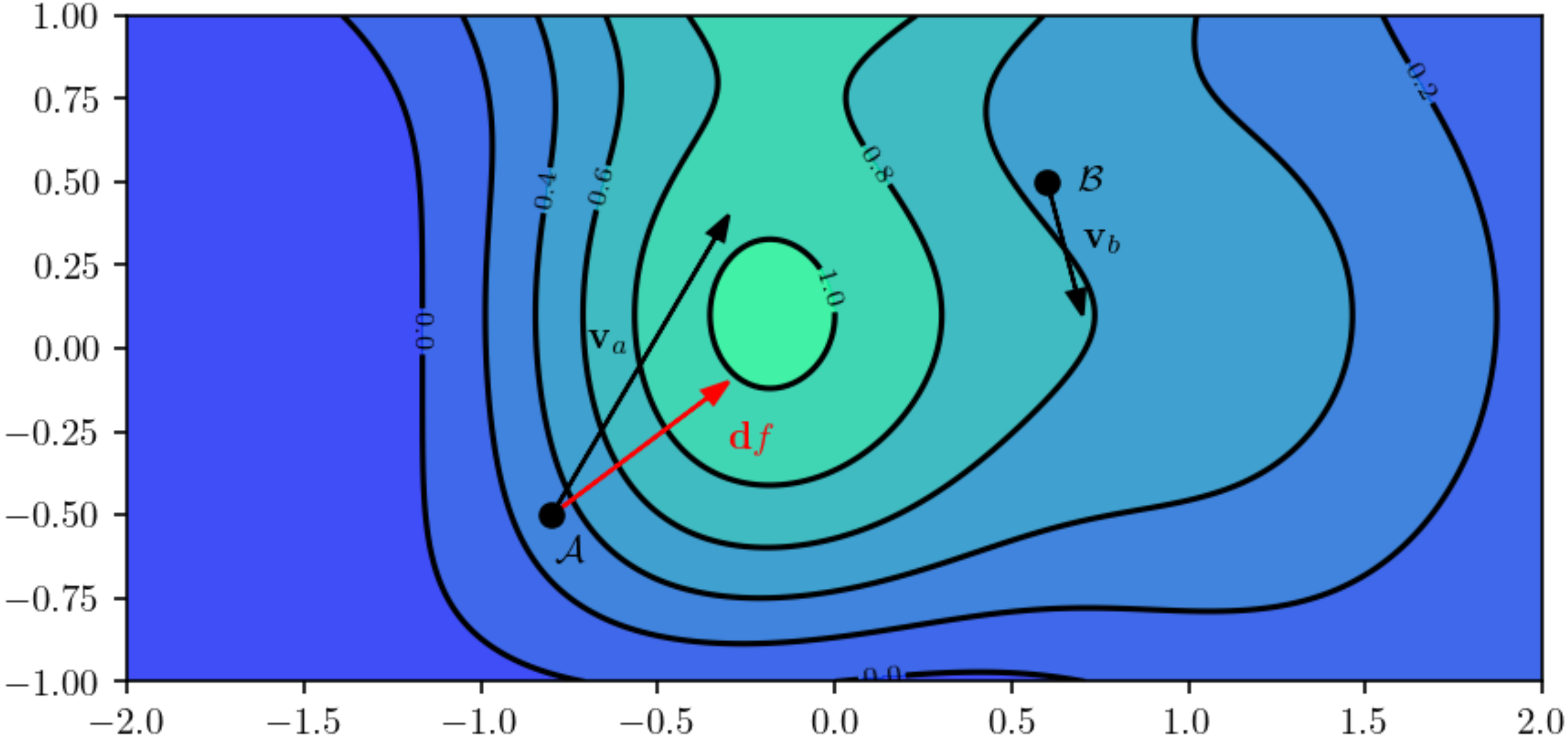
$$\partial_{\mathbf{v}} f \equiv \underbrace{\left.\frac{df}{d\lambda}\right|_{\lambda=0}}_{\text{This is a number}} = \underbrace{\left.\frac{df}{d\mathcal{P}}\right|_{\mathcal{P}_0}}_{\text{Gradient}} \underbrace{\left.\frac{d\mathcal{P}}{d\lambda}\right|_{\lambda=0}}_{\text{Vector tangent to a curve}} = \tilde{\mathbf{d}} f \ \mathbf{v}.$$

The gradient applied to a vector results in a number = one-form

$$\mathcal{O}: \ \partial_{\mathbf{v}} f = \left.\frac{df}{d\lambda}\right|_{\lambda=0} = \left.\frac{\partial f}{\partial x^\mu}\right|_{x_0^\mu} \left.\frac{dx^\mu}{d\lambda}\right|_{\lambda=0} = (\tilde{\mathbf{d}} f)_\mu v^\mu$$

Same expression but in components

Example of one-form: the gradient



Interpretation of the gradient

Example of one-form: the gradient

The gradient of the coordinates is the dual basis:

$$(\tilde{\mathbf{d}}x^\mu)_\nu = x^\mu{}_{,\nu} = \delta^\mu{}_\nu = (\tilde{\boldsymbol{\omega}}^\mu)_\nu \rightarrow \quad \tilde{\mathbf{d}}x^\mu = \tilde{\boldsymbol{\omega}}^\mu$$

Consequently:

$$\tilde{\mathbf{d}}f = \underbrace{\frac{\partial f}{\partial x^\mu}}_{\text{Components}} \overbrace{\tilde{\mathbf{d}}x^\mu}^{\text{Dual basis of } \mathcal{O}}$$

General tensors

(Tensor of type $\binom{M}{N}$ or tensor N times covariant and M times contravariant)

Definition: Linear function of N vectors and M one-forms over the reals.

$$\mathbf{R} : \underbrace{V \times V \times \cdots \times V}_{N} \times \underbrace{\tilde{V} \times \tilde{V} \times \cdots \times \tilde{V}}_{M} \to \mathbb{R}$$

Rank of a tensor: N+M

Example: tensor of type $\binom{2}{1}$

$\mathbf{R}(\tilde{\mathbf{p}}, \tilde{\mathbf{q}}, \mathbf{u})$ is a number

Basis of $\mathcal{O}$: $\mathbf{e}_\alpha \otimes \mathbf{e}_\beta \otimes \tilde{\boldsymbol{\omega}}^\mu$

Exterior product

$$R^{\alpha\beta}{}_{\mu} \neq R^{\beta\alpha}{}_{\mu} \neq R^{\alpha}{}_{\mu}{}^{\beta} \neq R_{\mu}{}^{\alpha\beta}$$

Components in $\mathcal{O}$: $\mathbf{R}(\tilde{\boldsymbol{\omega}}^\alpha, \tilde{\boldsymbol{\omega}}^\beta, \mathbf{e}_\mu) = R^{\alpha\beta}{}_{\mu}$

Caution with the notation

Tensor expressed in the basis of $\mathcal{O}$: $\mathbf{R} = R^{\alpha\beta}{}_{\mu}\, \mathbf{e}_\alpha \otimes \mathbf{e}_\beta \otimes \tilde{\boldsymbol{\omega}}^\mu$

Components in the basis of $\mathcal{O}'$: $R^{\gamma'\delta'}{}_{\nu'} = \Lambda^{\gamma'}{}_{\alpha} \Lambda^{\delta'}{}_{\beta} \Lambda^{\mu}{}_{\nu'} R^{\alpha\beta}{}_{\mu}$

(It can be proved using the transformation properties of the basis)

General tensors

(Tensor of type $\binom{M}{N}$ or tensor N times covariant and M times contravariant)

Symmetric tensor: $\mathbf{R}(\mathbf{u},\mathbf{v}) = \mathbf{R}(\mathbf{v},\mathbf{u}) \quad \forall\, \mathbf{u},\mathbf{v} \in V. \xrightarrow{\mathcal{O}} R_{\mu\nu} = R_{\nu\mu}$

Antisymmetric tensor: $\mathbf{R}(\mathbf{u},\mathbf{v}) = -\mathbf{R}(\mathbf{v},\mathbf{u}) \quad \forall\, \mathbf{u},\mathbf{v} \in V. \xrightarrow{\mathcal{O}} R_{\mu\nu} = -R_{\nu\mu}$

Construction of symmetric and antisymmetric tensors:

$$\left.\begin{aligned} \mathbf{R}_{(\mathrm{sim})}(\mathbf{u},\mathbf{v}) &= \frac{1}{2}\left(\mathbf{R}(\mathbf{u},\mathbf{v}) + \mathbf{R}(\mathbf{v},\mathbf{u})\right) \\ \mathbf{R}_{(\mathrm{asim})}(\mathbf{u},\mathbf{v}) &= \frac{1}{2}\left(\mathbf{R}(\mathbf{u},\mathbf{v}) - \mathbf{R}(\mathbf{v},\mathbf{u})\right) \end{aligned}\right\} \xrightarrow{\mathcal{O}} \left\{\begin{aligned} R_{(\mathrm{sim})\mu\nu} &= \frac{1}{2}\left(R_{\mu\nu} + R_{\nu\mu}\right) \equiv R_{(\mu\nu)} \\ R_{(\mathrm{asim})\mu\nu} &= \frac{1}{2}\left(R_{\mu\nu} - R_{\nu\mu}\right) \equiv R_{[\mu\nu]} \end{aligned}\right.$$

Any tensor of rank 2 can be expressed as:

$$R_{\mu\nu} = R_{(\mu\nu)} + R_{[\mu\nu]}$$

Map between vectors and one-forms

Given the vector $\mathbf{v}$ and the metric $\mathbf{g}$ let us consider the following one-form:

$$\tilde{\mathbf{v}}() \equiv \mathbf{g}(\mathbf{v},) = \mathbf{g}(, \mathbf{v})$$

Contraction of $\mathbf{v}$ with a vector $\mathbf{a}$:

$$\tilde{\mathbf{v}}(\mathbf{a}) = \mathbf{g}(\mathbf{v}, \mathbf{a}) = \mathbf{v} \cdot \mathbf{a}$$

Components of $\tilde{\mathbf{v}}$:

$$\mathbf{e}_\alpha \cdot \mathbf{e}_\beta = \mathbf{g}(\mathbf{e}_\alpha, \mathbf{e}_\beta) = g_{\alpha\beta}$$

$$v_\alpha = \tilde{\mathbf{v}}(\mathbf{e}_\alpha) = \mathbf{v} \cdot \mathbf{e}_\alpha = \mathbf{e}_\alpha \cdot \underbrace{(v^\beta \mathbf{e}_\beta)}_{\mathbf{v}\text{ expressed in the basis of }\mathcal{O}} = \overbrace{(\mathbf{e}_\alpha \cdot \mathbf{e}_\beta)}^{\text{We know how to calculate this}} v^\beta$$

Map

$$\boxed{v_\alpha = g_{\alpha\beta}\, v^\beta}$$

The components of $\mathbf{v}$ and $\tilde{\mathbf{v}}$ are related by the metric.

The inverse relation:

$$\boxed{v^\alpha = g^{\alpha\beta}\, v_\beta}$$

If $[g_{\alpha\nu}]$ is the matrix with the components of $\mathbf{g}$ the inverse matrix $[g^{\mu\nu}]$ is:

$$[g^{\mu\nu}] = \mathrm{diag}(-1, 1, 1, 1)$$

$$[g^{\mu\alpha}][g_{\alpha\nu}] = I \qquad g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu{}_\nu$$

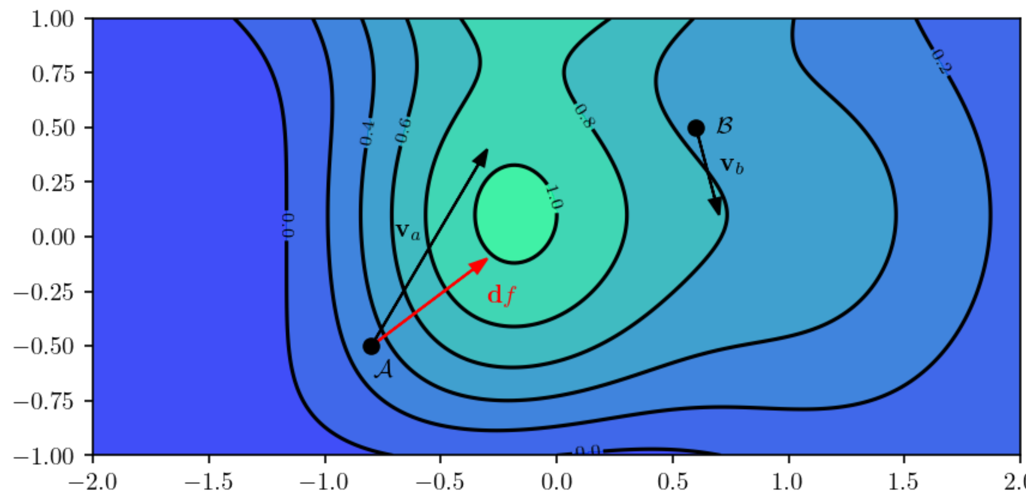
Map between vectors and one-forms

Example: gradient vector

$$\underbrace{\tilde{\mathbf{d}}f}_{\text{Gradient (one-form)}} \overset{\mathbf{g}}{\longleftrightarrow} \underbrace{\mathbf{d}f}_{\text{Gradient vector}}$$

$$\mathcal{O}: \quad (\mathbf{d}f)^{\alpha} = g^{\alpha\beta}\,(\tilde{\mathbf{d}}f)_{\beta} = g^{\alpha\beta} f_{,\beta}$$

This is the definition of gradient (as a vector) to which you are most accustomed.



Counter-example: basis and dual basis

$$\begin{array}{ccc} \{\mathbf{e}_{\alpha}\} & \overset{\mathbf{g}}{\longleftrightarrow} & \{\tilde{\mathbf{e}}_{\beta}\} \\ \cancel{\updownarrow} & & \\ \{\tilde{\boldsymbol{\omega}}^{\alpha}\} & \overset{\mathbf{g}}{\longleftrightarrow} & \{\boldsymbol{\omega}^{\beta}\} \end{array}$$

$$\tilde{\mathbf{e}}_{\beta} = g_{\alpha\beta}\,\tilde{\boldsymbol{\omega}}^{\alpha} \neq \tilde{\boldsymbol{\omega}}^{\beta} \quad ; \quad \boldsymbol{\omega}^{\beta} = g^{\alpha\beta}\mathbf{e}_{\alpha} \neq \mathbf{e}_{\beta}$$

Can be easily demonstrated **Ex. 1.5**

Inverse metric

Can we define a metric for the dual space? And a scalar product? Is it a Hilbert space?

We define the scalar product of one-forms so that: $\mathbf{v}^2 = \tilde{\mathbf{v}}^2$

This implies:

$$\begin{aligned} \tilde{\mathbf{v}}^2 &= \mathbf{v}^2 = g_{\alpha\beta} v^\alpha v^\beta = g_{\alpha\beta}(g^{\alpha\mu} v_\mu)(g^{\beta\nu} v_\nu) = (g_{\alpha\beta} g^{\alpha\mu}) g^{\beta\nu} v_\mu v_\nu = \\ &= \delta^\mu{}_\beta g^{\beta\nu} v_\mu v_\nu = g^{\mu\nu} v_\mu v_\nu, \end{aligned}$$

i.e.

$$\tilde{\mathbf{v}}^2 = g^{\mu\nu} v_\mu v_\nu$$

: Acting on two one-forms gives a number = tensor of type $\binom{2}{0}$

Matrix elements $[g^{\mu\nu}]$

Inverse metric: $\mathbf{g}^{-1}$ $\quad \mathbf{g}^{-1}(\tilde{\mathbf{v}}, \tilde{\mathbf{v}}) \equiv \tilde{\mathbf{v}}^{\mathbf{2}} = \mathbf{g}(\mathbf{v}, \mathbf{v})$

The **components of the inverse metric** are the elements of $[g^{\mu\nu}]$:

$$\mathbf{g}^{-1}(\tilde{\boldsymbol{\omega}}^\mu, \tilde{\boldsymbol{\omega}}^\nu) = g^{\mu\nu}$$

Inverse metric

Consequences:

- The metric and the inverse metric allow to make the map between vectors and one-shapes bidirectional.

- The inverse metric allows to define the scalar product between one-forms

$$\mathbf{g}^{-1}(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) \equiv \tilde{\mathbf{u}} \cdot \tilde{\mathbf{v}} = \mathbf{g}(\mathbf{u}, \mathbf{v})$$: **scalar product** of one-forms

$$\mathbf{g}^{-1}(\tilde{\mathbf{v}}, \tilde{\mathbf{v}}) \equiv \tilde{\mathbf{v}}^{\mathbf{2}}$$: **module squared** of a one-form

- The one-forms have structure of a Hilbert space

Rising and lowering indices

Consider a tensor $\mathbf{T}$ of type $\binom{0}{2}$:

$$\mathbf{T} : V \times V \to \mathbb{R} \quad ; \quad \mathbf{T}(\mathbf{e}_\alpha, \mathbf{e}_\beta) = T_{\alpha\beta}$$

We can use a procedure analogous to that used to map vectors and one-forms to define:

$$\begin{aligned}
\tilde{\mathbf{T}} &: \tilde{V} \times \tilde{V} \to \mathbb{R} \quad ; \quad \tilde{\mathbf{T}}(\tilde{\boldsymbol{\omega}}^\alpha, \tilde{\boldsymbol{\omega}}^\beta) = T^{\alpha\beta}, \\
\bar{\mathbf{T}} &: V \times \tilde{V} \to \mathbb{R} \quad ; \quad \bar{\mathbf{T}}(\mathbf{e}_\alpha, \tilde{\boldsymbol{\omega}}^\beta) = T_\alpha{}^\beta, \\
\bar{\bar{\mathbf{T}}} &: \tilde{V} \times V \to \mathbb{R} \quad ; \quad \bar{\bar{\mathbf{T}}}(\tilde{\boldsymbol{\omega}}^\alpha, \mathbf{e}_\beta) = T^\alpha{}_\beta,
\end{aligned}$$

Components of $\tilde{\mathbf{T}}$, $\bar{\mathbf{T}}$ and $\bar{\bar{\mathbf{T}}}$ as a function of those of $\mathbf{T}$:

$$\begin{aligned}
T^{\alpha\beta} &= g^{\alpha\mu} g^{\beta\nu} T_{\mu\nu}, \\
T^\alpha{}_\beta &= g^{\alpha\mu} T_{\mu\beta}, \\
T_\alpha{}^\beta &= g^{\mu\beta} T_{\alpha\mu},
\end{aligned}$$

To **raise the indices of a tensor** we use the inverse metric

Components of $\mathbf{T}$ as a function of those of $\tilde{\mathbf{T}}$, $\bar{\mathbf{T}}$ and $\bar{\bar{\mathbf{T}}}$

$$\begin{aligned}
T_{\alpha\beta} &= g_{\alpha\mu} g_{\beta\nu} T^{\mu\nu}, \\
T_{\alpha\beta} &= g_{\beta\mu} T_\alpha{}^\mu, \\
T_{\alpha\beta} &= g_{\alpha\mu} T^\mu{}_\beta,
\end{aligned}$$

To **lower the indices of a tensor** we use the metric

Map of the metric tensor:

$$\mathbf{g} \overset{\mathbf{g}}{\longleftrightarrow} \mathbf{g}^{-1}$$

The metric and its inverse are also related by such a map.

$$\mathcal{O}: \quad g_{\mu\nu} = g_{\mu\alpha}\, g_{\nu\beta}\, g^{\alpha\beta};$$

(The metric can be used to raise and lower indices of itself)

And the other two tensors that can be built?

Mixed metric:

$$\mathcal{O}: \quad g^{\alpha}{}_{\beta} = g^{\alpha\mu} g_{\mu\beta} = \delta^{\alpha}{}_{\beta} \qquad \text{y} \qquad g_{\alpha}{}^{\beta} = \delta_{\alpha}{}^{\beta}$$

Mixed metric can be used to rename indices:

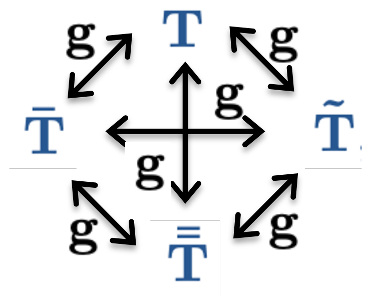
$$T^{\alpha\beta} = g^{\alpha}{}_{\mu}\, g^{\beta}{}_{\nu}\, T^{\mu\nu}.$$

Tensor maps and notation

Map of a rank 1 tensor

$$\mathbf{v} \overset{\mathbf{g}}{\longleftrightarrow} \tilde{\mathbf{v}}$$

Map of a rank 2 tensor



Map of a rank r tensor

2^r posibles tensores

- For higher rank tensors the geometric notation of tensors is complicated and not generic.

Component notation: The components of a tensor can be used as a representation of itself.

$$T^{\mu\nu},\ T_{\mu\nu},\ T^{\mu}{}_{\nu}\ \mathrm{y}\ T_{\mu}{}^{\nu}$$

- Allows to represent arbitrary range tensors with high precision
- Tensors and their components are distinguished by context.
- Language abuse: “tensor with indices up or down”

Index contraction

Generalization of the contraction of a vector and a one-form:

$$v^{\mu} u_{\mu} \qquad (1 + 1 \rightarrow 0)$$

Allows to generate tensors of rank lower than the sum of the contracted tensors:

$$A_{\mu}{}^{\alpha} = T_{\mu\nu}\, S^{\nu\alpha} \qquad (2 + 2 \rightarrow 2)$$

$$a = T_{\mu\nu}\, S^{\mu\nu} \qquad (2 + 2 \rightarrow 0)$$

The **tensor expressions**:

- They are tensors
- They are geometric entities independent of the observer(because they are tensors) (Expressions valid for any observer)

Scalar

Definition: tensor of type $\binom{0}{0}$

- Function without arguments: always returns the same value
- Its value is invariant under Lorentz transformations
- Examples: $T^{\mu\nu}u_\mu v_\nu$ $\quad u^\mu v_\mu$ $\quad A^{\alpha\beta\gamma}{}_\delta\, C_{\alpha\gamma}\, D_\beta\, E^\delta$ $\quad R^\mu{}_\mu$ $\quad T^{\mu\nu} g_{\mu\nu}$

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu$$

(interval)

Ex. 1.6

- **Observables**: Since all inertial observers agree on its value.

Note: The components of a tensor in a given basis are invariant: $\mathbf{T}(\mathbf{e}_\mu, \mathbf{e}_\nu)$

But the components in its own basis depend on the observer (since the basis changes):

$$\mathbf{T}(\mathbf{e}_\mu, \mathbf{e}_\nu) \neq \mathbf{T}(\mathbf{e}'_\mu, \mathbf{e}'_\nu)$$

Is the Lorentz transformation matrix related to the components of a tensor?

No. The matrix **relates components of two different bases**, but **its elements are not the components of a tensor**

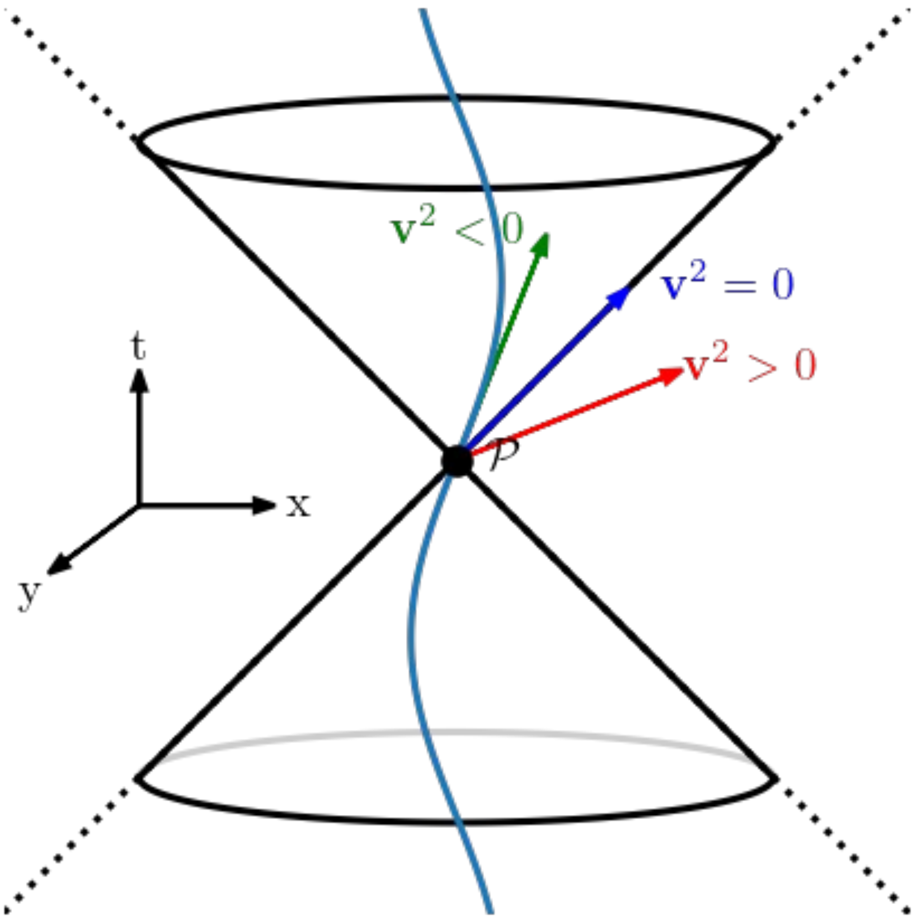
$$\frac{\partial x^{\mu'}}{\partial x^{\nu}} = \Lambda^{\mu'}{}_{\nu}$$

It is the Jacobian of
a transformation

1.2.12 - Space-like, time-like and null vectors

Since $\mathbf{g}(\mathbf{dx}, \mathbf{dx}) \equiv ds^2 \rightarrow \Delta s^2 = (\Delta \mathbf{x})^2$ (spacelike/timelike/null separation)

It is a vector



Given the vector $\mathbf{v}$ at the point $\mathcal{P}$

$\mathbf{v}^2 > 0$ Spacelike vector (outside the light cone)

$\mathbf{v}^2 < 0$ Timelike vector (inside the light cone)

$\mathbf{v}^2 = 0$ Null vector (on the light cone)

(Its components don't have to be zero)

1.2.13 - 4-velocity

Definition: vector tangent to the worldline of a particle such that: $\mathbf{u}^2 = -1$

- It is a timelike vector of temporal ($\mathbf{u}^2 < 0$): inside the light-cone of a particle that moves at a speed less than that of light.

- $\mathcal{O}$: Observer for which the particle is at rest in $\mathcal{P}$

  $\mathbf{u}$ has the direction of $t \xrightarrow{\mathbf{u}^2 = -1} \mathbf{u} = \mathbf{e}_0$

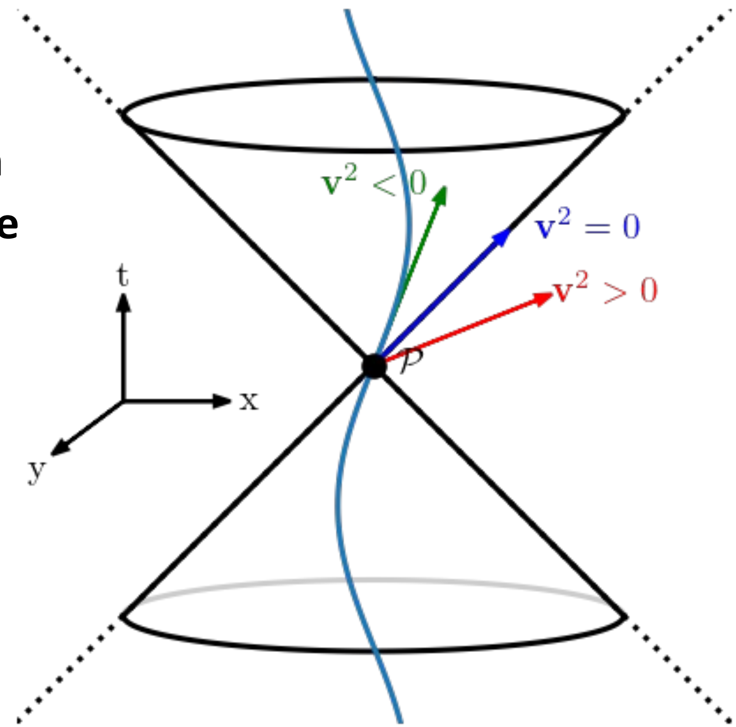
- **Accelerated particle:** the definition of $\mathcal{O}$ depends on The point → **momentarily comoving reference frame** (MCRF)

- $$\boxed{\mathbf{u} = \frac{d\mathbf{x}}{d\tau}}$$

Infinitesimal displacement along the worldline.

proper time interval in this displacement.

Tangent to the worldline



$$\left(\frac{d\mathbf{x}}{d\tau}\right)^2 = \frac{d\mathbf{x}\cdot d\mathbf{x}}{(d\tau)^2} = -1$$

(since $(d\tau)^2 = -(d\mathbf{x})^2$)

1.2.14 - 4-momentum

Definition: $\mathbf{p} = m\mathbf{u}$

Where m = **rest mass** = mass measured by an observer at rest with respect to the particle

$\mathcal{O}'$: Momentarily comoving reference frame (MCRF) → $\mathbf{u} = \mathbf{e}_0$ and mass = m

$$\mathbf{p} = m\mathbf{e}_0$$

$\mathcal{O}$: Observer moving relative to $\mathcal{O}'$ with velocity $\vec{v}$.

Change of observer:

$$\begin{aligned} u^\mu &= \Lambda^\mu{}_{\nu'} u^{\nu'} = \Lambda^\mu{}_{\nu'} (\mathbf{e}_0)^{\nu'} = \Lambda^\mu{}_{0'}, \\ p^\mu &= m\Lambda^\mu{}_0. \end{aligned}$$

In components (for a boost in the x direction):

$$\begin{aligned} u^0 &= W, & p^0 &= m\,W, \\ u^1 &= v\,W, & p^1 &= m\,v\,W, \\ u^2 &= 0, & p^2 &= 0, \\ u^3 &= 0, & p^3 &= 0. \end{aligned}$$

1.2.14 - 4-momentum

Small velocities ($v << 1$):

- Spatial part of $\mathbf{u}$: $(v, 0, 0)$ = classical 3-velocity
- Spatial part of $\mathbf{p}$: $(mv, 0, 0)$ = classical momentum
- Spatial part of $\mathbf{u}$: $u^0 \approx 1$
- Spatial part of $\mathbf{p}$: $p^0 = \dfrac{m}{\sqrt{1-v^2}} \approx \underbrace{m}_{\text{Mass}} + \underbrace{\frac{1}{2}mv^2}_{\text{Kinetic energy}}$ = total energy → $p^0 \equiv E$

In the classical limit the 4-velocity and the 4-momentum contain information about the velocity, momentum and energy of the particle measured by an observer.

1.2.14 - 4-momentum

Energy of a particle with 4-momentum $\mathbf{p}$ as measured by an observer $\mathcal{O}'$ with 4-velocity $\mathbf{u}_{\mathrm{obs}}$ (different to the particle): $E' = -\mathbf{p} \cdot \mathbf{u}_{\mathrm{obs}}$

Proof: The next scalar is an invariant: $\mathbf{p} \cdot \mathbf{u}_{\mathrm{obs}} = \underbrace{\mathbf{p} \cdot \mathbf{e}_{0'}}_{\text{Observador en reposo consigo mismo}} = -E'$

Any observer can compute the energy of a particle as measured by $\mathcal{O}'$ in a simple way.

Ex. 1.7

1.2.15 - photons

Particles moving at the speed of light

→ There is no inertial observer in which they are at rest

→ It is not possible to define rest mass

→ Null interval between any two events on their worldline

$$ds^2 = d\mathbf{x} \cdot d\mathbf{x} = 0 \quad \rightarrow \quad d\tau = 0 \quad \text{(proper time interval)}$$

→ The vector tangent to the trajectory is null ($\mathbf{u}^2 \neq -1$)

It is not possible to define the 4-velocity of a photon

4-momentum of a photon moving in the x direction:

$\mathcal{O}$: Energy: $E \equiv p^0$ y $E = h\nu$

y, z momentum : $p^y = p^z = 0$

x momentum : $\mathbf{p}^2 = -E^2 + (p^x)^2 = 0 \quad \rightarrow \quad p^x = E$

(4-momento es un vector nulo)

Zero mass

$m^2 = -\mathbf{p}^2 = 0$

$\mathcal{O}'$: $E' = h\nu' = WE - p^x vW = W(1-v)h\nu$ (Lorentz t. in the x direction)

$$\frac{\nu'}{\nu} = \frac{(1-v)}{\sqrt{1-v^2}} = \sqrt{\frac{1-v}{1+v}}$$

: Relativistic Doppler redshift (longitudinal effect)