MODULAR CHARACTERS, HALL SUBGROUPS, AND NORMAL COMPLEMENTS

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Abstract. If H is a Hall subgroup of a finite group G, it was proven in 1989 using the classification of finite simple groups that all the irreducible complex characters of H extend to G if and only if there is $N \trianglelefteq G$ such that $HN = G$ and $H \cap N = 1$. In this note we offer a modular version of this result under more general hypotheses.

1. Introduction

One of the first applications of character theory was to produce normal subgroups of finite groups. In many cases, the problem was: given a Hall subgroup H of a finite group G, when H does have a normal complement in G? In other words, if H is a subgroup of G such that $|H|$ and $|G : H|$ are coprime, when there is $N \trianglelefteq G$ such that $G = NH$ and $H \cap N = 1$? Originally, this problem was treated extensively in the literature, for instance by Baer, Frobenius, Higman, Sah, Thompson and Wielandt among many others ([B], [Fr], [Hi], [S], [T], [W]). If H is a Sylow p-subgroup of G (or more generally, if H is nilpotent), the famous Brauer–Suzuki theorem asserts that this happens if and only if $h^G \cap H = h^H$ for every $h \in H$ (where h^G is the conjugacy class of h in G). If H is not nilpotent, this is not necessarily true, as shown, for instance, by $H = \mathsf{S}_{p-1}$ and $G = \mathsf{S}_p$ for $p > 4$ prime.

If the attention is turned to complex characters, it is trivial to observe that if H has a normal complement in G, then every irreducible character $\alpha \in \mathrm{Irr}(H)$ extends to some $\beta \in \mathrm{Irr}(G)$. Of course, it was natural to ask about the converse, and this question was settled in 1989 by P. Ferguson [F], using the Classification of Finite Simple Groups (CFSG).

At first sight, this result does not seem to admit a modular version with p-Brauer characters, if only by considering the case where $H \in \mathrm{Syl}_p(G)$. Less trivially, for $p = 3$, notice that all irreducible 3-Brauer characters of A_4 extend to A_5. In fact,

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note that if $G(q)$ is a finite simply connected simple group of Lie type over the field of q elements, then all its irreducible representations in the defining characteristic extend even to the simple algebraic group.

The main result in this note is to prove the following.

Theorem A. *Let G be a finite group, let H be a Hall subgroup of G, and let p be a prime not dividing $|H|$. Then every $\alpha \in \mathrm{Irr}(H)$ extends to a p-Brauer character of G if and only if H has a normal complement in G.*

The easier case where H is solvable is proven in Section 2 below, without using the CFSG. In the case where H is not solvable, our new strategy here is not to assume that every character of H extends to G, but only that every *primitive* character does, together with the usual group theoretical conditions that follow from the character extension property. Then CFSG is used.

Of course, by choosing any prime p not dividing $|G|$, we see that Theorem A implies the main result of [F].

For ordinary characters and almost-simple groups G, we prove a stronger result: we replace the Hall subgroup assumption in Theorem A by the condition that H has odd index in G. (See Theorem 4.8.)

2. Solvable Hall subgroups

Our notation for characters is as in [I]. Our notation for Brauer characters is as in [N1]. Hence, we have chosen a fixed maximal ideal M of the ring of algebraic integers $\mathbf{R}$ containing p, worked in the algebraically closed field $F = \mathbf{R}/M$, and we have defined a set $\mathrm{IBr}(G)$ of the irreducible p-Brauer characters of every finite group G, for the rest of the paper.

In this section, we first dispose of the easy case of Theorem A where H is solvable.

If $H \leq G$ and every irreducible character of H extends to G, the following two are essentially the only known group theoretical consequences at our disposal. If G is a finite group, in this paper g^G denotes the conjugacy class of g in G. Recall that a subgroup H of G is *c*-closed in G if $h^G \cap H = h^H$ for every $h \in H$.

Lemma 2.1. *Let $H \leq G$, p a prime such that p does not divide $|H|$. Suppose that every character of H extends to a p-Brauer character of G. Then*

(i) For every $h \in H$ we have that $h^G \cap H = h^H$.

(ii) For every $L \trianglelefteq H$, there is $K \trianglelefteq G$ such that $K \cap H = L$, and such that every character of HK/K extends to a p-Brauer character of G/K.

Proof. Suppose that $h^g \in H$ for some $g \in G$. Let $\alpha \in \mathrm{Irr}(H)$ and let $\hat{\alpha} \in \mathrm{IBr}(G)$ be such that $\hat{\alpha}_H = \alpha$. Then $\alpha(h^g) = \hat{\alpha}(h^g) = \hat{\alpha}(h) = \alpha(h)$, and (a) follows from the orthogonality of characters.

For every $\alpha \in \mathrm{Irr}(H/L)$, choose $\hat{\alpha} \in \mathrm{IBr}(G)$ such that $\hat{\alpha}_H = \alpha$. Let

$$K = \bigcap_{\alpha \in \mathrm{Irr}(H/L)} \ker(\hat{\alpha}) \trianglelefteq G\,.$$

Notice that

$$K \cap H = \bigcap_{\alpha \in \mathrm{Irr}(H/L)} \ker(\alpha) = L\,.$$

Suppose that $\gamma \in \mathrm{Irr}(HK/K)$. Then $\gamma_H = \alpha \in \mathrm{Irr}(H/L)$. Since $K \subseteq \ker(\hat{\alpha})$, we have that $\hat{\alpha}_{KH} = \gamma$. □

Lemma 2.2. *Suppose that H is a Hall subgroup of G which is c-closed in G. Let $N \trianglelefteq G$ such that $N \subseteq H$ and H/N has a normal complement K/N in G/N. Then H has a normal complement in G.*

Proof. Let π be the set of primes dividing $|H|$. Now let $n \in N$ and $k \in K$. Then $n^G \cap H = n^H$ by hypothesis. Thus $n^k = n^h$ for some $h \in H$, and thus $n^{kh^{-1}} = n$ and thus $K \subseteq \mathbf{C}_G(n)H$. Hence $G = KH \subseteq \mathbf{C}_G(n)H$ and we have that $|G : \mathbf{C}_G(n)|$ is a π-number. Hence so is $|K : \mathbf{C}_K(n)|$. Since N is a Hall π-subgroup of K, we conclude that $K = N\mathbf{C}_K(n)$ for every $n \in N$. Hence all N-conjugacy classes are K-invariant. Let A be a complement of N in K. By coprime action we conclude that $A \trianglelefteq K$. (Apply for instance, Lemma 13.10 of [I].) Hence $A \trianglelefteq G$ since A is characteristic in $K \trianglelefteq G$. □

Theorem 2.3. *Let G be a finite group, let H be a Hall subgroup of G, and let p be a prime not dividing $|H|$. Assume that H is solvable. If every $\alpha \in \mathrm{Irr}(H)$ extends to a p-Brauer character of G, then H has a normal complement in G.*

Proof. We argue by induction on $|G|$. By Lemma 2.1(i), we have that H is c-closed in G. Let M be a minimal normal subgroup of H, so that M is an abelian q-group for some prime q. By Lemma 2.1, let $K \trianglelefteq G$ such that $K \cap H = M$ and such that all irreducible characters of HK/K extend to a p-Brauer character of G/K. Since $K > 1$, by induction, there is $L \trianglelefteq G$ such that $HL = G$ and $H \cap L = M$. If $M \trianglelefteq G$, then we apply Lemma 2.2, and we are done. So we may assume that $\mathbf{N}_G(M) < G$. By induction, we have that $\mathbf{N}_G(M) = HX$ where $X \trianglelefteq \mathbf{N}_G(M)$ and $H \cap X = 1$. Now, $\mathbf{N}_L(M) = M \times X$. Since H is a Hall subgroup, we have that M is a Sylow q-subgroup of L. Also $M \subseteq \mathbf{Z}(\mathbf{N}_L(M))$. Therefore M has a normal q-complement Y in L by Burnside's Theorem. Then $Y \trianglelefteq G$ is a complement of H in G. □

3. Primitive Characters

Recall that an irreducible complex character $\chi \in \mathrm{Irr}(G)$ is primitive if there is no $U < G$ and $\gamma \in \mathrm{Irr}(U)$ such that $\gamma^G = \chi$.

Lemma 3.1. *Let G be a finite group, let H be a Hall subgroup of G, let N be a normal subgroup of G such that $H \cap N = 1$. Let p be a prime not dividing $|H|$. Let $\alpha \in \mathrm{Irr}(H)$ be primitive. If α extends to some p-Brauer character β of G, then β_N is $\beta(1)\nu$ for some linear character $\nu \in \mathrm{IBr}(N)$. Moreover, there is an extension $\gamma \in \mathrm{IBr}(G)$ of α that contains N in its kernel.*

Proof. Let $\beta \in \mathrm{IBr}(G)$ such that $\beta_H = \alpha$. We claim that $\beta_N = \beta(1)\nu$, for some linear $\nu \in \mathrm{IBr}(N)$. Let $\mathcal{X}$ be an F-representation affording β. Let $\mathbf{Z}(\beta)$ be the subgroup $\{g \in G \mid \mathcal{X}(g) = \lambda_g I\}$, where $\lambda_g \in F$ is a scalar. For each prime $q \neq p$ dividing $|N|$, by coprime action, there exists an H-invariant Sylow q-subgroup Q of N. Let $U = HQ$. Let $\chi = \beta_U \in \mathrm{Irr}(U)$. Let $\lambda \in \mathrm{Irr}(Q)$ be under χ. Then $\lambda(1)$ divides $\chi(1)$, which divides $|H|$. Since $\lambda(1)$ divides $|Q|$, we have that λ is linear. By the Clifford correspondence (Theorem 6.11 of [I]), there is $\tau \in \mathrm{Irr}(U_\lambda)$ lying over λ such that

$$\tau^U = \chi\,.$$

By Mackey, $(\tau_{H_\lambda})^H = \alpha$. Since α is primitive, then we conclude that $H = H_\lambda$ and that λ is H-invariant. Therefore $\beta_Q = \beta(1)\nu$, for some linear character $\nu \in \mathrm{Irr}(Q)$, and we conclude that $Q \subseteq \mathbf{Z}(\beta_U) \subseteq \mathbf{Z}(\beta)$. If $W = \mathbf{Z}(\beta) \cap N$, it then follows that N/W is a p-group and that $\beta_W = \beta(1)\delta$ for some linear character $\delta \in \mathrm{IBr}(W)$. By Green's theorem (8.11 of [N1]), we have that $\beta_N = \beta(1)\nu$ for some linear $\nu \in \mathrm{IBr}(N)$, as claimed.

Next, we claim that ν extends to G. By Theorem 8.29 of [N1], it suffices to show that ν extends to R, where R/N is a Sylow r-subgroup of G/N for all primes r. If r divides $|H|$, this follows from Theorem 8.23 of [N1]. Suppose that r does not divide $|H|$. The r does not divide $\alpha(1) = \beta(1)$. Write $\beta_R = e_1\delta_1 + \ldots + e_t\delta_r$, where $\delta_i \in \mathrm{Irr}(R|\nu)$. By Theorem 8.30 of [N1], then notice that for some i we necessarily have that $\delta_i(1)/\nu(1) = 1$. This proves the claim. Let $\mu \in \mathrm{Irr}(G)$ be an extension of ν. If π is the set of primes dividing $|N|$, we may assume that $o(\mu)$ is a π-number, by considering the π-part μ_π of μ. In particular, $\mu_H = 1_H$. Now, consider $\gamma = \beta\bar{\mu} \in \mathrm{IBr}(G)$, which has N in its kernel and extends α. □

If G is π-separable, H is a Hall π-subgroup and $N \trianglelefteq G$ is a π'-subgroup, it is also not difficult to prove that if $\alpha \in \mathrm{Irr}(H)$ extends to some p-Brauer character of G, then there is an extension containing N in its kernel. (See the proof of Lemma 2.1 of [MN].) Using this and Lemma 2.2, we could easily prove Theorem A in the case where G is π-separable.

We shall use the following extension theorem.

Theorem 3.2. *Suppose that $N = S^n$ is a minimal non-abelian normal subgroup of G, where S is simple, and let $\tau \in \mathrm{Irr}(S)$. If τ extends to some $\beta \in \mathrm{Irr}(\mathrm{Aut}(S))$, then $\gamma = \tau \times \cdots \times \tau$ extends to some $\chi \in \mathrm{Irr}(G)$. Furthermore, if τ is primitive, then χ is primitive.*

Proof. See, for instance, Lemma 5 of [BCLP], or Corollary 10.5 of [N2]. We have that γ is primitive by Theorem 4.5 of [H]. By Mackey (see Problems 5.6 and 5.7 of [I]), we have that χ is primitive. $\square$

We notice the following variation of Lemma 4.2 of [M].

Lemma 3.3. *If S is a finite nonabelian simple, group then there exists a primitive complex character $\chi \neq 1$ such that χ extends to* $\mathrm{Aut}(S)$.

Proof. If S is sporadic, this follows by an easy inspection. In general, the minimal degree character extends and this is primitive, except in the following cases: If $S = M_{12}$, we take the character of degree 45; if $S = J_2$, we take the character of degree 63; if $S = J_3$, then we take the character of degree 324; if $S = He$, we take the character of degree 680; if $S = HN$, we take the character of degree 760.

If $S = \mathsf{A}_n$, $n \neq 6$, let χ be the character of degree $n-1$. If $S = \mathsf{A}_6$, then we take the character of degree 9. If S is of Lie type in characteristic p, then it is known that the Steinberg character extends to the full automorphism group. Since the degree is a power of p, this character could only be imprimitive if S has a subgroup of index a power of p. This only happens for $S = \mathrm{PSL}(3,2)$, $\mathrm{PSp}(4,3)$ or $\mathrm{PSL}(2,11)$ (see [Se, Theorem A] and [G, Theorem 1]). In the last case, the subgroups of index a power of p is perfect, the Steinberg character is primitive. In the remaining cases, we take a degree 6 character. $\square$

4. Almost Simple Groups

Our objective in this Section is to prove Theorem 4.7 for almost simple groups G with socle N. We start with the following.

Lemma 4.1. *Let G be a finite group and $N \trianglelefteq G$. Let H be a Hall subgroup of G and assume that $Q = H \cap N$ is an abelian q-group. Suppose that N does not have a normal q-complement. Then there exists $a \in Q$ such that $a^G \cap Q \neq a^H$.*

Proof. Notice that $Q = H \cap N \in \mathrm{Syl}_q(N)$ using that H is a Hall subgroup. Write $L = \mathbf{N}_G(Q)$. Suppose that $a^G \cap Q = a^H$ for every $a \in Q$. Fix $a \in Q$. If $l \in L$, then there is $h \in H$ such that $a^{lh^{-1}} = a$. Thus $L = H\mathbf{C}_L(a)$. Hence $|L : \mathbf{C}_L(a)|$ is coprime with $\mathbf{N}_N(Q)/Q$. Since $\mathbf{N}_N(Q) \trianglelefteq L$, we have that $\mathbf{N}_N(Q) \subseteq \mathbf{C}_L(a)$. Therefore $Q \subseteq \mathbf{Z}(\mathbf{N}_N(Q))$. By Burnside's theorem, we have that N has a normal q-complement, but this is not possible. $\square$

We first consider the case where N is a sporadic group, with a slightly more general hypothesis.

Lemma 4.2. *Let G be an almost simple group, with socle N, a sporadic simple group. Suppose that H is a subgroup of G of odd index and $H \cap N$ is a minimal normal subgroup of H with $H < HN$. Then $G = N$ and either H has odd prime order or $G = M_{23}$ and $H = M_{22}$. In both cases, H is not c-closed in G.*

Proof. Recall that $|G/N| \le 2$. If $H \cap N$ is a 2-group, then H is a 2-group and since $|H \cap N| \ge 4$, $H \cap N$ is not a minimal normal subgroup of H. Since H contains a Sylow 2-subgroup of G, $H \cap N$ cannot be a p-group for p odd.

So we may assume that $L = H \cap N$ is a direct product of nonabelian simple groups. Let M be a maximal subgroup of N containing L. Since L has odd index in N, $\mathbf{O}_2(M) \le L$ and so $\mathbf{O}_2(M) \le \mathbf{O}_2(L) = 1$. Since L is a perfect and contains a Sylow 2-subgroup of N, M modulo the last term of its derived series has odd order.

The maximal subgroups of the sporadic simple groups are all given in [Wi] aside from the case of $N = Fi_{24}'$. See [LW] for this case. Aschbacher also studies the overgroups of Sylow 2-subgroups [A]. See also [DLP].

By inspection of the maximal subgroups of odd index in a sporadic simple group, the only cases where $\mathbf{O}_2(M) = 1$ and $M/[M, M]$ has odd order are $N = M_{23}$ and $M = M_{22}$ or $N = McL$ and $M = M_{22}$ or $\mathrm{PSU}(4, 3)$. All the proper subgroups of odd index of M_{22} and $\mathrm{PSU}(4, 3)$ have nontrivial $\mathbf{O}_2$ and so $H \cap N$ is maximal in M. Checking the character tables shows these subgroups are not c-closed. $\square$

We next consider symmetric and alternating groups.

Lemma 4.3. *Let $N = \mathsf{A}_n$ with $n > 4$. Assume that $G = HN \le \mathsf{S}_n$ and H is a proper subgroup of G containing a Sylow 2-subgroup of G.*

- (i) *If any two involutions of $H \cap \mathsf{A}_n$ conjugate in G are conjugate in H, then $H = \mathsf{A}_{n-1}$ or S_{n-1} with n odd.*
- (ii) *If H satisfies (i) and is a Hall subgroup, then $H = \mathsf{A}_{n-1}$ or S_{n-1} with n prime.*
- (iii) *If any two A_n conjugate elements of $H \cap \mathsf{A}_n$ are H-conjugate, then $H = \mathsf{S}_{n-1}$.*

Proof. We first prove (i). We induct on n. If $n \le 8$, this follows by inspection. First suppose that H is not transitive. If H has an invariant subset Γ of size d with $2 \le d \le n/2$, then H contains two involutions each moving exactly 4 points – one trivial on Γ and one nontrivial on Γ and its complement which is a contradiction.

Thus H has precisely 2 orbits of sizes 1 and $n - 1$ and so n is odd. Note that we can work in $G \cap \mathsf{S}_{n-1}$ and by induction there are no examples.

Suppose H is transitive but not primitive and let d be the size of a block. If $d \ge 4$, there are two involutions in H moving exactly 4 points with one non-trivial on two blocks and one trivial on all but one block and the result holds. If $d = 2$, there are two such involutions with one fixing all blocks and another switching two blocks. If $d = 3$, then since $n > 8$, we similarly find two involutions moving exactly 8 points acting differently on blocks unless $n = 9$. In that case the stabilizer of the blocks does not contain a Sylow 2-subgroup.

If H is primitive and contains a double transposition by a classical theorem of Manning (see [W]), H contains A_n for $n > 8$, a contradiction.

Clearly (ii) follows. Now we prove (iii). By (i) we may assume $H = \mathsf{A}_{n-1}$ with n odd. Let $x \in H$ be an $n - 2$ cycle. Then there exists $x' \in H$ another $n - 2$ cycle not conjugate to $x \in H$. Since x commutes with a transposition in S_n, x and x' are conjugate in A_n. $\square$

In the one family above where $N = \mathsf{A}_n$, $G = HN$, H is a subgroup of odd index and $H \cap N$ is a minimal normal subgroup of H, we need to prove the existence of a primitive irreducible character of $H \cap N$ that extends to H but not to N. So $H = \mathsf{S}_{n-1}$ and $G = \mathsf{S}_n$ with $n > 5$ odd. The only prime dividing $|G|$ but not $|H|$ is n if n is prime. Aside from that case, we can take the irreducible representation of $H \cap N$ of degree $n-2$. In the case of characteristic n, we need to produce a different character. We pick a particular one but in fact the argument below shows that there are many such.

Lemma 4.4. *Let $p > 5$ be prime. Let $G = \mathsf{A}_p$ and $H = \mathsf{A}_{p-1}$. Let χ be the irreducible character of H corresponding to the partition $(p-3, 2)$. The following hold:*

(i) *χ extends to* $\mathrm{Aut}(H)$*;*
(ii) *$\chi(1) = (p-1)(p-4)/2$;*
(iii) *χ is primitive for H; and*
(iv) *χ does not extend to an irreducible p-Brauer character ψ of G.*

Proof. Since p does not divide $|H|$, it suffices to prove the first three results over $\mathbb{C}$. Since the partition is not equal to its transpose the first condition holds (for $p = 7$, $\mathrm{Aut}(H)$ properly contains S_6 and one checks directly). The formula for dimensions of S_{n-1} modules yields the second result.

Suppose $\chi = \phi_L^H$ with L a proper maximal subgroup of H and ϕ an irreducible character of L. If $L = \mathsf{A}_{p-2}$, then $\phi(1) = (p-4)/2$. Since the smallest irreducible character of A_n is $n-1$, this is impossible. If $p > 23$, it follows by [L, Lemma 1.1] that there is no other proper subgroup of index less than $\chi(1)$ and so χ is primitive.

The only possibilities for L are primitive subgroups (intransitive and imprimitive groups have index that is too large). By inspection of the maximal subgroups and the (complex) character tables for the remaining cases, we see that there are no possibilities.

We prove the last statement. Let ψ' be the irreducible p-Brauer character of S_p with ψ a constituent of ψ'_G. If ψ' is a hook, then by [P], $\psi'(1) = \binom{p-2}{k}$ for some $0 \le k \le p-2$ and so $\psi(1) \ne \chi(1)$. If not, then ψ' has no p-hooks and so is a p-core and so ψ' is the just the reduction modulo p of the corresponding complex character (by the hook formula for character degrees, p divides $\psi(1)$ and since the Sylow p subgroup has order p, these characters remain irreducible in characteristic p – see also [JK, p. 268]). In particular, $\chi(1) \ne \psi(1)$, a contradiction. $\square$

Note that for $p = 7$, $\mathrm{Aut}(H)$ properly contains S_6. In this case χ is the Steinberg character (viewing $\mathsf{A}_6 = \mathrm{PSL}_2(9)$) and extends to the full automorphism group.

Lemma 4.5. *Let $G = HN$ be an almost simple group with socle N, a finite group of Lie type over the field of q elements and H a proper subgroup of G of odd index.*

Assume that $R = H \cap N$ is a minimal normal subgroup of H. Then one of the following holds:

(i) $N = \mathrm{PSL}(2, q)$, $q \equiv \pm 1 \pmod 5$, $q \equiv \pm 3 \pmod 8$ *and* $R = \mathsf{A}_5$*; or*
(ii) $N = \Omega(7, q)$ *with* $q \equiv \pm 3 \pmod 8$ *and* $R = \Omega(7, 2)$*; or*
(iii) $N = \Omega^+(8, q)$ *with* $q \equiv \pm 3 \pmod 8$ *and* $R = \Omega^+(8, 2)$.
(iv) $R = G(q_0)$ *where q is odd and* $q = q_0^c$ *for some odd c.*

In all cases, there exists a primitive irreducible character γ of R that extends to a complex character of $\mathrm{Aut}(R)$ *and does not extend to an irreducible complex character of N.*

Proof. We use the classification of maximal subgroups of odd index [LS]. Let M be a maximal subgroup of N containing R. Since $\mathbf{O}_2(R) = 1$ and $R/[R, R]$ has odd order, the same is true for M. By the main result of [LS], the only possibilities are for R are as listed above. If R is proper in M, then we must be in the last case and the result follows by induction.

The last fact follows in the first three cases by taking γ of degree $4, 7$ or 28 respectively. These do extend to the automorphism group and by [SZ] are too small to extend to N.

In the last case aside from the possibility that $R = \mathrm{PSL}(2, 7)$ or $\mathrm{PSp}(4, 3)$, let χ be the Steinberg character of R. Then χ is primitive as in the proof of Lemma 3.1. By the main result of [MZ], χ does not extend to N. If the remaining cases, let χ be the unique irreducible character of degree 6. □

Lemma 4.6. *Let $G = HN$ be an almost simple group with socle N, a finite group of Lie type over the field of q elements and H a proper Hall subgroup of G of odd index. Let p be a prime not dividing $|H|$. Assume that $R = H \cap N$ is a minimal normal subgroup of H. Then one of the following holds:*

(i) $N = \mathrm{PSL}(2, q)$, $q \equiv \pm 1 \pmod 5$, $q \equiv \pm 3 \pmod 8$ *and* $R = \mathsf{A}_5$*; or*
(ii) $N = \Omega(7, q)$ *with* $q \equiv \pm 3 \pmod 8$ *and* $R = \Omega(7, 2)$*; or*
(iii) $N = \Omega^+(8, q)$ *with* $q \equiv \pm 3 \pmod 8$ *and* $R = \Omega^+(8, 2)$.

In all cases, there exists a primitive irreducible character character γ of R that extends to a complex character of $\mathrm{Aut}(R)$ *and does not extend to an irreducible p-Brauer character of N.*

Proof. This follows immediately from the previous lemma noting that in the last case there H is not a Hall subgroup. The only additional assertion is in the case of p-Brauer characters. Then we take γ of degree $4, 7$ or 28 respectively. These do extend to the automorphism group and by [SZ] are too small to extend to N in any characteristic p unless possibly p divides q. In that case, take the characters of degree $4, 15$ or 50, respectively and use [Lu]. □

Theorem 4.7. *Let p be a prime. Suppose that G is an almost simple group with socle N and a Hall subgroup H of order not divisible by p such that $H < HN$. Assume that*

$R = H \cap N$ is a minimal normal subgroup of H. Assume that $r^G \cap R = r^H$ for every $r \in R$. Then R is simple non-abelian and has a primitive character $1 \neq \gamma \in \mathrm{Irr}(R)$ that extends to a complex character of $\mathrm{Aut}(R)$ but not to an irreducible p-Brauer character of N.

Proof. We may assume that $G = NH$. By Lemma 4.1, $N \cap H$ is a direct product of nonabelian simple groups. In particular H contains a Sylow 2-subgroup of G. The case where N is of Lie type follows from Lemma 4.6. If N is sporadic, this follows by Lemma 4.2. If N is alternating, the result follows by Lemmas 4.3 and 4.4. □

Essentially the same proof gives:

Theorem 4.8. *Suppose that G is an almost simple group with socle N and a subgroup H of odd index such that $H < HN$. Then there exists a character $1 \neq \gamma \in \mathrm{Irr}(H)$ that does not extend to an irreducible character of G.*

Proof. We may assume that $G = HN$. Suppose that every irreducible character of H extends. By Lemma 2.1, H is c-closed in G and $H \cap N$ is a minimal normal subgroup of H. Now argue precisely as in the previous proof. □

5. Main result

Theorem 5.1. *Let G be a finite group, let H be a Hall subgroup of G which is c-closed in G, with $|G : H|$ odd. Assume that for every $L \trianglelefteq H$ there exists $K \trianglelefteq G$ such that $K \cap H = L$. Let p be a prime not dividing $|H|$. If every primitive character of H extends to some p-Brauer character of G, then H has a normal complement in G.*

Proof. Let G be a counterexample minimizing $|G|$. Let π be the set of divisors of $|H|$. We may assume that $H < G$. Let N be a minimal normal subgroup of G. If $N \subseteq H$, then H/N is c-closed in G/N by Lemma 2.2(c) in [S]. By minimality, we have that H/N has a normal complement K/N in G/N. Then we apply Lemma 2.2. Hence, we may assume that N is not contained in H.

Suppose first that N is an abelian q-group. Then q does not divide $|H|$, and therefore $H \cap N = 1$. We claim that HN/N satisfies the hypothesis of the theorem. We have that HN/N is c-closed in G/N (for instance, by Lemma 2.2(d) in [S]). Suppose that R/N is normal in HN/N. Let $L = R \cap H \trianglelefteq H$. By the hypothesis, there is $K \trianglelefteq G$ such that $K \cap H = L$. Since KN/K is π', then $H \cap KN = H \cap K = L$. Then $KN \cap HN = LN = R$. Finally, let $\alpha \in \mathrm{Irr}(HN/N)$ be primitive. Then α_H is primitive, and therefore it has an extension to G by hypothesis. By Lemma 3.1, there is an extension $\beta \in \mathrm{IBr}(G)$ of α_H that contains N in its kernel. Necessarily we have that $\alpha = \beta_{HN}$. We have that HN/N satisfies the hypothesis, as claimed, and by minimality we are done.

So we may assume that N is a direct product of non-abelian simple groups. Let $W = HN$. We have that H is c-closed in W and that every primitive character of

H extends to some p-Brauer character of W. Also, if $L \trianglelefteq H$, then there is $K \trianglelefteq G$ such that $K \cap H = L$, and therefore $K \cap W \cap H = L$. Suppose that $W < G$. By minimality, we have that H has a normal complement F_0 in W. Now N has π-index in W, so it contains F_0. But $F_0 \trianglelefteq N$ so it is a direct product of some of the factors of N. However, $|F_0| = |W : H|$ is odd. This is a contradiction.

So we may assume that $G = HN$, where N is any minimal normal subgroup of G. Since G/N is a π-group, we have that N is the unique minimal normal subgroup of G (since If $M \neq N$ are minimal normal subgroups, $G \leq G/N \times G/M$ is a π-group).

Using that every normal subgroup of H is the intersection of a normal subgroup of G with H, we deduce that $H \cap N$ is a minimal normal subgroup of H. In fact, it is the unique one.

Let $S \trianglelefteq N$ be simple. Then $N \leq \mathbf{N}_G(S)$. If $H = \mathbf{N}_H(S)h_1 \cup \ldots \cup \mathbf{N}_H(S)h_t$, then notice that $N = S^{h_1} \times \cdots \times S^{h_t}$, using that N is a minimal normal subgroup of G. Also, $H \cap N = (H \cap S)^{h_1} \times \cdots \times (H \cap S)^{h_t}$, using that $H \cap N$ is a minimal normal subgroup of H. Since $S \trianglelefteq\trianglelefteq G$, we have that $S \cap H$ is a Hall π-subgroup of S. Also, we have that $S \cap H$ is a minimal normal subgroup of $\mathbf{N}_H(S)$. Write $J = S \cap H$. We claim that $x^{\mathbf{N}_G(S)} \cap J = x^{\mathbf{N}_H(S)}$ for every $x \in J$. Suppose that $x, x^v \in J$, where $v \in \mathbf{N}_G(S)$, and $x \neq 1$. By hypothesis, we have that $x^v = x^h$ for some $h \in H$. Then $1 \neq x \in S \cap S^{h^{-1}}$, and therefore $S = S^h$. Hence $x^v = x^h$ for some $h \in \mathbf{N}_H(S)$, as claimed.

Now, we claim that the same property holds in JC/C with respect to $\mathbf{N}_G(S)/C$, where $C = \mathbf{C}_G(S)$. Let $x \in J$ and suppose that $x^v \in JC$, where $v \in \mathbf{N}_G(S) = N\mathbf{N}_H(S)$. Write $N = T \times S$ and $v = wst$, where $w \in \mathbf{N}_H(S)$, $s \in S$ and $t \in T$. Then $x^v = (x^{ws})^t$. Since $J \trianglelefteq \mathbf{N}_H(S)$, we have that $x^v = x^{ws}$. Write $x^{ws} = jc$, where $j \in J$ and $c \in C$. Hence $j^{-1}x^{ws} = c \in S \cap C = 1$. Therefore $x^v \in J$. But then $x^v = x^k$ for some $k \in \mathbf{N}_H(S)$. This proves the claim.

We claim now that the almost simple group $\mathbf{N}_G(S)/C$ satisfies the hypothesis of Theorem 4.7 with respect to the socle SC/C and the Hall π-subgroup $\mathbf{N}_H(S)C/C$. Notice first that the socle SC/C is not a π-group. Otherwise, S is a π-group, N is a π-group and $H = G$, which is not possible. Let $R = (\mathbf{N}_H(S)C/C) \cap (SC/C)$. We need to prove that R is a minimal normal subgroup of $\mathbf{N}_H(S)C/C$ and that

$$r^{\mathbf{N}_G(S)/C} \cap R = r^{\mathbf{N}_H(S)C/C}$$

for $r \in R$. Notice now that $J = H \cap S$ is a Hall π-subgroup of $S \trianglelefteq\trianglelefteq G$. Then JC/C is Hall π-subgroup of SC/C (since the natural map $S \to SC/C$ is an isomorphism). Also $\mathbf{N}_H(S)$ is a Hall π-subgroup of $\mathbf{N}_G(S)$ and therefore $\mathbf{N}_H(S) \cap SC$ is a Hall π-subgroup of SC. Thus $(\mathbf{N}_H(S) \cap SC)C/C$ is a Hall π-subgroup of SC/C. Therefore

$$(\mathbf{N}_H(S) \cap SC)C/C = (H \cap S)C/C = JC/C\,.$$

By Dedekind's lemma, $(\mathbf{N}_H(S) \cap SC)C = \mathbf{N}_H(S)C \cap SC$. And we conclude that $R = JC/C$, which is naturally isomorphic to J. Since J is a minimal normal subgroup of $\mathbf{N}_H(S)$, we conclude that R is a minimal normal subgroup of $\mathbf{N}_H(S)C/C$. The equality

$$r^{\mathbf{N}_G(S)/C} \cap R = r^{\mathbf{N}_H(S)C/C}$$

follows now from the previous paragraph.

By Theorem 4.7, we have that R is simple non-abelian and has a primitive character $1 \neq \gamma \in \mathrm{Irr}(R)$ that extends to a character of $\mathrm{Aut}(R)$ but not to a p-Brauer character of SC/C. Using the natural isomorphism $S \to SC/C$, this easily implies that $J = H \cap S$ is a non-abelian simple group, and J has a primitive character $1 \neq \gamma \in \mathrm{Irr}(J)$ that extends to some character of $\mathrm{Aut}(J)$ but not to a p-Brauer character of S. By Theorem 3.2, there is a primitive $\alpha \in \mathrm{Irr}(H)$ such that $\alpha_{H\cap S} = \gamma$. By hypothesis, there is $\tau \in \mathrm{IBr}(G)$ such that $\tau_H = \alpha$. Then $\tau_{H\cap S} = \gamma$, and therefore τ_S extends γ. This is a contradiction. □

We are now ready to prove Theorem A.

Corollary 5.2. *Let G be a finite group and let H be a Hall subgroup of G. Let p be a prime not dividing $|H|$. If every irreducible character of H extends to some p-Brauer character of G, then H has a normal complement in G.*

Proof. By Theorem 2.3, we may assume that H is non-solvable. Hence, $|G : H|$ is odd. Now we apply Lemma 2.1 and Theorem 5.1. □

Notice that the conclusion of Theorem 5.1 does not hold if the hypothesis that every normal subgroup L of H is the intersection of a normal subgroup of G with H is dropped, as shown by S_4 and S_5.

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