Notes on Mathematics II

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Content

Unit 1. An introduction to optimization

Below we introduce various concepts related to optimization as well as several basic theorems and results we will use in later topics.

Optimization problems

Optimization problems arise in numerous fields, including logistics, economics, public services and sports. An example of a typical optimization problem, specifically a production planning problem, is as follows:

An engineering factory makes 4 products (PROD1 to PROD4) on the following machines: 4 grinders, 2 drills and 1 planer. Each product yields a certain contribution to profit (defined as €/unit selling price minus cost of raw materials). These quantities together with the unit production times (hours) required on each process are given in the table below.

	PROD 1	PROD 2	PROD 3	PROD 4
Contribution to profit	10	6	8	4
Grinding	0.5	0.7	0.2	-
Drilling	0.1	0.2	-	0.3
Planer	-	-	0.1	-

If there are 8 working hours in a day, how many units of each product should the factory produce (daily) in order to maximize the total profit?

We have an objective (maximize profit), several unknowns (the number of units of each product we need to produce), and several conditions or limitations (the number of available hours for each machine).

To solve these problems, we will formulate a Mathematical Programming (MP) problem. An MP problem comprises the following elements:

- Variables: these represent the unknown quantities whose optimal values we wish to find.
- Objective function: this mathematical function represents the value we wish to maximize or minimize (typically, profit or costs).
- Constraints: these are usually written as equalities or inequalities involving the variables of the problem; these constraints represent the conditions the solution must satisfy.

For the above example, we would formulate the following MP problem:

$$
\begin{array}{lrcl}
Max & 10x_1 + 6x_2 + 8x_3 + 4x_4 & & \\
s.t.: & 0.5x_1 + 0.7x_2 + 0.2x_3 & \leq & 32 \\
 & 0.1x_1 + 0.2x_2 + 0.3x_4 & \leq & 16 \\
 & 0.1x_3 & \leq & 8 \\
 & x_1, x_2, x_3, x_4 & \geq & 0
\end{array}
$$

In this MP problem, we have four variables (x_1, x_2, x_3 and x_4), which represent the quantities of products (1, 2, 3, and 4), respectively, that we need to produce.

The objective function is $10x_1 + 6x_2 + 8x_3 + 4x_4$, which we want to maximize and which corresponds to total profit.

The letters “s.t.” stand for “subject to” and are always written after the objective functions.

We then have several inequalities, which are the constraints of the problem. The first three inequalities express the limit on the maximum number of hours available for each type of machine (we have one inequality for each machine). The left part of the inequalities (or left-hand side) calculates the total number of hours the machine will be used, while the number on the right (right-hand side or, simply, RHS) is the total number of hours available (8 hours a day times the number of available machines). The last four inequalities (written on a single line) are called non-negativity inequalities, which are needed to avoid negative values for the variables, which would make no sense in this context.

A formulation such as this is also often called a model.

Our goal is to find, from all possible combinations of variables that satisfy the constraints of the model, the combination that gives us maximum profit. This combination of values is called the optimal solution of the problem.

In this example, the optimal solution is $x_1 = 32, x_2 = 0, x_3 = 80, x_4 = 42$, and the optimal value of the objective function is 1128. This means that, to maximize its profit, the company needs to produce 32 units of the first product, 80 units of the third, and 42 units of the fourth (but none of the second) to yield a total profit of 1128€.

Below we will learn several basic concepts, study the classifications of MP problems, introduce several basic theorems, and learn some basic tools for solving this type of problem.

Types of MP problems

We will classify MP problems according to the variables and functions involved and study four of the numerous types of MP problems that exist.

- Non-Linear Programming problems (NLP): This is the largest class of problems we will study. In NLP problems we can find any type of function (including powers, polynomials, logarithms and trigonometric functions). The variables must be continuous, i.e., they must belong to the set of real numbers (they can therefore take any value, including fractional values).
- Linear Programming problems (LP): In LP problems all functions of the model (both the objective functions and the constraints) must be linear and the variables must be continuous. Note that an LP problem can also be considered an NLP problem.

- Classical Programming problems (CP): These problems can contain any type of function but the constraints must all be equalities. As with NLP and LP problems, the variables must be continuous. CP problems can also be considered NLP problems. Some CP problems can also be considered LP problems.
- Integer Linear Programming problems (ILP): In ILP problems all functions of the model must be linear. The variables must be integers, which means that they can take only integer values.

Here we have several examples of MP problems and the class they belong to:

$$\begin{array}{lrcl} Max & x_1 + x_2 & & \\ s.t.: & 3x_1 + 2x_2 & \leq & 12 \\ & x_1 + 2x_2 & \leq & 10 \\ & x_1, x_2 & \geq & 0 \end{array}$$

This is an LP problem (all functions are linear and the variables are continuous). However, it can also be considered an NLP problem.

$$\begin{array}{lrcl} Max & x_1 + x_2 & & \\ s.t.: & x_1^2 + x_2^2 & = & 9 \end{array}$$

This second problem is both a CP problem (the variables are continuous and the only constraint is an equality) and an NLP problem. It is not an LP problem, however, since the function of the constraint is not linear (it contains powers).

$$\begin{array}{lrcl} Max & x_1 + x_2 & & \\ s.t.: & 4x_1 + 2x_2 & \leq & 12 \\ & x_1 - 3x_2 & \leq & 1 \\ & x_1, x_2 & \geq & 0 \ integer \end{array}$$

This third problem is an ILP problem since all the constraints are linear and the variables are integers. When the variables are integers, this is indicated in the model with the word *integer* or with the expressions $x_1, x_2 \in \mathbb{Z}$ (both variables are integers) or $x_1, x_2 \in \mathbb{Z}^+$ (both variables are integers and greater than or equal to 0).

Basic concepts

Feasible and infeasible solutions

A solution is a set of values for the variables of the problem (they do not need to satisfy the constraints). If a solution satisfies all the constraints of the model, it is called a feasible solution. If the solution does not satisfy one or several constraints (we say that it violates the constraints), it is called an infeasible solution. The set of all feasible solutions of the problem is called the solution set (sometimes also called the choice set or feasible set).

If an MP problem has at least one feasible solution, we say that it is a feasible problem. If it has no feasible solution at all, we call it an infeasible problem.

Examples:

$$Min. x - y$$

$$s.t.\ x + y \leq 5$$

$$x \geq 0, y \geq 0$$

In this problem, points (0,0), (2,1), and (5,0) (and many others) are feasible solutions. The problem is feasible since it contains feasible solutions. Points (-1,0) or (2,4), for example, are infeasible solutions. Point (-1,0) violates constraint $x \geq 0$, while point (2,4) violates constraint $x + y \leq 5$.

The following problem is infeasible:

$$Min. x - y$$

$$s.t.\ x + y \leq 5$$

$$x \geq 3, y \geq 3$$

There are no feasible solutions because it is impossible to satisfy all three constraints simultaneously.

Binding constraints and interior/boundary solutions

When a solution satisfies a constraint with equality (LHS=RHS), we say that the constraint is binding for this solution. If the constraint is satisfied, but not with equality, it is non-binding. A feasible solution for which there is at least one binding constraint is called a boundary solution. If the constraints are all non-binding, it is called an interior solution.

Example:

$$Min. x - y$$

$$s.t.\ x + y \leq 5$$

$$x \geq 0, y \geq 0$$

Point (4,1) is a boundary solution because it satisfies the first constraint, $x + y \leq 5$, with equality (5=5). Constraint $x + y \leq 5$ is binding for point (4,1), but constraints $x \geq 0$ and $y \geq 0$ are non-binding (they are satisfied by the point, but not with equality). Other examples of boundary points are (0,0), (0,1), and (5,0).

Point (1,1) is an interior solution. It satisfies all the constraints but none of them with equality. All constraints of the problem are non-binding for this point.

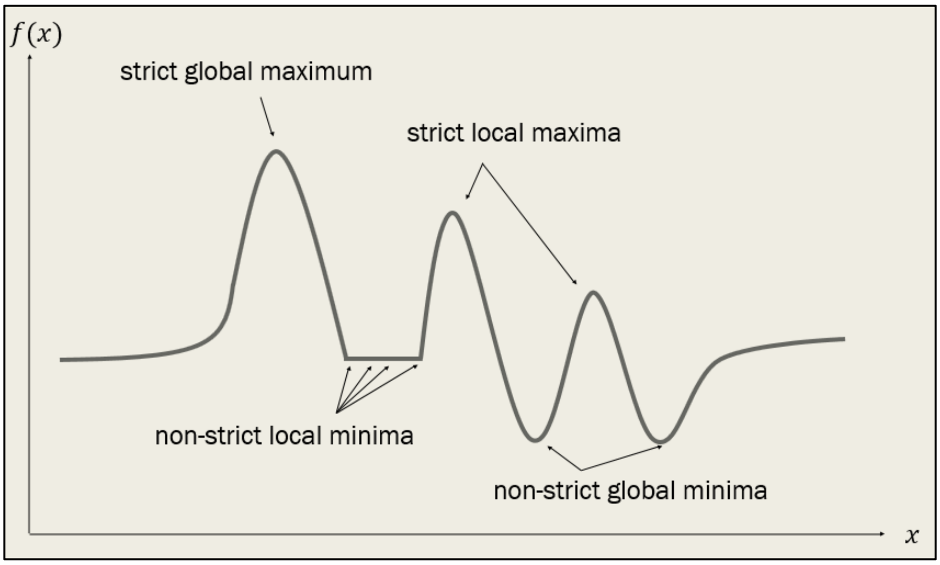
Maxima and minima

In an optimization problem we are looking for a feasible solution that maximizes or minimizes the objective function. We therefore need to define the concept of maximum and minimum points of a function. These are the formal definitions of strict/non-strict local/global maxima:

Given a function $f : \mathbb{R}^n \to \mathbb{R}$, a solution set $S \subset \mathbb{R}^n$ and a feasible solution $x^* \in S$:

- x^* *is a non-strict global maximum of* f *in* S *if for all* $x \in S$ $f(x^*) \geq f(x)$
- x^* *is a strict global maximum of* f *in* S *if for all* $x \in S,\ x \neq x^*$ $f(x^*) > f(x)$
- x^* *is a non-strict local maximum of* f *in* S *if there is an* $\epsilon > 0$ *such that for all* $x \in S$ *satisfying* $||x - x^*|| < \epsilon$, $f(x^*) \geq f(x)$.
- x^* *is a strict local maximum of* f *in* S *if there is an* $\epsilon > 0$ *such that for all* $x \in S,\ x \neq x^*$ *satisfying* $||x - x^*|| < \epsilon$, $f(x^*) > f(x)$.

The strict/non-strict local/global minima can be defined similarly. In the figure below we can see examples of minima and maxima:



A global maximum (minimum) is the highest (lowest) point of the whole function, while a local maximum (minimum) is the highest (lowest) point of a certain region of the function. Note that a global maximum (minimum) is always a local maximum (minimum) too, but not vice versa. A strict maximum (minimum) is a point that is neither surpassed nor equalled by any other point in a certain region of the function, while a non-strict maximum (minimum) is a point that is not surpassed, but is equalled, by other points in its neighbourhood.

Optimal solutions and unbounded problems

In a maximization problem, a global maximum of the objective function in the solution set is called the optimal solution; in a minimization problem, the global minimum is the optimal solution.

If we have a feasible problem without optimal solutions, we say that the problem is unbounded. This occurs when, no matter which point we choose in the solution set, there is always another point in the set that provides a better value of the objective function.

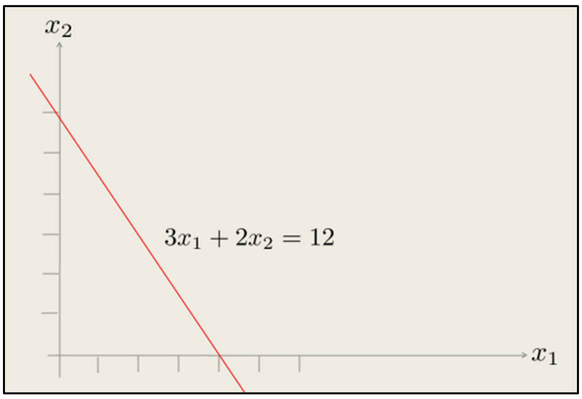
Graphical method for solving MP problems

When the problem has only two variables, we can draw the solution set and solve the problem graphically. We will illustrate this process with an example of an LP problem. This method can be extended to NLPs but it gets more complicated in some cases.

Let us consider the following LP problem:

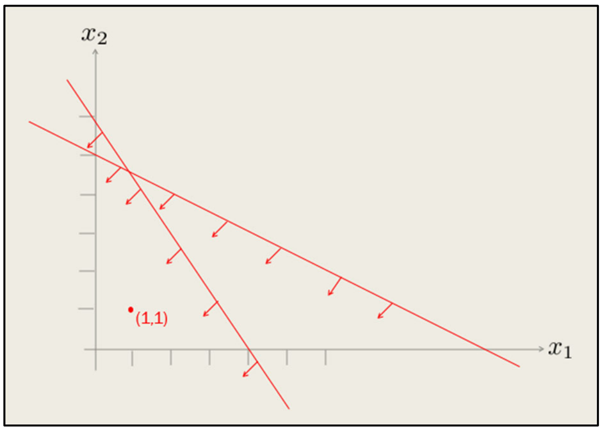
$$\begin{array}{rrcl} Max & x_1 + x_2 & & \\ s.t.: & 3x_1 + 2x_2 & \leq & 12 \\ & x_1 + 2x_2 & \leq & 10 \\ & x_1, x_2 & \geq & 0 \end{array}$$

We will draw each constraint one by one. Let us begin with constraint $3x_1 + 2x_2 \leq 12$. We first consider the equality $3x_1 + 2x_2 = 12$. Since this is a linear function, its graphical representation is a straight line, so we need only two points that satisfy the equality to draw it. Let us take, for example, points (4,0) and (0,6), and draw the line that passes through these points:

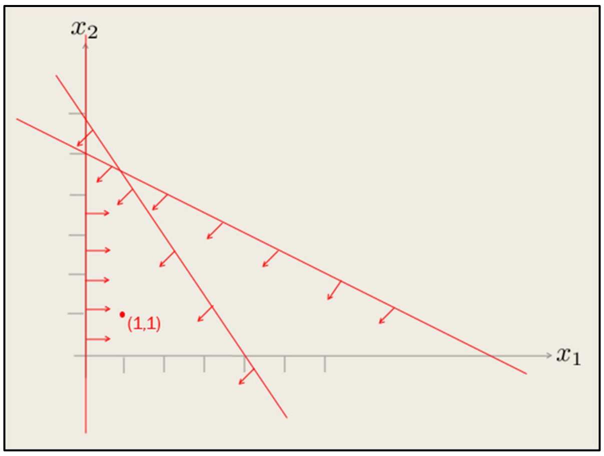


This line divides the space into two halves (two semispaces). The inequality $3x_1 + 2x_2 \leq 12$ therefore defines one of these two halves and we just need to check which one, i.e., the lower half or the upper half. To make sure which one it is, we simply choose one random point (that does not lie on the line we have drawn) – for example, point (1,1) – and check the inequality. Once we put the values into the inequality, we get $5 \leq 12$, which is true, so this point satisfies the constraint. This means that (1,1) belongs to the correct half of the space, so we take the lower half and discard the upper one.

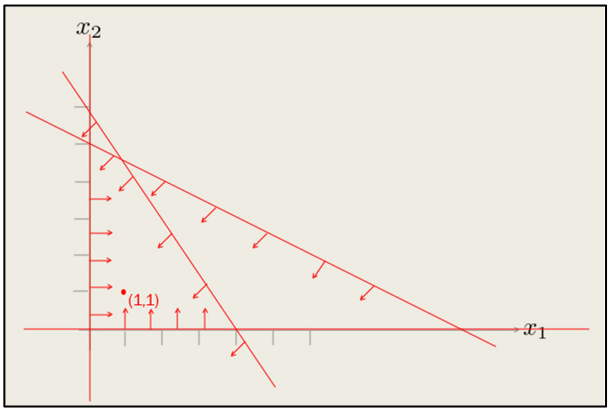
We now proceed in the same way with the second constraint, $x_1 + 2x_2 \leq 10$:



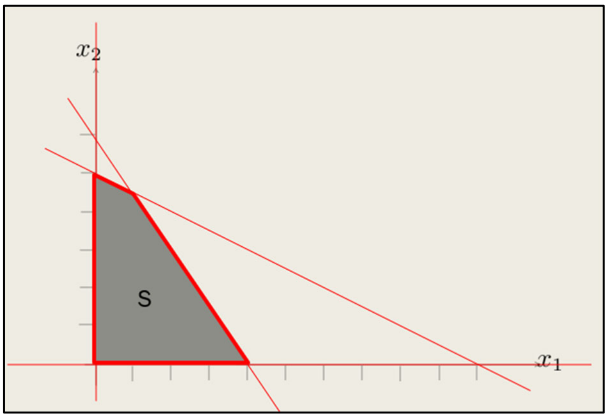
and with the non-negativity inequalities, $x_1 \geq 0$



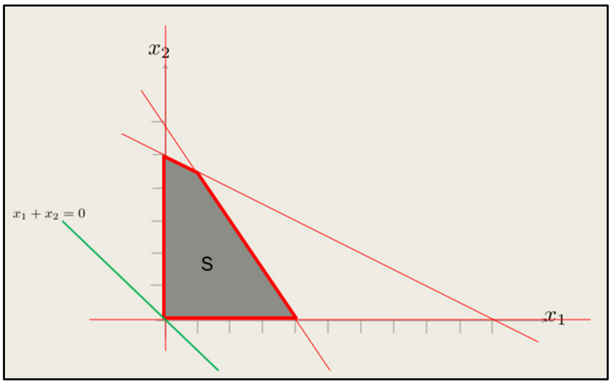
and $x2 \geq 0$



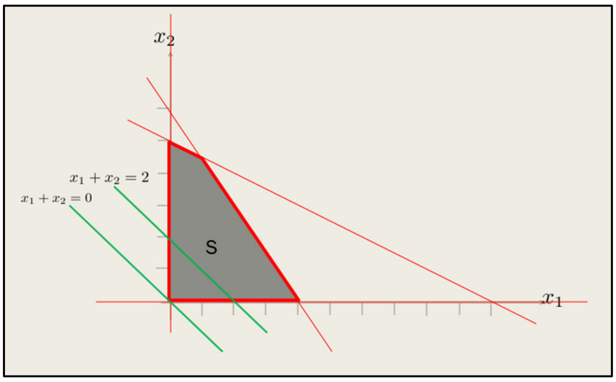
The area resulting from the intersections of all these semispaces is the solution set of the problem:



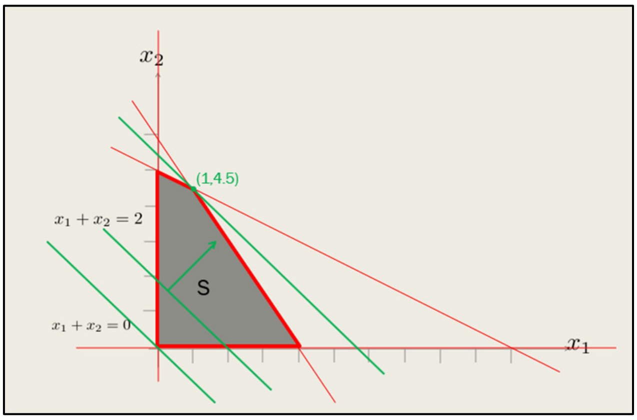
Now we need to draw the objective function to find the global maximum inside the solution set. We therefore need to draw several level curves of the objective function $x_1 + x_2$. For example, we can draw the curve of level 0, i.e., the equality $x_1 + x_2 = 0$:



And now we draw the curve of level 2, $x_1 + x_2 = 2$:



If we draw more level curves, we see that they are all parallel and that the higher the value of the level, the higher the drawing of the line. This means that to maximize the value of the objective function, we must push the line upwards as far as we can while ensuring that at least one point of the solution set is still touched.



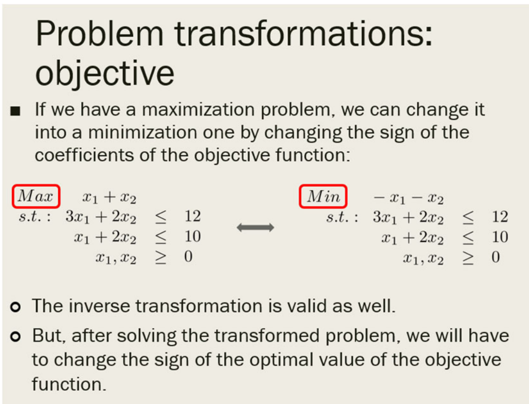
In the above picture we can see the last line we can draw before leaving the solution set completely. The last point of the solution set that is touched by the line is thus the optimal solution of the problem.

Note that if we had had a minimization problem, we would have needed to minimize the value of the objective function and we would have pushed the line in the opposite direction. The optimal solution would therefore have been point (0,0).

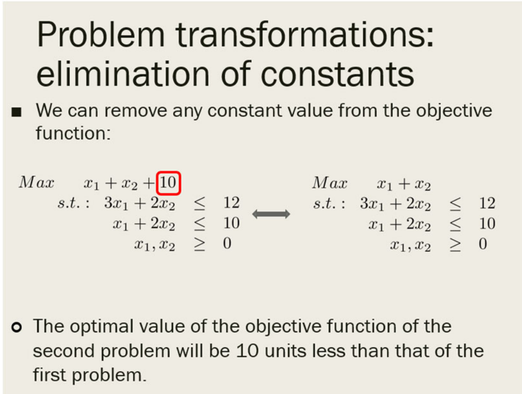
Problem transformations

In some situations we need our problem to be written in a very specific way. If our problem does not meet these requirements, we can apply certain transformations to obtain an equivalent problem that does. This new problem will have the same optimal solution as the original one (or its optimal solution can be easily transformed into the optimal solution of the original one). Below we will study several transformations that can be applied to the objective function, constraints and variables.

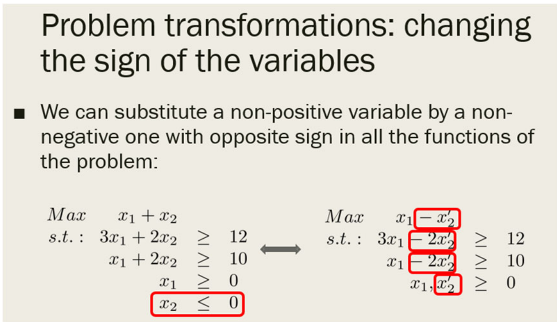
- The objective of the problem can be changed from minimization to maximization or vice versa. All we need to do is multiply the objective function by -1 (i.e., change the sign of all its terms). Remember that once we solve the transformed problem, the optimal value of the objective function will have the opposite sign. If, for example, the value of the objective function in the new problem is 5, this means that the value of the objective function in the original problem is -5.



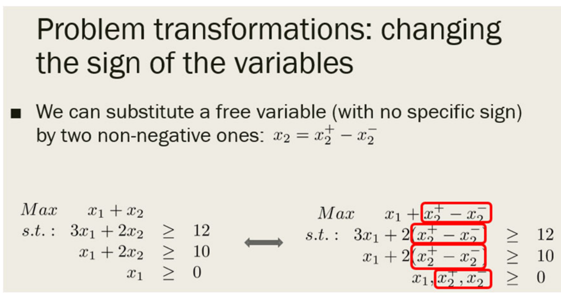
- Any constant term can be removed from the objective function. Just remember to add it back to the value of the objective function if you want to obtain the value of the original problem.



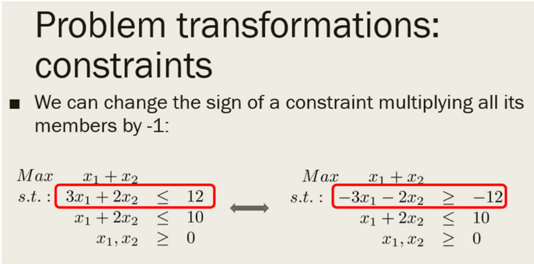
- Variables that are non-positive (≤ 0) can be transformed into non-negative ones (≥ 0) by replacing them with a new variable with the opposite sign in all the functions of the problem. For the sake of simplicity, we can use the same name for the new variable but we must remember that we have changed its sign. For example, if, after solving the transformed problem, this variable in the optimal solution takes the value -2, in the original problem before the transformation, it will take the value 2.



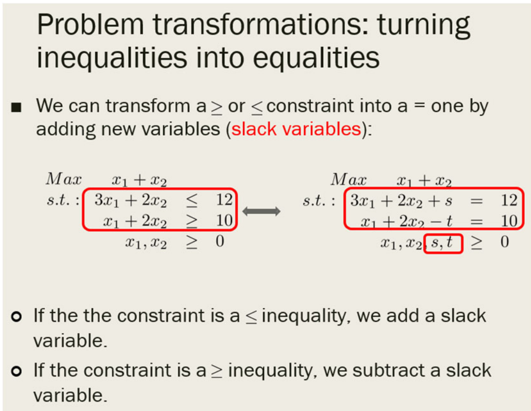
- Free variables (i.e., variables with no constraint specifying their sign) can be replaced by two new variables that are non-negative (≥ 0). Variable x_2, for example, can be replaced by $x_2 = x_2^+ - x_2^-$ in all functions of the problem, where $x_2^+, x_2^- \geq 0$. This is because any (positive or negative) number can always be expressed as the difference of two non-negative numbers. Moreover, it can be expressed as the difference of two non-negative numbers one of which takes the value 0. For example, 2=2-0 or -2=0-2. As before, remember that if you have a solution to the transformed problem, you must undo this transformation if you want to obtain the value of the original variable.



- The sign of constraints can be changed from $\geq$ to $\leq$ or conversely by changing the signs of all its terms (on both sides of the constraint). This is the same as multiplying the whole constraint by -1.



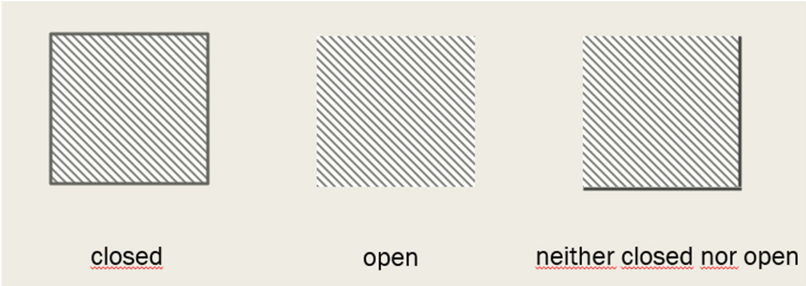
- Inequalities can be transformed into equalities by adding or subtracting a new non-negative variable that we call the slack variable. If the constraint is $\leq$, we must add a slack variable; if it is $\geq$, we must subtract one.



Closed sets

Formally we define a closed set as a set whose complement is an open set. However, since the definition of an open set is not trivial, here we will give an intuitive idea of what a closed set is, as well as a characterization of closed sets that can easily be applied to many mathematical programming problems.

The basic idea is that a set is closed if the border of the set also belongs to it, while the set is open if it does not. If some parts of the border belong to the set while others do not, the set is neither closed nor open.



In the above figure, we have three sets. The first set contains all the points inside the square plus the points on the outline of the square and is therefore a closed set. The second set contains only the points inside the square but not those on the outline and is therefore an open set. The third set contains only two of the edges and is therefore neither closed nor open.

This can be expressed more formally with the following theorem:

Every set defined by non-strict inequalities ($\leq$ or $\geq$) or equalities and continuous functions is a closed set.

Example:

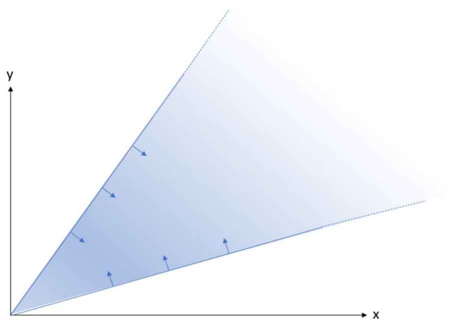
Set $S = \{(x, y, z) \in \mathbb{R}^3 : x + y + z \geq 3, x - y - z = 2\}$ is a closed set, since it is defined by an inequality of type $\geq$ and an equality, and both functions are continuous.

Note that if we have an inequality $<$ or $>$, the set is not closed. Set $S = \{(x, y, z) \in \mathbb{R}^3 : x + y + z > 3, x - y - z = 2\}$ is not a closed set.

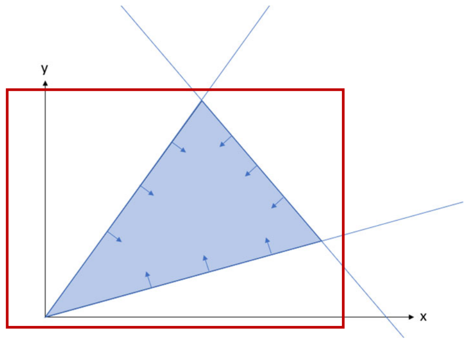
In practice, this means that all solution sets we find in the mathematical programming problems we will solve here are closed sets since they always satisfy this theorem.

Bounded sets

A set is called bounded if there is a certain distance *d* such that all the points of the set are within this distance of each other. Visually, if we can draw the set, it will be bounded if we can enclose it inside a rectangle.



Unbounded set

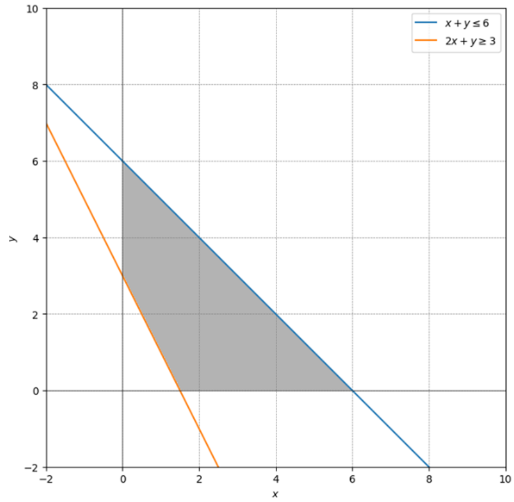


Bounded set

The set on the left is unbounded because there are no limits on the upper-right part of the set, while the set on the right is bounded (it is limited in all directions). Notice, however, that both sets are closed if we consider the edges that limit them are part of the set.

Example:

Let us consider the set $S_1 = \{(x, y) \in \mathbb{R}^2 / x + y \leq 6,\ 2x + y \geq 3, x \geq 0, y \geq 0\}$. We can draw this set to check visually whether it is bounded:



Set S_1 corresponds to the shaded area, which is bounded as it is limited on all sides.

An alternative way to check whether a set is bounded without having to draw it involves finding lower and upper bounds for all variables. If this is possible, the set is bounded. In this example, variables x and y have a lower bound of 0 (because of the non-negativity constraints). Given that both variables are non-negative, from the first constraint $x + y \leq 6$ we can conclude that both variables cannot be greater than 6, and so 6 is a valid upper bound for x and y:

$$0 \leq x \leq 6$$

$$0 \leq y \leq 6$$

Since both variables have lower and upper bounds, the set is bounded. Note that we do not need to find the tightest possible bounds, just some bounds that are valid.

This method is particularly important for situations where we cannot draw the solution set (i.e., problems with three or more variables).

Let us now consider set $S_2 = \{(x, y) \in \mathbb{R}^2 / x + y \leq 6,\ 2x + y \geq 3, x \geq 0\}$. There is a lower bound for variable x, 0, but not for variable y. Variable y can take as low a value as we like. For example, we can take y=-100000. All we need is to take a large value for x (for example, x=100000) and the point will belong to set S_2 (in the first constraint, the left-hand side is 0, and in the second constraint, it is 100000, so both constraints are satisfied).

Compact sets

A set is compact if and only if it is closed and bounded.

Example:

$S_1 = \{(x, y) \in \mathbb{R}^2 / x + y \leq 6,\ 2x + y \geq 3, x \geq 0, y \geq 0\}$ from the previous example is compact because we have already seen that it is bounded and, from the theorem we saw earlier, it is also closed.

Weierstrass theorem

Weierstrass theorem gives us conditions under which we can ensure that an MP problem has at least one optimal solution.

A continuous function defined over a non-empty compact solution set has at least one global maximum and one global minimum in this set

We will apply this theorem to the objective function of MP problems. We must then check whether the objective function is continuous (in fact, all the functions we will consider in this subject are continuous) and whether the solution set is non-empty (the set must contain at least one feasible point) and compact (it is closed and bounded).

To check that the solution set is non-empty, we just have to find one feasible point, i.e., a point that satisfies all constraints of the problem. Then we must check that the solution set is compact. Since all the solution sets we are considering in the MP problems studied here are defined by $\leq$ or $\geq$ inequalities, or equalities, and since all the functions are continuous, we can always conclude that the solution set is closed (from the theorem on closed sets we saw above). We therefore only need to check that the set is bounded for it to be compact. This is the only difficult part we encounter when applying Weierstrass theorem.

To see whether the set is bounded, we can either draw it and check visually (but we can only do this if we have two variables) or try to find lower and upper bounds for all variables. Lower and upper bounds are minimum and maximum values that the variables can never exceed for any feasible solutions. We do not need to find the tightest possible bounds; any valid bound will do.

If all the hypotheses of Weierstrass theorem are satisfied, we can conclude that the objective function will have at least one global minimum and one global maximum in the solution set, which means that our MP problem will have at least one optimal solution.

Note that if any of the theorem's hypotheses does not hold (in practice, this can only happen with the condition that the set must be bounded), we cannot conclude that the problem has an optimal solution. The problem may still have an optimal solution but we cannot prove it using Weierstrass theorem.

Example:

Theoretically prove, if possible, the existence of a global optimum for the following problem:

$$\begin{aligned} Max \quad & x + y \\ s.t.: \quad & 3x + y \geq 4 \\ & x + y \leq 5 \\ & x \geq 0, y \geq 0 \end{aligned}$$

The objective function is continuous (it is a linear function). The solution set is not empty (for example, point (1,1) is a feasible solution). It is also closed since it is defined by continuous functions and constraints that include equality (they are all $\leq$ or $\geq$). Moreover, the set is bounded because all variables have lower and upper bounds; for example:

$$\begin{gathered} 0 \leq x \leq 5 \\ 0 \leq y \leq 5 \end{gathered}$$

We have a continuous objective function and a non-empty compact (closed and bounded) solution set. We can therefore apply Weierstrass theorem and conclude that the problem will have an optimal solution.

Now consider this problem:

$$\begin{aligned} Min \quad & x + y \\ s.t.: \quad & xy \geq 4 \\ & x, y \geq 0 \end{aligned}$$

In this case, the solution set is not bounded, since the variables have no upper bound (x and y can take values as large as we like). Therefore, we cannot apply Weierstrass theorem. This simply means that we cannot guarantee that the problem has an optimal solution, though it could have. In fact, it does and the optimal solution is (2,2).

Local-Global theorem

The Local-Global theorem can be used whenever we have a local minimum or maximum and we want to check whether it is also a global minimum or maximum (and, therefore, an optimal solution of the problem).

If the solution set S is convex and the objective function is (strictly) convex in S, then every local minimum is a (strict) global minimum.

If the solution set S is convex and the objective function is (strictly) concave in S, then every local maximum is a (strict) global maximum.

To apply this theorem, we must first learn what convex/concave functions are and what a convex set is. To identify convex and/or concave functions, we first must learn how to classify matrices.

Classification of matrices

A square matrix (i.e. one with the same number of rows as columns) can be classified as:

- a Positive Definite matrix (PD)
- a Negative Definite matrix (ND)
- a Positive Semidefinite matrix (PSD)
- a Negative Semidefinite matrix (NSD)
- an Indefinite matrix

Classification of diagonal matrices

If the matrix we are considering is diagonal (i.e., all values outside the main diagonal are 0), the matrix can be classified easily as follows:

- If all values of the main diagonal are strictly greater than 0 (>0), then the matrix is PD.
- If all values of the main diagonal are strictly lower than 0 (<0), then the matrix is ND.
- If all values of the main diagonal are greater than or equal to 0 (≥0), then the matrix is PSD.
- If all values of the main diagonal are lower than or equal to 0 (≤0), then the matrix is NSD.
- If the main diagonal contains positive and negative values, the matrix is indefinite.

Examples:

$$\begin{pmatrix}1&0&0\\0&2&0\\0&0&1\end{pmatrix}\ PD\ matrix \qquad \begin{pmatrix}-2&0&0\\0&-3&0\\0&0&-1\end{pmatrix}\ ND\ matrix \qquad \begin{pmatrix}1&0&0\\0&0&0\\0&0&4\end{pmatrix}\ PSD\ matrix$$

$$\begin{pmatrix}0&0&0\\0&-2&0\\0&0&-1\end{pmatrix}\ NSD\ matrix \qquad \begin{pmatrix}1&0&0\\0&-2&0\\0&0&1\end{pmatrix}\ indefinite\ matrix$$

Principal minors of a matrix

Given a square matrix of size $n \times n$, a k-th order principal minor is the determinant of the submatrix that results from removing $n-k$ rows and the same $n-k$ columns of the matrix.

This means that if, for example, we have a 3×3 matrix, it will have three first-order principal minors (the values of the main diagonal), three second-order principal minors (the determinants of the matrix resulting from removing the first, second and third row and column, respectively), and one third-order principal minor (the determinant of the whole matrix).

Example:

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 3 & -1 \\ 1 & -1 & 4 \end{pmatrix}$$

First-order principal minors:

$$A_1 = 1 \quad A_2 = 3 \quad A_3 = 4$$

Second-order principal minors:

$$A_{12} = \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} = 3 \quad A_{13} = \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1 \quad A_{23} = \begin{vmatrix} 3 & -1 \\ -1 & 4 \end{vmatrix} = 11$$

Third-order principal minors:

$$A_{123} = \begin{vmatrix} 1 & 0 & 3 \\ 2 & 3 & -1 \\ 1 & -1 & 4 \end{vmatrix} = -4$$

Leading principal minors of a matrix

Given a square matrix of size $n \times n$, the k-th order **leading** principal minor is the determinant of the submatrix that results from removing the last n−k rows and columns of the matrix. In this case, there is only one leading principal minor of each order.

Example:

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 3 & -1 \\ 1 & -1 & 4 \end{pmatrix}$$

First-order leading principal minor:

$$A_1 = 1$$

Second-order leading principal minor:

$$A_{12} = \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} = 3$$

Third-order leading principal minor:

$$A_{123} = \begin{vmatrix} 1 & 0 & 3 \\ 2 & 3 & -1 \\ 1 & -1 & 4 \end{vmatrix} = -4$$

Jacobi method

When the matrix we wish to classify is not diagonal, we can use the Jacobi method. This method is based on calculation of the principal minors of the matrix.

Suppose we have a square $n \times n$ matrix A. First, we calculate the determinant of A. If $|A| \neq 0$, we say that the matrix is regular. In this case, we must calculate only the leading principal minors. If all leading principal minors are strictly positive, the A is positive definite. If the leading principal minors of an odd order (1st, 3rd, etc.) are strictly negative and those of an even order (2nd, 4th, etc.) are strictly positive, A is negative definite. In any other case, A is indefinite.

If $|A| = 0$, we say that the matrix is singular and we must calculate all principal minors of A. If all principal minors are greater than or equal to 0, A is positive semidefinite. If all odd-order principal minors are less than or equal to 0 and all the even-order ones are greater than or equal to zero, A is negative semidefinite. In any other case, A is indefinite.

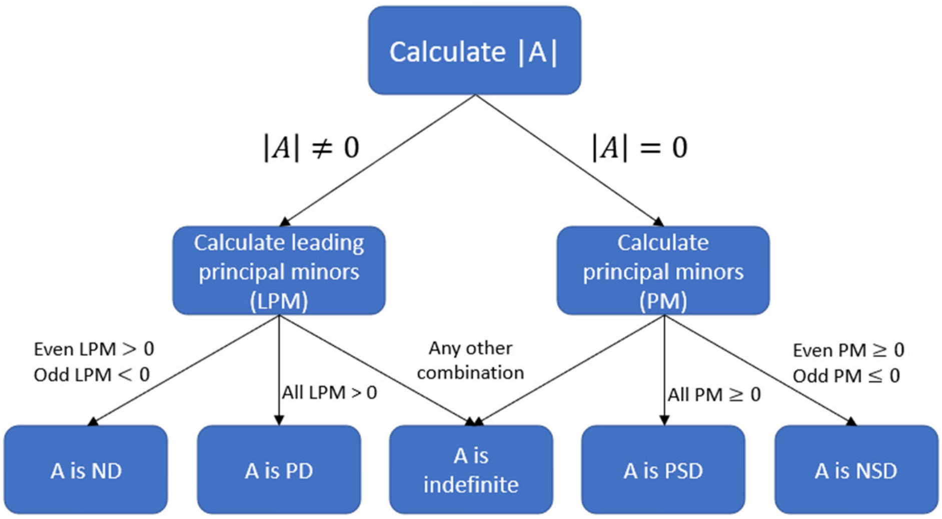


Figure 1. Jacobi method

Examples:

Classify the following matrix:

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 3 & -1 \\ 1 & 1 & 4 \end{pmatrix}$$

First, we calculate $|A| = 10 \neq 0$, so A is regular. This means that we must calculate only the leading principal minors:

$$A_1 = 1 \quad A_{12} = \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} = 3 \qquad A_{123} = |A| = 10$$

All principal minors are strictly positive, so A is PD.

Classify:

$$B = \begin{pmatrix} -1 & 1 & -2 \\ -2 & -1 & 2 \\ 1 & 2 & -4 \end{pmatrix}$$

$|A| = 0$, and the matrix is singular, so we need to calculate all principal minors.

First-order principal minors:

$$B_1 = -1 \quad B_2 = -1 \quad B_3 = -4 \text{ , all are} \leq 0$$

Second-order principal minors:

$$B_{12} = \begin{vmatrix} -1 & 1 \\ -2 & -1 \end{vmatrix} = 3 \quad B_{13} = \begin{vmatrix} -1 & -2 \\ 1 & -4 \end{vmatrix} = 6 \qquad B_{23} = \begin{vmatrix} -1 & 2 \\ 2 & -4 \end{vmatrix} = 0\text{, all are} \geq 0$$

Third-order principal minors:

$$B_{123} = \begin{vmatrix} -1 & 1 & -2 \\ -2 & -1 & 2 \\ 1 & 2 & -4 \end{vmatrix} = 0\text{, which is} \leq 0 \text{ (and also} \geq 0\text{)}$$

We can therefore conclude that B is NSD.

Convex and concave functions

Now we will define concave and convex functions and various methods for determining whether a function is convex, concave, both, or neither.

Convex functions

Formally, a convex function is defined as follows:

Definition

Let $f : D \subset \mathbb{R}^n \to \mathbb{R}$ be a function defined on a convex set D. Then f is convex if for every pair $\overline{x}, \overline{y} \in D$ and $0 \leq \lambda \leq 1$ it holds:

$$f((1-\lambda)\overline{x} + \lambda\overline{y}) \leq (1-\lambda)f(\overline{x}) + \lambda f(\overline{y})$$

This definition says that if we take any pair of values $\bar{x}$ and $\bar{y}$ in the domain of the function and draw a segment joining the points of the graph of the function $f(\bar{x})$ and $f(\bar{y})$, the points of this segment must always lie above the graph of the function.

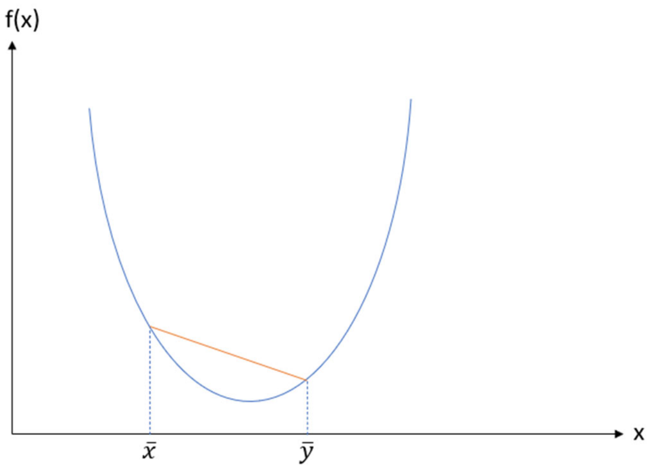
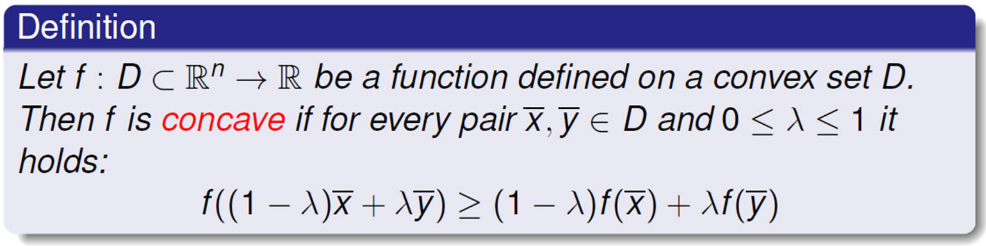


Figure 2. Convex function

Concave functions

Formally, a concave function is defined as follows:



Definition

Let $f : D \subset \mathbb{R}^n \to \mathbb{R}$ be a function defined on a convex set D. Then f is concave if for every pair $\overline{x}, \overline{y} \in D$ and $0 \leq \lambda \leq 1$ it holds:

$$f((1-\lambda)\overline{x} + \lambda\overline{y}) \geq (1-\lambda)f(\overline{x}) + \lambda f(\overline{y})$$

This definition says exactly the opposite of the definition of a convex function, i.e., if we take any pair of values $\bar{x}$ and $\bar{y}$ in the domain of the function and draw a segment joining the points of the graph of the function $f(\bar{x})$ and $f(\bar{y})$, the points of this segment must always lie below the graph of the function.

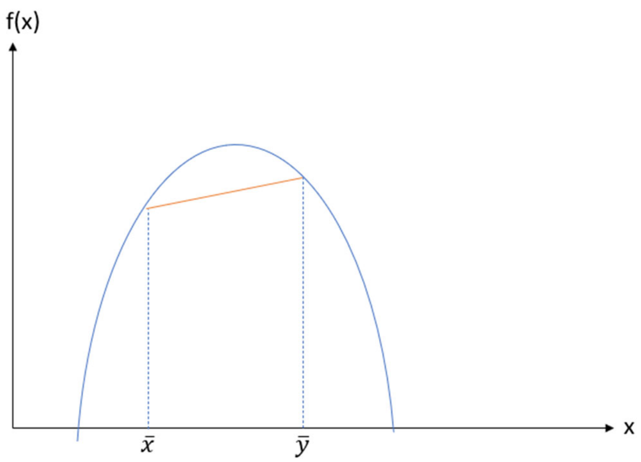


Figure 3. Concave function

Studying the convexity and/or concavity of a function

A function can be convex, concave, both or neither. This can be determined by calculating its Hessian matrix Hf (the matrix containing the second partial derivatives of the function) and classifying it using the methods we saw earlier. Once we know whether the Hessian matrix is PD, PSD, ND, NSD or indefinite, we can apply the following theorem:

Theorem

Let $f : D \subset \mathbb{R}^n \to \mathbb{R}$ be a class C^2 function on a convex open set D.

1. *If $Hf(\overline{x})$ is positive definite (or semi-definite) for every point $\overline{x} \in D$, f is convex.*
2. *If $Hf(\overline{x})$ is negative definite (or semi-definite) for every point $\overline{x} \in D$, f is concave.*

From this theorem, we can conclude that linear functions are both convex and concave (we can use this result directly, i.e., there is no need to calculate the Hessian matrix in these cases).

Examples:

Study whether the following functions are convex and/or concave:

$$f(x, y) = -4x^2 - y^2$$

First we calculate the Hessian matrix:

$$H_f = \begin{pmatrix} -8 & 0 \\ 0 & -2 \end{pmatrix}$$

This is a diagonal matrix and, since all values in the main diagonal are strictly less than 0, it is an ND matrix. This means that the function $f(x, y)$ is concave.

$$f(x, y, z) = 4x^2 + 2y^2 + 3z^2 - 2xy + 4xz$$

Its Hessian matrix is:

$$H_f = \begin{pmatrix} 8 & -2 & 4 \\ -2 & 4 & 0 \\ 4 & 0 & 6 \end{pmatrix}$$

Now we apply the Jacobi method to classify H_f:

$|H_f| = 104$, so H_f is regular. Now we calculate the leading principal minors:

$$H_1 = 8 \quad H_{12} = \begin{vmatrix} 8 & -2 \\ -2 & 4 \end{vmatrix} = 24 \quad H_{123} = |H_f| = 104$$

These are all strictly greater than 0, so H_f is PD and $f(x, y, z)$ is convex.

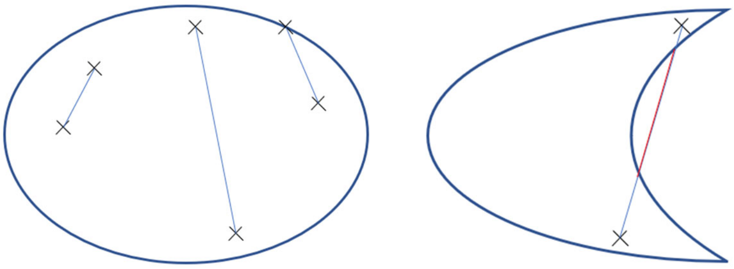
Convex sets

Sets can be convex or not convex but there are no concave sets. We define convex sets as follows:

Definition

A set $C \subset \mathbb{R}^n$ is convex if for any pair of points $\overline{x}, \overline{y} \in C$ and $0 \leq \lambda \leq 1$, then $(1-\lambda)\overline{x} + \lambda\overline{y} \in C$.

The idea behind this definition is that if for every pair of points in the set the segment that joins both points fall inside the set completely, then the set is convex. If there is a pair of points such that the segment that joins them has some points outside the set, then the set is not convex.



A convex set   Not a convex set

The empty set Ø and sets containing one single point are considered convex sets.

Characterisation of convex sets

A hyperplane is a set defined by an equation with a linear function, i.e.:

Definition

An hyperplane in $\mathbb{R}^n$ is a subset of the form

$$H = \{\overline{x} \in \mathbb{R}^n | c_1 x_1 + c_2 x_2 + \ldots + c_n x_n = \alpha\}$$

where $c_i, \alpha \in \mathbb{R}$.

For example, the set defined by the equality $2x + 3y - z = 3$ is a hyperplane. In $\mathbb{R}^2$, hyperplanes are lines, while in $\mathbb{R}^3$, hyperplane are planes.

Theorem

Hyperplanes are convex sets.

A semispace is a set defined by an inequality $\leq$ or $\geq$ and a linear function, i.e.:

Definition

A *semispace* *in* $\mathbb{R}^n$ *is a subset of the form*

$$S = \{\overline{x} \in \mathbb{R}^n | c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \leq \alpha\}$$

where $c_i, \alpha \in \mathbb{R}$.

For example, the set defined by $3x - 4y + 6z \leq 3$ is a semispace.

Theorem

Semispaces are convex sets.

Theorem

The intersection of convex sets is a convex set.

For example, the set $S = \{(x, y, z) \in \mathbb{R}^3 : 2x + 3y - z = 3, 3x - 4y + 6z \leq 3\}$ is convex, since it is the intersection of a hyperplane and a semispace, which are both convex sets.

If the function is not linear, for example $2x^2 + 3y - z = 3$ or $3xy - 4yz + 6z \leq 3$, the set is neither a hyperplane nor a semispace, so we cannot use the previous results to show that the set is convex. In these cases, to check whether the set is convex, we can use the following property

Property

Let $f : D \subset \mathbb{R}^n \to \mathbb{R}$ *be a function defined on a convex set* D.

4. *If* f *is a convex function and* $\alpha \in \mathbb{R}$, *the lower level set*

$$D_\alpha = \{\overline{x} \in D | f(\overline{x}) \leq \alpha\}$$

is convex.

5. *If* f *is a concave function and* $\alpha \in \mathbb{R}$, *the upper level set*

$$D^\alpha = \{\overline{x} \in D | f(\overline{x}) \geq \alpha\}$$

is convex.

For example, the set defined by the inequality $x^2 + y^2 \leq 2$ is convex, since the function $f(x, y) = x^2 + y^2$ is convex. However, the set defined by $x^2 + y^2 \geq 2$ is not convex.

Equalities whose function is not linear are not convex sets.

If we have a set that is defined by several equalities or inequalities and at least one of these is not a convex set, we cannot conclude that the set is convex. Nor can we claim that it is not a convex set, however, since the intersection of a set that is not convex with other sets can sometimes produce convex sets.

Examples:

$$S_1 = \{(x, y) \in \mathbb{R}^2: x + 2y \leq 4, 3x - 2y \geq 7\}$$

Both inequalities that define S_1 are semispaces, so S_1 is the intersection of convex sets and is also a convex set.

$$S_2 = \{(x, y, z) \in \mathbb{R}^3: -4x^2 - y^2 - 2z^2 \geq 7\}$$

The function of this inequality is not linear, so we must study whether it is convex or concave. We calculate its Hessian matrix, which is:

$$H_f = \begin{pmatrix} -8 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$

This matrix is diagonal and its diagonal values are strictly negative, so it is ND and the function is concave. Set S_2 is defined by a $\geq$ inequality and a concave function, so it is a convex set.

$$S_3 = \{(x, y, z) \in \mathbb{R}^3: x + 2y + z = 4, -4x^2 - y^2 - 2z^2 \leq 7\}$$

Set S_3 is defined by two inequalities. The first one is a hyperplane, because the function is linear, so this part is a convex set. However, the second inequality is non-linear of type $\leq$, and its function is concave (we saw this in the previous example). This part is therefore not convex. So we have the intersection of a convex set and a non-convex set and we cannot conclude that the resulting set S_3 is convex (nor can we conclude that it is not convex).

Now that we know how to study convexity and concavity, we can use the Local-Global theorem we saw earlier.

Examples of the Local-Global theorem:

Given the problem

$$\begin{array}{ll} Min & x^2 + y^2 + 2x + 4y + 10 \\ s.t.: & x + y = 21 \end{array}$$

and knowing that (11,10) is a local minimum apply the Local-Global.

To apply the Local-Global theorem to this point, we need the objective function to be convex (because the point is a local minimum) and the solution set to be convex.

First we study the objective function $f(x, y) = x^2 + y^2 + 2x + 4y + 10$. Its Hessian matrix is $H_f = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, which is PD, so $f(x, y)$ is convex. The solution set is convex because it is defined by a hyperplane. Therefore, we can apply the Local-Global theorem and conclude that the point (11,10) is a global minimum (and, therefore, an optimal solution) of this problem.

Unit 2. Non-Linear Programming

As we saw in Unit 1, in this subject we will consider three types of mathematical programming problems: Non-linear, Linear and Integer Linear. In this unit we will study Non-Linear Programming (NLP) problems.

In Unit 1 we learnt that any problem whose variables are continuous can be considered an NLP problem. In this unit we will concentrate on problems that contain one or more non-linear functions (the objective function or some of the constraints) since problems in which all functions are linear are LP problems and can best be addressed by using the tools we will see in units 3, 4, and 5 (though, theoretically, they can also be solved as NLP problems).

Example

Let us consider a problem similar to the first one we saw in unit 1 where x_1, x_2, x_2, x_4 are the units of each product:

An engineering factory makes 4 products (PROD1 to PROD4) on the following machines: 4 grinders, 2 drills and 1 planer. Each product yields a certain contribution to profit (defined as €/unit selling price minus cost of raw materials). These quantities together with the unit production times (hours) required on each process are given in the table below.

	PROD 1	PROD 2	PROD 3	PROD 4
Contribution to profit	$8+0.1x_1$	6	8	$6-0.2x_4$
Grinding	0.5	0.7	-	-
Drilling	0.1	0.2	-	0.3
Planing	-	-	0.01	-

If there are 8 working hours in a day, what should the factory produce (daily) in order to maximize the total profit?

Note that the profit associated with one unit of each product is now not constant but depends on the number of units produced. For example, the profit from one unit of product 1 increases when we produce a larger amount of it but the profit from one unit of product 4 decreases when its production is larger.

This problem is formulated as follows:

$$\begin{array}{llrcl} Max & (8+0.1x_1)x_1+6x_2+8x_3+(6-0.2x_4)x_4 & & & \\ s.t.: & & 0.5x_1+0.7x_2 & \leq & 32 \\ & & 0.1x_1+0.2x_2+0.3x_4 & \leq & 16 \\ & & 0.01x_3 & \leq & 8 \\ & & x_1,x_2,x_3,x_4 & \geq & 0 \end{array}$$

The objective function of the problem is then $(8 + 0.1x_1)x_1 + 6x_2 + 8x_3 + (6 - 0.2x_4)x_4 = 8x_1 + 0.1x_1^2 + 6x_2 + 8x_3 + 6x_4 - 0.2x_4^2$, which is non-linear, so this is an NLP problem (the non-linear function could also be part of a constraint and the problem would also be NLP).

In what follows, we will learn several basic concepts, study various classifications of MP problems, introduce some basic theorems, and learn some basic tools for solving this type of problem.

Kuhn-Tucker points

When we have an NLP problem, we should look for a special type of points in the solution set called Kuhn-Tucker (KT) points since, as we will see later, the optimal solution of the problem is very often (but not always) a KT point.

Lagrangian function

To define KT points we must first introduce the Lagrangian function, denoted by L, which is written as follows:

$$L(x_1, x_2, \dots, x_n, \lambda_1, \lambda_2, \dots, \lambda_m) = f(x_1, x_2, \dots, x_n) + \sum_{i=1}^{m} \lambda_i (b_i - g_i(x_1, x_2, \dots, x_n)$$

This Lagrangian function has $n + m$ variables, where n is the number of variables in the problem and m is the number of constraints. The first n variables $x_1, x_2, \dots, x_n$ are the same as those in the problem, while the m remaining ones, $\lambda_1, \lambda_2, \dots, \lambda_m$, are called Lagrange multipliers or Kuhn-Tucker multipliers (or simply, multipliers). One λ multiplier is associated with each constraint of the problem.

The first part of the Lagrangian function, $f(x_1, x_2, \dots, x_n)$ is the objective function of the problem. The second part, $\sum_{i=1}^{m} \lambda_i (b_i - g_i(x_1, x_2, \dots, x_n)$, is built using the constraints, where b_i is the right-hand side of constraint i and $g_i(x_1, x_2, \dots, x_n)$ is the function on the left-hand side of the constraint.

For example:

$$\begin{aligned} &Max \quad xyz \\ &s.t. \quad x + y + z \le 3 \\ &\qquad\ \ x^2 + y^2 + z^2 = 9 \\ &\qquad\ \ y \ge 0 \end{aligned}$$

$$L(x, y, z, \lambda_1, \lambda_2, \lambda_3) = xyz + \lambda_1(3 - x - y - z) + \lambda_2(9 - x^2 - y^2 - z^2) - \lambda_3 y$$

Canonical and non-canonical constraints

In a maximization problem, constraints of the form $\leq$ are called canonical, while constraints of the form $\geq$ are called non-canonical. In minimization problems it is exactly the opposite, i.e., $\geq$ constraints are canonical while $\leq$ constraints are non-canonical. Equalities are neither canonical nor non-canonical.

Example:

$$\begin{aligned} &Max \quad xyz \\ &s.t. \quad x+y+z \leq 3 \\ &\qquad\quad x^2+y^2+z^2=9 \\ &\qquad\quad y \geq 0 \end{aligned}$$

In this problem, constraint 1 is canonical, constraint 3 is non-canonical, and constraint 2 is neither canonical nor non-canonical.

Kuhn-Tucker conditions

Kuhn-Tucker points are important because the optimal solution of an NLP problem is often (but not always) one of them. We must therefore be able to check whether a given point is a K-T point or not.

A point is a K-T point if it satisfies the following four conditions:

- Feasibility: the point must be feasible, i.e., it must satisfy all the constraints of the problem.
- Stationary point: the partial derivatives of the Lagrangian function with respect to the problem variables must take the value 0 at this point: $\frac{\partial L}{\partial x_i}(\bar{x}) = 0$.
- Sign: The K-T multiplier associated with canonical constraints must be non-negative ($\lambda \geq 0$) and the multiplier of non-canonical constraints must be non-positive ($\lambda \leq 0$). If the constraint is an equality, its multiplier has no sign condition.
- Complementary slackness: Each constraint is either binding or its associated K-T multiplier takes the value 0. This is the same as saying $\lambda_i\big(b_i - g_i(x_1, x_2, \dots, x_n)\big) = 0$. Equality constraints do not have complementary slackness conditions (they do not need them since they are always binding for feasible points, so the complementary slackness is always be satisfied).

Example:

Write the K-T conditions of the following problem:

$$\begin{aligned} &Max \;\; xyz \\ &s.t. \quad x+y+z \le 3 \\ &\qquad\; x^2+y^2+z^2=9 \\ &\qquad\; y \ge 0 \end{aligned}$$

$$Feasibility: \begin{cases} x+y+z \le 3 \\ x^2+y^2+z^2=9 \\ y \ge 0 \end{cases}$$

$$L(x,y,z,\lambda_1,\lambda_2,\lambda_3) = xyz + \lambda_1(3-x-y-z) + \lambda_2(9-x^2-y^2-z^2) - \lambda_3 y$$

$$Stationary\ point: \begin{cases} \dfrac{\partial L}{\partial x} = yz - \lambda_1 - 2x\lambda_2 = 0 \\ \dfrac{\partial L}{\partial y} = xz - \lambda_1 - 2y\lambda_2 - \lambda_3 = 0 \\ \dfrac{\partial L}{\partial z} = xy - \lambda_1 - 2z\lambda_2 = 0 \end{cases}$$

$$Sign: \begin{cases} \lambda_1 \ge 0 \\ \lambda_3 \le 0 \end{cases}$$

$$Complementary\ slackness: \begin{cases} \lambda_1(3-x-y-z) = 0 \\ \lambda_3 y = 0 \end{cases}$$

Checking the K-T conditions

If we are given a point and we want to check whether it is a K-T point, we must apply the K-T conditions in the following order:

1. Feasibility: we simply put the values of the variables into the constraints of the problem and check whether they are satisfied. If they are, we continue with the second condition; if they are not, the point is not feasible and cannot be a K-T point, so we stop.
2. Complementary slackness: we put the values of the variables into the equations $\lambda_i\big(b_i - g_i(x_1, x_2, \dots, x_n)\big) = 0$. For each equation, we can reach one of two conclusions: either it is trivially satisfied (when the term in the brackets takes the value 0) or it can only be satisfied when $\lambda_i = 0$. We now continue to condition 3.
3. Stationary point: we calculate the partial derivatives of the Lagrangian function and put the values of variables into them. We also remove all the λ_i that we know are 0 (from the previous condition). We now have a system of equations (which is usually linear but could also be non-linear), in which the unknown multipliers λ_i are the variables, that we have to solve. If the system is incompatible (i.e., there is no solution), the point we are

studying will not be a K-T point, so we stop. If we obtain a solution for the system, we now have all the values for the multipliers λ_i.

4. Sign: we need to check whether the values obtained for the multipliers λ_i satisfy the sign conditions. If they do, our point is a K-T point; if they don't, it is not.

Example (continuation of the previous one):

Check whether (0,3,0) is a K-T point ✓

$$Feasibility:\begin{cases} x+y+z\leq 3 & 0+3+0\leq 3 \\ x^2+y^2+z^2=9 & 0+3^2+0=9 \checkmark \\ y\geq 0 & 3\geq 0 \checkmark \end{cases}$$

$$Complementary\,slackness:\begin{cases} \lambda_1(3-0-3-0)=0 & \lambda_1\cdot 0=0 \checkmark \\ \lambda_3 y=0 & \lambda_3\cdot 3=0\ \rightarrow \lambda_3=0 \end{cases}$$

$$Stationary\ point:\begin{cases} \dfrac{\partial L}{\partial x}=yz-\lambda_1-2x\lambda_2=0 & 3\cdot 0-\lambda_1-2\cdot 0\cdot\lambda_2=0 & -\lambda_1=0 \\ \dfrac{\partial L}{\partial y}=xz-\lambda_1-2y\lambda_2-\lambda_3=0 & 0\cdot 0-\lambda_1-2\cdot 3\lambda_2-0=0 & -\lambda_1-6\lambda_2=0 \\ \dfrac{\partial L}{\partial z}=xy-\lambda_1-2z\lambda_2=0 & 0\cdot 3-\lambda_1-2\cdot 0\lambda_2=0 & \lambda_1=0 \end{cases}$$

$$\rightarrow \lambda_1=\lambda_2=\lambda_3=0$$

$$Sign:\begin{cases} \lambda_1\geq 0 \checkmark \\ \lambda_3\leq 0 \checkmark \end{cases}$$

We can therefore conclude that point (0,3,0) is a K-T point.

Another example:

Given the following problem, check whether (8,0) is a K-T point:

$$\begin{aligned} &Max\ \ (x-2)^2+(y-2)^2 \\ &s.t.:\quad x-y\leq 8 \\ &\qquad\ \ x+y\geq -4 \\ &\qquad\ \ x,y\geq 0 \end{aligned}$$

Feasibility:

$$\begin{array}{lr} x-y\leq 8 & 8-0\leq 8 \checkmark \\ x+y\geq -4 & 8+0=8\geq -4 \checkmark \\ x\geq 0 & 8\geq 0 \checkmark \\ y\geq 0 & 0\geq 0 \checkmark \end{array}$$

Complementary Slackness:

$\lambda_1(8 - x + y) = 0 \qquad \lambda_1(8 - 8 + 0) = \lambda_1 0 = 0$ ✓

$\lambda_2(-4 - x - y) = 0 \qquad \lambda_2(-4 - 8 - 0) = -12\lambda_2 = 0 \quad \rightarrow \quad \lambda_2 = 0$

$\lambda_3(0 - x) = 0 \qquad \lambda_3(0 - 8) = -8\lambda_3 = 0 \quad \rightarrow \quad \lambda_3 = 0$

$\lambda_4(0 - y) = 0 \qquad \lambda_4(0 - 0) = \lambda_4 0 = 0$ ✓

Stationary Point:

$$L(x, y, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = (x-2)^2 + (y-2)^2 + \lambda_1(8 - x + y) + \lambda_2(-4 - x - y) + \\ +\lambda_3(0 - x) + \lambda_4(0 - y)$$

$$\frac{\partial L}{\partial x} = 2(x-2) - \lambda_1 - \lambda_2 - \lambda_3 = 2(8-2) - \lambda_1 - 0 - 0 = 12 - \lambda_1 = 0 \quad \rightarrow \quad \lambda_1 = 12$$

$$\frac{\partial L}{\partial y} = 2(y-2) + \lambda_1 - \lambda_2 - \lambda_4 = 2(0-2) + \lambda_1 - 0 - \lambda_4 = -4 + 12 - \lambda_4 = 8 - \lambda_4 = 0$$

$$\rightarrow \quad \lambda_4 = 8$$

Sign:

$\lambda_1 \geq 0$ ✓, $\lambda_2 \leq 0$ ✓, $\lambda_3 \leq 0$ ✓, $\lambda_4 \leq 0$ ✗

Therefore, (8,0) is not a K-T point.

Obtaining all the Kuhn-Tucker points of a problem

We have seen how to check whether a given point is a K-T point, but how do we obtain the Kuhn-Tucker points of a problem if we are given no candidates?

We must start by writing the complementary slackness conditions of the problem. Each complementary slackness equation has the form $\lambda_i\big(b_i - g_i(x_1, x_2, \dots, x_n)\big) = 0$, which means that either the constraint is binding or the multiplier takes the value 0. There are therefore two possible solutions for each complementary slackness condition. One of these implies that $\lambda_i = 0$ while the other implies that the constraint is satisfied with equality, $g_i(x_1, x_2, \dots, x_n) = b_i$.

If we have m constraints, and each constraint has two possible solutions, by combining all possibilities we get a total of 2^m situations or cases. In each case, we will have some multipliers that take the value 0 and some equations that must be satisfied.

Now, for each case, we write the stationary point conditions and remove those multipliers that are 0 in that case. In this way we obtain a set of equalities that, together with the equalities corresponding to the binding constraints of the complementary slackness condition, form a system of equations. We now solve this system and obtain the values for the variables and for the multipliers that were still unknown (if the system is incompatible, we discard this case and move to the next one). Finally, we check whether the values obtained satisfy the feasibility and sign K-T conditions. If they do, the point we have obtained is a K-T point. We proceed in this way with all the cases we have derived from the complementary slackness conditions and, once we have finished, we will have all the K-T points of the problem.

Example:

$$Max \quad -(x-4)^2-(y-4)^2$$
$$x+y \leq 4$$
$$x+3y \leq 9$$

Feasibility: $x+y \leq 4 \quad x+3y \leq 9$

Stationary point: $-2(x-4)-\lambda_1-\lambda_2=0 \qquad -2(y-4)-\lambda_1-3\lambda_2=0$

Sign: $\lambda_1 \geq 0 \quad \lambda_2 \geq 0$

Complementary slackness: $\lambda_1(4-x-y)=0 \qquad \lambda_2(9-x-3y)=0$

We obtain the cases:

$$\lambda_1(4-x-y)=0 \quad \begin{cases} \lambda_1=0 \\ \ \ or \\ 4-x-y=0 \end{cases} \qquad \lambda_2(9-x-3y)=0 \quad \begin{cases} \lambda_2=0 \\ \ \ or \\ 9-x-3y=0 \end{cases}$$

Case 1: $\lambda_1=0 \quad \lambda_2=0$

We use this information in the stationary point conditions:

$$\left.\begin{array}{r} -2(x-4)=0 \\ -2(y-4)=0 \end{array}\right\} \Longrightarrow x=4 \quad y=4$$

We check feasibility, but the values obtained do not satisfy $x+y \leq 4$

Case 2: $\lambda_1=0 \quad 9-x-3y=0$

$$\left.\begin{array}{r} 9-x-3y=0 \\ -2(x-4)-\lambda_2=0 \\ -2(y-4)-3\lambda_2=0 \end{array}\right\} \Longrightarrow x=\frac{33}{10} \quad y=\frac{19}{10} \quad \lambda_2=\frac{7}{5}$$

The values do not satisfy $x+y \leq 4$

Case 3: $4-x-y=0 \quad \lambda_2=0$

$$\left.\begin{array}{r} 4-x-y=0 \\ -2(x-4)-\lambda_1=0 \\ -2(y-4)-\lambda_1=0 \end{array}\right\} \Longrightarrow x=2 \quad y=-2 \quad \lambda_1=4$$

These values satisfy all the feasibility and sign conditions, so we have a K-T point (2-2).

Case 4: $4-x-y=0 \quad 9-x-3y=0$

$$\left.\begin{array}{r} 4-x-y=0 \\ 9-x-3y=0 \end{array}\right\} \Longrightarrow x=\frac{3}{2} \quad y=\frac{5}{2}$$

$$\left.\begin{array}{r} 5-\lambda_1-\lambda_2=0 \\ 3-\lambda_1-3\lambda_2=0 \end{array}\right\} \Longrightarrow \lambda_1=6 \quad \lambda_2=-1$$

The values of x and y satisfy the feasibility conditions, but the value of λ_2 does not satisfy the sign condition.

Finally, we have obtained one single K-T point: (2,-2)

Obtaining K-T points in Classical Programming problems

In CP problems all constraints are equalities. This means that there are no sign or complementary slackness conditions. Therefore, the only K-T conditions we have in these problems are Feasibility and Stationary point conditions. Since these conditions are all equalities, once we write them all we will have a system of equations. To obtain the K-T points, we just need to solve this system. All solutions of the system will be K-T points of the problem.

Example:

$$\begin{aligned} &Min\ \ x^2 + y^2 + 2x + 4y + 10 \\ &s.t. \quad x + y = 21 \end{aligned}$$

$$L(x, y, \lambda_1) = x^2 + y^2 + 2x + 4y + 10 + \lambda_1(21 - x - y)$$

$$\left.\begin{array}{l} \textit{Stationary point:} \begin{cases} \dfrac{\partial L}{\partial x} = 2x + 2 - \lambda_1 = 0 \\ \dfrac{\partial L}{\partial y} = 2y + 4 - \lambda_1 = 0 \end{cases} \\ \\ \textit{Feasibility}: x + y = 21 \end{array}\right\} \quad x = 11 \quad y = 10 \quad \lambda_1 = 24$$

By solving the system with the stationary point and feasibility conditions, we obtain the K-T point (11,10).

Studying the optimality of Kuhn-Tucker points

Below we will study several conditions that will help us to determine whether the K-T points of the problem are optimal solutions.

Constraint qualifications and Kuhn-Tucker necessary condition

Constraint qualifications are conditions that, when met, guarantee that a local optimum (and also the global optimum) of a problem must be a Kuhn-Tucker point. There are several constraint qualifications but we will consider just two of them:

1. There are no constraints in the problem.
2. All the constraints of the problem are linear (linearity constraint qualification).

If we have an NLP problem where all functions are of class C^2 (which means that they have partial derivatives up to, at least, second order and that these partial derivatives are continuous) and either of these two conditions hold, we can apply the following theorem (necessary condition):

For each solution x* that is a local optimum, there is a vector of Lagrangian multipliers λ such that (x*, λ) is a K-T point.

This implies that any optimal solution must be a K-T point. Consequently, a point that is not K-T cannot be an optimal solution.

A common mistake is to think that, by applying the necessary condition we can conclude that a K-T point will be an optimal solution, but this is not true in general. Even with the necessary condition, there can be K-T points that are not optimal solutions (there may even be no K-T points that are optimal at all and the problem has no optimal solution). To ensure that a K-T point is an optimal solution of the problem, we can use the sufficient condition below.

Kuhn-Tucker sufficient condition

If we have a Kuhn-Tucker point and we want to prove that it is an optimal solution of the problem, we can apply the following theorem:

Let us consider an NLP problem in which:

- all functions are of class C^2,
- the solution set is convex, and
- we are minimizing and the objective function is convex, or we are maximizing and the objective function is concave

Every Kuhn-Tucker point of the problem is then a global optimal solution.

This sufficient condition is similar to the Local-Global theorem we saw in unit 1 (in fact, the practical way of checking it is the same). The difference is that the Local-Global theorem is applied to local minima or maxima, while the K-T sufficient condition is applied to K-T points.

The conditions of this problem are not always met. For example, we may have a concave function in a maximization problem. However, this does not mean that our Kuhn-Tucker point is not an optimal solution. It could still be optimal but we are simply not able to prove it using the sufficient condition.

Example:

Consider the following problem:

$$
\begin{gathered}
Max \quad -(x-4)^2-(y-4)^2 \\
x+y \leq 4 \\
x+3y \leq 9
\end{gathered}
$$

We know that (2,-2) is a K-T point (we saw this in a previous example), now we want to determine whether it is an optimal solution of the problem.

We will apply the sufficient condition.

All functions are of class C^2 (they are polynomials).

Both constraints are semispaces (they are non-strict inequalities with linear functions), so they define convex sets. Therefore, the solution set is convex since it is the intersection of convex sets.

We must determine whether the objective function is convex or concave, so we calculate its Hessian matrix:

$$H_f = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

This matrix is ND, so the objective function is concave. Since we are maximizing, we can apply the sufficient condition, which says that all K-T points of the problem are optimal solutions. Therefore, (2-2) is an optimal solution of the problem.

Necessary condition + Weierstrass theorem for optimality

Remember that the necessary condition states that, if a constraint qualification is satisfied, any optimal solution of the problem must be a Kuhn-Tucker point. This means that, if we know all the K-T points of the problem (we must be sure that there are no other unknown K-T points) and if we can guarantee that the problem has an optimal solution, the optimal solution must be one of these K-T points. To prove that the problem has an optimal solution, we can use the Weierstrass theorem we saw in unit 1. If the hypotheses of the necessary condition and of the Weierstrass theorem hold, we simply have to evaluate the objective function for all the Kuhn-Tucker points. The one that gives the best result for the problem will be the optimal solution.

Example:

Consider the following problem:

$$\begin{array}{rl} Max & x^2 + y^2 \\ s.t. & x + 2y \leq 5 \\ & x, y \geq 0 \end{array}$$

We can see that this problem has four K-T points: (0,0), (1,2), (5,0), and (0,2.5). Study the optimality of these points.

Since we have four K-T points, it would be useless to try to apply the sufficient condition. If it could be applied, all four of them would be optimal solutions but this is not true because they give different values when evaluated in the objective function. So we must try to apply the necessary condition and Weierstrass theorem.

First, we check Weierstrass theorem to prove that the problem has an optimal solution. All the functions are continuous. The solutions set is closed for the usual reasons and it is bounded ($0 \leq x \leq 5$ and $0 \leq y \leq 2'5$ for example), so it is compact. It is also non-empty (for example, (0,0) is a feasible solution). We can therefore apply Weierstrass theorem and be sure that the problem has an optimal solution.

All constraints of the problem are linear, so it satisfies one constraint qualification (linearity). We can therefore apply the necessary condition and conclude that the optimal solution of the problem must be a K-T point.

Since we know all the K-T points, we simply have to evaluate them in the objective function:

F(0,0)=0, F(1,2)=5, F(5,0)=25, F(0,2.5)=6.25

Since we are maximizing, (5,0) must be the optimal solution of the problem.

When should we apply the necessary or the sufficient condition?

When we have several K-T point(s) and wish to study their optimality (i.e., find out if they are optimal solutions), we should try one of the two methods we have seen, but which one?

If we have only one K-T point but are not sure whether there are other K-T points, we should use the sufficient condition since the other method works only if we are sure that we have all the K-T points.

If we have all the K-T points and there is more than one of them, we should use the necessary condition with Weierstrass theorem. If there are several K-T points, it is unlikely that all of them will be optimal solutions (unless they all give the exact same value for the objective function, which is not very common, only the best solution can be optimal). The sufficient condition would imply that all the K-T points are optimal but this would not make sense in this case. This does not mean that the sufficient condition is false but if we try to use it, some of its hypotheses will not hold and we will have wasted our time. It is best, therefore, to go directly to the necessary condition.

If we have one K-T point and we can guarantee that there are no more K-T points in the problem, we can try both methods (though there is no guarantee with either of them). We try one of them and, if that fails, we try the other one. If both fail, we can't prove that our point is an optimal solution (it could be but we just can't prove it).

Economic meaning of the K-T multipliers

The value of the K-T multipliers can be obtained to find an approximate value of the objective function if we modify the right-hand side of one of the constraints. For each unit that we increase the right-hand side, the value of the objective function will increase by approximately λ_i units. So, if we increase the right-hand side by Δb units, we just have to multiply $\Delta b \times \lambda_i$ and add the result to the current optimal value of the objective function to obtain an estimation of the new optimal value. This also works when we decrease the value of the right-hand side (in this case, Δb will be negative).

Note that this is just an approximation not the real exact value of the objective function we would obtain if we solved the problem with the new right-hand side. If the increase in the right-hand side is small, the approximation will be very close to the exact value, but this will get worse as the increase in the right-hand side gets larger. We should therefore use this estimation only for small increases on the right-hand side.

There is one situation in which we should not use this approximation method at all. If there is a constraint that is binding in the optimal solution and its multiplier takes the value 0, we should not try to use multipliers to estimate new values of the objective function since the results obtained can be very different from the exact solution.

In practical problems, this method can be used, for example, to estimate how a change in the available amounts of raw materials will affect a company's profits. Or how an increase in the production requirements will affect its production costs.

Example:

Consider the following problem:

$$Min.\ \ x^2 + 2y^2 + 2xy + 3z$$

$$s.t.: -x + 2y + z = 4$$

$$x + 3z \leq 6$$

$$x \geq 0$$

We know that (0,3/2,1) is a K-T point with multipliers $\lambda_1 = 3, \lambda_2 = 0, \lambda_3 = 6$, and it is an optimal solution of the problem, whose optimal value of the objective function is F(0,3/2,1)=15/2.

If the right-hand side of the first constraint decreases by 1/6, can you tell, approximately, what the new optimal value of the objective function will be in the new problem?

The multiplier associated with the first constraint is $\lambda_1 = 3$, so the optimal value of the objective function will increase by approximately $\left(-\frac{1}{6}\right) \times 3 = -\frac{1}{2}$, i.e., it will decrease by $\frac{1}{2}$. The new optimal value of the objective function will then be approximately $\frac{15}{2} - \frac{1}{2} = \frac{14}{2} = 7$.

Unit 3. An introduction to linear programming

In this unit we will solve problems in which all functions are linear (in both the objective function and the constraints).

Special forms of linear programming problems

To solve Linear Programming (LP) problems, we need the problems to have a specific form. Two forms of LP problems are useful:

1. Canonical form

In the canonical form of a problem, all variables must be non-negative (≥ 0) and all the other constraints must be canonical. This means that the inequalities must be $\leq$ if we are maximizing and $\geq$ if we are minimizing.

$$\begin{aligned} Max. \quad & 2x + 3y + z \\ s.t. \quad & x + 2y + z \leq 30 \\ & x + y \leq 20 \\ & x, y, z \geq 0 \end{aligned}$$

Example of a maximization LP problem in canonical form

2. Standard form

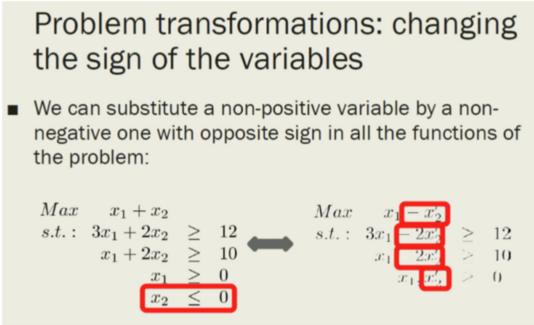
In the standard form of a problem, all variables must be non-negative (≥ 0) and all other constraints must be equalities.

$$\begin{aligned} & Max. x + y \\ & s.t. \ x = 4 \\ & x \geq 0, y \geq 0 \end{aligned}$$

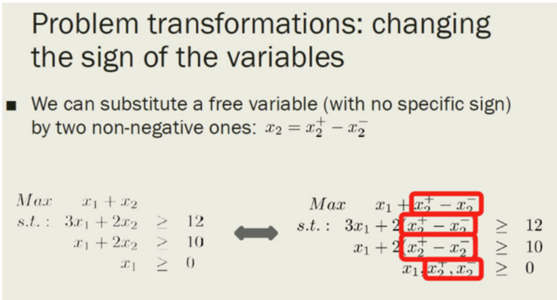
Example of a maximization LP problem in standard form

Any LP problem can be transformed into an equivalent one in canonical or standard form by applying some of the transformations seen in unit 1:

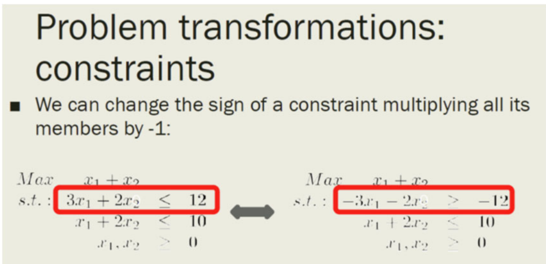
- Variables that are ≤ 0 can be transformed into variables that are ≥ 0 by changing their signs everywhere:



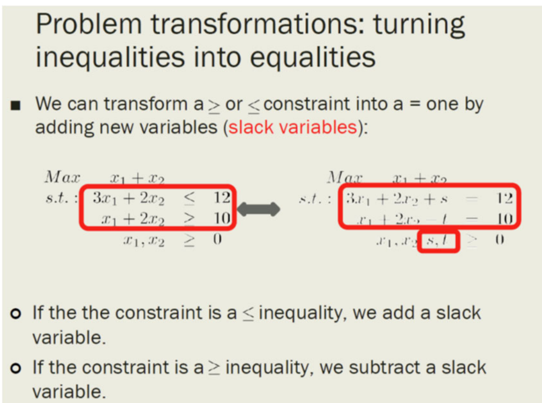
- Variables that are free can be replaced by two new variables that are ≥ 0:



- The sign of constraints can be changed from $\geq$ to $\leq$ or vice versa by changing all the signs:



- Inequalities can be transformed into equalities by adding or subtracting a slack variable:



Examples:

$$\begin{aligned} \text{a) } Max. \quad & x + y \\ s.t. \quad & -x + y = 2 \\ & x + 2y \le 6 \\ & 2x + y \le 6 \\ & x \ge 0, y \ge 0 \end{aligned}$$

<u>Canonical form:</u>

We need to transform all the constraints into <= inequalities.

Constraint 1:

$$-x + y = 2$$

First we replace the equality with two inequalities:

$$\begin{aligned} -x + y \le 2 \\ -x + y \ge 2 \end{aligned}$$

The first inequality is OK. To change the second one, we change the signs:

$$\begin{aligned} -x + y \le 2 \\ x - y \le -2 \end{aligned}$$

The remaining inequalities are already in canonical form, so the canonical form of the problem is:

$$\begin{aligned} Max \quad & x + y \\ s.t.: \quad & -x + y \le 2 \\ & x - y \le -2 \\ & x + 2y \le 6 \\ & 2x + y \le 6 \\ & x, y \ge 0 \end{aligned}$$

<u>Standard form:</u>

We need to transform the inequalities into equalities by adding slack variables:

$$\begin{aligned} x + 2y + s_1 = 6 \\ 2x + y + s_2 = 6 \end{aligned}$$

The standard form of the problem is:

$$\begin{aligned} Max \quad & x + y \\ & -x + y = 2 \\ & x + 2y + s_1 = 6 \\ & 2x + y + s_2 = 6 \\ & x, y, s_1, s_2 \ge 0 \end{aligned}$$

f) $Min.\ 3x+2y-z$

$$\begin{aligned} s.t.\quad & x-2y+z=2 \\ & 2x+y-3z \leq 3 \\ & x \text{ free},\ y\geq 0,\ z\leq 0 \end{aligned}$$

<u>Canonical form:</u>

First we transform constraint 1 into two inequalities:

$$x - 2y + z \leq 2$$
$$x - 2y + z \geq 2$$

Since this is a minimization problem, in the canonical form the inequalities must be >=, so we change the first inequality:

$$-x + 2y - z \geq -2$$
$$x - 2y + z \geq 2$$

We also need to change the other constraint of the problem:

$$-2x - y + 3z \geq -3$$

At this point, our problem is:

$$\begin{aligned} & Min\ \ 3x + 2y - z \\ & s.t.: -x + 2y - z \geq -2 \\ & \qquad x - 2y + z \geq 2 \\ & \qquad -2x - y + 3z \geq -3 \\ & \qquad x\, free, y \geq 0, z \leq 0 \end{aligned}$$

Now we must make all variables ≥ 0. First, we turn z from negative to positive by changing its sign everywhere in the problem:

$$\begin{aligned} & Min\ \ 3x + 2y + z \\ & s.t.: -x + 2y + z \geq -2 \\ & \qquad x - 2y - z \geq 2 \\ & \qquad -2x - y - 3z \geq -3 \\ & \qquad x\, free, y \geq 0, z \geq 0 \end{aligned}$$

Finally, we change x from free to positive by replacing $x = x_1 - x_2$:

$$\begin{aligned} & Min\ \ 3(x_1 - x_2) + 2y + z \\ & s.t.: -(x_1 - x_2) + 2y + z \geq -2 \\ & \qquad x_1 - x_1 - 2y - z \geq 2 \\ & \qquad -2(x_1 - x_2) - y - 3z \geq -3 \\ & \qquad x_1, x_2, y, z \geq 0 \end{aligned}$$

<u>Standard form:</u>

We begin by adding a slack variable to the second constraint to change it into an equality:

$$2x + y - 3z + s_1 = 3$$

Then we change the variables as in the canonical form, so we obtain:

$$Min\ \ 3(x_1 - x_2) + 2y + z$$

$$s.t.: \quad x_1 - x_2 - 2y - z = 2$$
$$2(x_1 - x_2) + y + 3z + s_1 = 3$$
$$x_1, x_2, y, z \geq 0$$

Representing LP problems with matrices and vectors

To represent LP problems in a more abbreviated form, we use matrices and vectors. This representation is used for problems in canonical or standard form.

The coefficients of the variables in the objective function are represented by vector $\bar{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$.

The right-hand side of the constraints are represented by vector $\bar{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$.

The variables $x_1, x_2, \dots, x_n$ are represented by vector $\bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$.

The coefficients of the variables in the constraints are represented by matrix A

$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$, which is called the technical matrix, where each row corresponds to a constraint and each column corresponds to a variable.

Example:

$$Max\ x - y + 2z$$
$$s.t.:\ x + 3y - z \leq 2$$
$$-2x - y \leq 5$$
$$x, y, z \geq 0$$

In this LP problem we have the following vectors and matrices:

$$\bar{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \bar{c} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \quad \bar{b} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 3 & -1 \\ -2 & -1 & 0 \end{pmatrix}$$

With these vectors and matrices, a maximization LP problem in canonical form can be represented as:

$$\begin{array}{ll} Max & \overline{c}^{t}\overline{x} \\ s.t. & A\overline{x} \leq \overline{b} \\ & \overline{x} \geq \overline{0} \end{array}$$

A minimization problem would be similar, but constraints would be ≥.

A maximization LP problem in standard form is represented as:

$$\begin{array}{ll} Max & \overline{c}^{t}\overline{x} \\ s.t. & A\overline{x} = \overline{b} \\ & \overline{x} \geq \overline{0} \end{array}$$

From now on, we will consider that all the problems we will solve are in standard form. If they are not in standard form, we will transform them so that they are.

Basic Feasible Solutions

In LP problems, there is a special type of solutions called Basic Feasible Solutions, or BFS.

Basic and non-basic variables

In a BFS solution, we classify the variables of the problem into two types:

- Non-basic variables: these variables must take the value 0.
- Basic variables: these variables can take any value (including 0).

The set of basic (and non-basic variables) can be different for each BFS.

Definition of a BFS

The BFS of a problem is a solution that satisfies the following condition:

Definition: A solution $\overline{X}$ is a basic feasible solution if:

- $A\overline{x} = \overline{b}$.
- At least n-m of the variables are zero.
- The submatrix of A associated with the other m variables is regular (its determinant is not zero). This submatrix is called basic matrix and denoted by B
- All the variables satisfy the non-negativity constraints

Non-basic variables

Basic variables

The first condition says that all constraints of the problem must be satisfied (remember that the problem must be in standard form, so all constraints are equalities).

In the second condition, n is the total number of variables and m is the number of constraints (not counting non-negativity constraints). Of the n variables, n-m of them must be equal to zero. These 0 variables will be the non-basic variables of the solution. The m remaining variables will be the basic variables, which can have any value (including 0).

For the third condition, from the technical matrix A we must remove the columns that correspond to non-basic variables. The remaining columns (those that correspond to the basic variables) form a new matrix called basic matrix B. Matrix B must be regular, i.e., its determinant must be different from 0.

Finally, the fourth condition says that the value of all variables must be ≥ 0.

Checking whether a given solution is a BFS

If we are given a solution of our LP problem, we can check whether it is a BFS by checking the conditions explained above.

Example:

Suppose we have the following problem:

$$\begin{array}{ll} Max & 4x+5y \\ s.t. & 2x+y \leq 8 \\ & y \leq 5 \\ & x, y \geq 0 \end{array}$$

This problem is not in standard form. The variables are non-negative (which is required in standard form) but the constraints are not equalities. To transform the constraints into equalities, we must add slack variables s_1 and s_2 to the constraints:

$$\begin{array}{ll} Max & 4x+5y \\ s.t. & 2x+y+s_1=8 \\ & y+s_2=5 \\ & x, y, s_1, s_2 \geq 0 \end{array}$$

This problem has n=4 variables and m=2 equalities, and its technical matrix A is:

$$\begin{array}{cc} & \begin{array}{cccc} x & y & s_1 & s_2 \end{array} \\ A= & \begin{pmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

Now suppose that we are given the solution (0,0,8,5) of the standard form, i.e., $x=0, y=0, s_1=8, s_2=5$, and we want to check whether this solution is a BFS.

First we check whether $A\bar{x}=\bar{b}$, i.e., whether the constraints of the standard form are satisfied. All we have to do is put the values of the variables into the equalities and check them:

$$2x + y + s_1 = 2 \times 0 + 0 + 8 = 8$$
$$y + s_2 = 0 + 5 = 5$$

Both constraints are therefore satisfied. For the second condition, we must find at least n-m=4-2=2 variables that take the value 0 in the solution. In this case, we have variables x and y. These will be the non-basic variables of the solution (and s_1, s_2 will be the basic ones).

Now we must check the determinant of the basic matrix B. To obtain B, we remove from A the columns of the non-basic variables x and y:

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We calculate |B|=1, therefore B is regular and this condition is also satisfied.

Finally, since all the variables in the solution are ≥ 0, the last condition is also satisfied.

Therefore, (0,0,8,5) is a BFS of the problem.

More examples:

Given the following LP problem:

$$\begin{aligned} Max. \quad & 2x + 3y + z \\ s.t. \quad & x + 2y + z \leq 30 \\ & x + y \leq 20 \\ & x, y, z \geq 0 \end{aligned}$$

Which of the following points is/are BFS? Justify your answer.

a) (10,10,0,0,0)

b) (0,0,0,30,20)

c) (20,0,5,5,0)

d) (0,0,30, 0,20)

e) (20,0,5,0,0)

First we must transform the problem into its standard form:

$$\begin{aligned} Max. \quad & 2x + 3y + z \\ s.t. \quad & x + 2y + z + s_1 = 30 \\ & x + y + s_2 = 20 \\ & x, y, z, s_1, s_2 \geq 0 \end{aligned}$$

The standard form of the problem has n=5 variables and m=2 equations. Its technical matrix is:

$$A = \begin{pmatrix} 1 & 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Now we can start checking each point:

a) (10,10,0,0,0)

We need to check 4 points:

- Since n=5 and m=2, we need to find n-m=3 variables that have the value 0. In this case, variables z, s_1, s_2 take the value 0, so these will be the non-basic variables of the problem, and x, y will be the basic ones.
- We check that the point satisfies the equations of the problem:
$$x + 2y + z + s_1 = 10 + 20 + 0 + 0 = 30$$
$$x + y + s_2 = 10 + 10 + 0 = 20$$
Both equations are satisfied.
- We also check that all variables are ≥ 0. This is clearly true for this point.
- We obtain the basic matrix B, which is formed by the columns of technical matrix A that correspond to the basic variables (in this case, variables x, y, which are columns 1 and 2 of A):
$$B = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$
Now we calculate the determinant of B, which should be different from 0. In this case |B|=-1, so this condition is also satisfied by this solution.

All conditions are satisfied by the solution (10,10,0,0,0), so this solution is a BFS.

b) (0,0,0,30,20)

- We have 3 variables with the value 0 (x, y, z), which will be the non-basic variables. The basic variables will be s_1, s_1.
- We check the equations:
$$x + 2y + z + s_1 = 0 + 0 + 0 + 30 = 30$$
$$x + y + s_2 = 0 + 0 + 20 = 20$$
These are satisfied.
- The value of all variables is ≥ 0.
- The basic matrix is:
$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
And |B|=1, which is not 0.

All conditions are satisfied by the solution (0,0,0,30,20), so this solution is a BFS.

c) (20,0,5,5,0)

- We need 3 variables with the value 0, but there are only 2, so we conclude that this solution is not a BFS.

d) (0,0,30,0,20)

- The 3 variables with the value 0 are x, y, s_1. These are the non-basic variables. The basic variables are z, s_2.
- We check the equations:
$$x + 2y + z + s_1 = 0 + 0 + 30 + 0 = 30$$

$$x + y + s_2 = 0 + 0 + 20 = 20$$

These are satisfied.

- All variables take the value ≥ 0.
- The basic matrix here is

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

And |B|=1 is not 0.

All conditions are therefore satisfied, so this solution is a BFS.

e) (20,0,5,0,0)

- The 3 non-basic variables with value 0 are y, s_1, s_2. The basic variables are x, z.
- We check the equations:

$$x + 2y + z + s_1 = 20 + 0 + 5 + 0 = 25 \neq 30$$
$$x + y + s_2 = 20 + 0 + 0 = 20$$

Since the first equation is not satisfied, this solution is not a BFS.

Degenerate solutions

Basic variables usually take the value 0. However, there may be problems with solutions that have "too many" zeroes. If, for example, we have n=5 variables and m=2 equations, we need 2 basic variables and 3 non-basic variables. If we have a solution with values (20,0,0,0,0), we have 4 zeroes, but we only need 3 non-basic variables. This means that one of the variables with the value 0 must be basic (if it is a BFS). Such solutions where some basic variables take the value 0 are called degenerate solutions.

Obtaining Basic Feasible Solutions

Here we will learn how to obtain the BFS of a problem. Let us consider the same problem as in the first part of this unit.

As always, we must obtain the standard form of the problem:

$$\begin{aligned} Max \quad & 4x + 5y \\ s.t. \quad & 2x + y + s_1 = 8 \\ & y + s_2 = 5 \\ & x, y, s_1, s_2 \geq 0 \end{aligned}$$

This problem has n=4 variables and m=2 equations, so we need 2 basic and 2 non-basic variables. To obtain a BFS, we must choose which variables will be basic. Since we have 4 variables, there are 6 possible combinations of basic variables. Some of these will provide BFS and some will not. Let us try, for example, with variables y, s_2 as basic variables.

If y and s_2 are basic, this means that x and s_1 are non-basic. All the non-basic variables take the value 0, so $x = 0$ and $s_1 = 0$. If we use this information in the equations of the problem, we have:

$$y = 8$$
$$y + s_2 = 5$$

This is a linear system that we need to solve (you can use any method for solving linear problems). The solution to this system is:

$$y = 8 \quad s_2 = -3$$

However, in this solution, one of the variables takes a negative value, so this is not a BFS.

Since we have been unable to obtain a BFS from our initial choice of basic variables, we will try another combination. Let us now try with a different combination of basic variables: x and s_1.

Now the non-basic variables are y and s_2, so they must be equal to 0, and the constraints of the problem give us the following linear system:

$$2x + s_1 = 8$$
$$0 = 5$$

However, this system has no solution, since the second equation is wrong. This means that we cannot get a BFS from these basic variables either.

Now let us try with basic variables x and y. Since s_1 and s_2 will be non-basic, their value will be 0, and the constraints of the problem will give us this linear system:

$$2x + y = 8$$
$$y = 5$$

The solution of this system is therefore:

$$x = 1.5 \quad y = 5$$

All variables are ≥ 0, so (1.5,5,0,0) is a BFS of the problem.

Example

Calculate all the basic feasible solutions of this LP problem and state the basis associated with each one.

$$Max.\ 2x - 3y$$

$$s.t. \quad x + y \leq 1$$

$$x - y \leq 0$$
$$x, y \geq 0$$

First we need to obtain the standard form of the problem. We just need to add slack variables to the constraints to change them into equalities:

$$Max.\ 2x - 3y$$
$$\begin{aligned} s.t. \quad & x + y + s_1 = 1 \\ & x - y + s_2 = 0 \\ & x, y, s_1, s_2 \geq 0 \end{aligned}$$

We have n=4 variables and m=2 equalities, so there will be 2 basic and 2 non-basic variables. Now we need to study each possible combination of basic variables. There are a total of six combinations. Remember that once you choose which variables are basic, the remaining variables will be non-basic and take the value 0. This means they can be removed from the constraints to obtain a solvable linear system.

Basic variables: $\boldsymbol{x}, \boldsymbol{y}$ Non-basic variables: s_1, s_2

Linear system: $x + y = 1$
$x - y = 0$

Solution: $x = \frac{1}{2}$ $y = \frac{1}{2}$
BFS: $(\frac{1}{2}, \frac{1}{2}, 0,0)$

Basic variables: $\boldsymbol{x}, \boldsymbol{s_1}$ Non-basic variables: y, s_2

Linear system: $x + s_1 = 1$
$x = 0$

Solution: $x = 0$ $s_1 = 1$
BFS: $(0,0,1,0)$

Note: in this BFS there is a basic variable with the value 0 (i.e., x), so this is called a degenerate solution.

Basic variables: $\boldsymbol{x}, \boldsymbol{s_2}$ Non-basic variables: y, s_1

Linear system: $x = 1$
$x + s_2 = 0$

Solution: $x = 1$ $s_2 = -1$
This is not a BFS because variable s_2 is negative.

Basic variables: $\boldsymbol{y}, \boldsymbol{s_1}$ Non-basic variables: x, s_2

Linear system: $y + s_1 = 1$
$-y = 0$

Solution: $y = 0$ $s_1 = 1$
BFS: $(0,0,1,0)$ This BFS already appeared in an earlier case.

Basic variables: $\boldsymbol{y}, \boldsymbol{s_2}$ Non-basic variables: x, s_1

Linear system: $y = 1$
$-y + s_2 = 0$

Solution: $y = 1$ $s_2 = 1$
BFS: $(0,1,0,1)$

Basic variables: $\boldsymbol{s_1, s_2}$ Non-basic variables: x, y

Linear system: $s_1 = 1$
$s_2 = 0$

Solution: $s_1 = 1$ $s_2 = 0$
BFS: $(0,0,1,0)$ This BFS already appeared in an earlier case.

We now create a table with the basis (set of basic variables) and BFS for each basis.

Basis	**BFS**
x, y	$(\frac{1}{2}, \frac{1}{2}, 0,0)$
x, s_1	$(0,0,1,0)$
x, s_2	-
y, s_1	$(0,0,1,0)$
y, s_2	$(0,1,0,1)$
s_1, s_2	$(0,0,1,0)$

Fundamental theorem of Linear Programming

The fundamental theorem of LP says:

- If an LP problem has at least one feasible solution, then it will also have at least one BFS.
- If an LP problem has optimal solution(s), then at least one optimal solution will be a BFS.

The first part of this theorem says that if a problem is not infeasible, there will be at least one BFS. The second part says that if the problem is feasible and has optimal solutions (i.e., it is not unbounded), there must be at least one optimal solution that is a BFS.

As a consequence of this theorem, we obtain a method for solving linear problems:

1. Use Weierstrass theorem to check whether the problem has an optimal solution.
2. If the problem has an optimal solution, obtain all the BFS of the problem.
3. Evaluate the BFS in the objective function. The one that provides the maximum/minimum value (depending on the objective of the problem) will be an optimal solution.

Let us use this method to solve the example we are using in this unit.

1. We must check Weierstrass theorem. We can do this in the standard form or in the original form depending on our preference. Let us do it in the original problem. For example:

   Since we are solving a linear problem, the objective function is continuous. Since the solution set S is defined by continuous functions and $\leq$ or $\geq$ inequalities, S is closed. To check that S is bounded, we must find lower and upper bounds for all the variables. From the constraints of the problem, it is easy to deduce that

$$0 \leq x \leq 4$$
$$0 \leq y \leq 5$$

   S is therefore bounded. Finally, S is not empty (it contains point (0,0), for example), so we can use Weierstrass theorem to conclude that the problem has an optimal solution.

2. We must now obtain all the BFS of the problem. To do so, we must try all possible combinations of basic variables. We already studied some of them in the previous examples. The table below shows all possible combinations for the basic variables and the results we have obtained so far:

Basic variables	**BFS**
x, y	(1.5,5,0,0)
x, s_1	No BFS
x, s_2	
y, s_1	
y, s_2	No BFS
s_1, s_2	(0,0,8,5)

   There are therefore still two combinations that we must study: x, s_2 and y, s_1

   Let us try x, s_2 as basic variables. Then $y = s_1 = 0$ because they will be non-basic, and the linear system we have from the constraints is:

$$2x = 8$$
$$s_2 = 5$$

   The solution is therefore $x = 4, s_2 = 5$. Since all values are ≥ 0, we have found another BFS: (4,0,0,5).

   We have only one combination of basic variables left: y, s_1. The non-basic variables in this case are x, s_2, so we have the following linear system:

$$y + s_1 = 8$$
$$y = 5$$

   The solution of this system is $y = 5, s_1 = 3$, so we have a new BFS: (0,5,3,0).
   Now we must use the objective function of the problem $F(x, y) = 4x + 5y$ to evaluate all the BFS we have found. Note that to evaluate the objective function, we need only the values of x and y, i.e., the values of the slack variables are not necessary. The results are shown in the following table:

Basic variables	BFS	F(x,y)
x, y	(1.5,5,0,0)	31
x, s_1	No BFS	-
x, s_2	(4,0,0,5)	16
y, s_1	(0,5,3,0)	25
y, s_2	No BFS	-
s_1, s_2	(0,0,8,5)	0

Since we are maximizing the objective function, we must look for the greatest value, which is 31 and corresponds to the BFS (1.5,5,0,0).
The optimal solution of the original problem (without the slack variables) is therefore:

$$x = 1.5 \quad y = 5$$

Example

Consider the problem

$$\begin{aligned} Max.\quad & 4x + 2y + 4z \\ s.t.\quad & x \leq 5 \\ & 2x + y + z = 4 \\ & y + 2z = 3 \\ & x, y, z \geq 0 \end{aligned}$$

a) Determine whether there is a basic feasible solution with the basic variables x, y, z.

b) Check whether (1/2, 3, 0, 9/2), (1, 1, 1, 4), (0, 5,−1, 5) are basic feasible solutions.

c) Obtain all basic feasible solutions and, for each one, obtain the value of the objective function.

d) Justify why the problem has an optimal solution and calculate it.

a) Standard form:

$$\begin{aligned} Max.\quad & 4x + 2y + 4z \\ s.t.\quad & x + s_1 = 5 \\ & 2x + y + z = 4 \\ & y + 2z = 3 \\ & x, y, z, s_1 \geq 0 \end{aligned}$$

Basic variables: x, y, z

Linear system: $$\begin{aligned} x &= 5 \\ 2x + y + z &= 4 \\ y + 2z &= 3 \end{aligned}$$

Solution: $x = 5 \quad y = -15 \quad z = 9$

There is no BFS because the value of y is negative.

b) (1/2, 3, 0, 9/2)

We need n-m=4-3=1 variables with the value 0. Since z=0, we have enough non-basic variables.

Basic variables: x, y, s_1

Non-basic variables: z

We check the constraints:

$$x + s_1 = \frac{1}{2} + \frac{9}{2} = 5$$
$$2x + y + z = 2 \times \frac{1}{2} + 3 + 0 = 4$$
$$y + 2z = 3 + 0 = 3$$

All constraints are satisfied.

Now we obtain technical matrix A and take the columns of the basic variables to form basic matrix B.

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

We calculate |B|=2. Since this is not 0, this condition is also satisfied.

Finally, all variables are ≥ 0, so this solution is a BFS.

(1, 1, 1, 4)

We need one variable with the value 0 but since there is none, this is not a BFS.

(0,5,-1,5)

One variable has a negative value, so this cannot be a BFS.

c) Since n=4 and m=3, we need three basic variables. The possible combinations of these three basic variables are:

- x, y, z
- x, y, s_1
- x, z, s_1
- y, z, s_1

Basic variables: x, y, z
There is no BFS (see question (a))

Basic variables: x, y, s_1
BFS: (1/2, 3, 0, 9/2) (see question (b))

Basic variables: x, z, s_1
Linear system: $x + s_1 = 5$
$2x + z = 4$
$2z = 3$
Solution: $x = \frac{5}{4}$ $z = \frac{3}{2}$ $s_1 = \frac{15}{4}$
BFS: $(\frac{5}{4}, 0, \frac{3}{2}, \frac{15}{4})$

Basic variables: y, z, s_1

Linear system: $s_1 = 5$

$$y + z = 4$$

$$y + 2z = 3$$

Solution: $y = 5 \quad z = -1 \quad s_1 = 5$

There is no BFS with these basic variables because $z \leq 0$.

BFS	$\boldsymbol{F(x,y,z) = 4x + 2y + 4z}$
$(\frac{1}{2}, 3, 0, \frac{9}{2})$	8
$(\frac{5}{4}, 0, \frac{3}{2}, \frac{15}{4})$	11

d) To prove that the problem has an optimal solution, we must use Weierstrass theorem:

- The objective function is continuous.
- The constraints are defined by continuous functions and ≥, ≤, or = signs, so the solution S is closed.
- The following lower and upper bounds are valid, so the solution set is bounded:

$$0 \leq x \leq 5$$
$$0 \leq y \leq 4$$
$$0 \leq z \leq 4$$

- (1/2, 3, 0) belongs to S, so S is not empty.

Therefore, all conditions for Weierstrass theorem are satisfied and we can conclude that the problem has an optimal solution.

Since we can guarantee that the problem has an optimal solution and thanks to the fundamental theorem of linear programming, we can state that a BFS exists that will be an optimal solution. In question (c) we obtained all BFS of the problem. Since this is a maximization problem, we must find the BFS with the greatest value of the objective function, which in this case is $(\frac{5}{4}, 0, \frac{3}{2}, \frac{15}{4})$, with a value of 11. This is therefore the optimal solution of the standard form of the problem and $(\frac{5}{4}, 0, \frac{3}{2})$ is the optimal solution of its original form.

Final observations

- Since the original problem has only two variables, it could have been solved using the graphical method. If we draw the solution set S and calculate all vertices of S, we see that these vertices are the BFS we obtained.

- The method we have seen for solving linear problems using the BFS has two main drawbacks:

- We need to use Weierstrass theorem to prove that the problem has an optimal solution. If this does not work (because the solution set is not bounded), the problem may still have an optimal solution but we cannot rely on the BFS to find it.
- Even if Weierstrass theorem works, we need to calculate all the BFS of the problem, which means we have to study all possible combinations of basic variables. In problems with more variables and constraints, the number of possible combinations can be quite large, so it takes a lot of time to obtain all the BFS.

We therefore need an alternative method that does not need Weierstrass theorem and can find the optimal solution without calculating all the BFS. This alternative method is called the SIMPLEX method, which we will study in Unit 4.

Unit 4. The simplex method

Introduction

In this unit we will learn the Simplex method, a procedure that enables us to solve Linear Programming problems. In unit 3 we saw a procedure for finding the optimal solution of an LP problem. However, that procedure was not very efficient as it involved using Weierstrass theorem and calculating all BFSs of the problem.

With Simplex we don't need to use Weierstrass theorem. After using this procedure we will know whether the problem has an optimal solution or is unbounded. The Simplex method is also based on obtaining BFSs but we only need to calculate a few of them, not all of them.

Simplex tableau

The Simplex method is based on calculating *tableaus*. A simplex tableau looks like this:

A simplex tableau is associated with a BFS of the problem. We need to fill this tableau with information from the standard form of the problem and the BFS.

Creating a simplex tableau

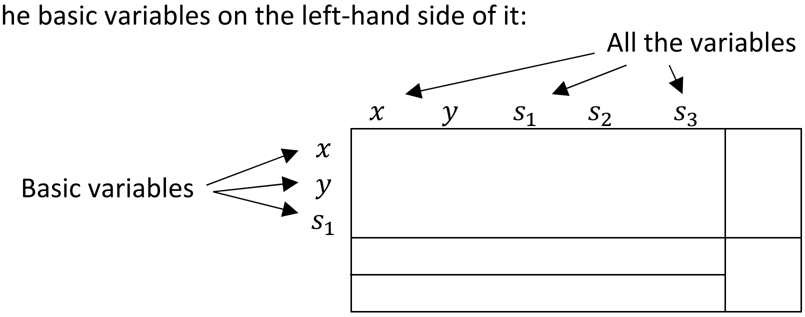
Below we will see how to create a simplex tableau using the following LP as an example:

$$
\begin{aligned}
&Max\ \ 2x+y\\
&s.t.\quad x+y-s_1=1\\
&\qquad\ \ x-2y-s_2=0\\
&\qquad\ \ x-y+s_3=1\\
&\qquad\ \ x,y,s_1,s_2,s_3\geq 0
\end{aligned}
$$

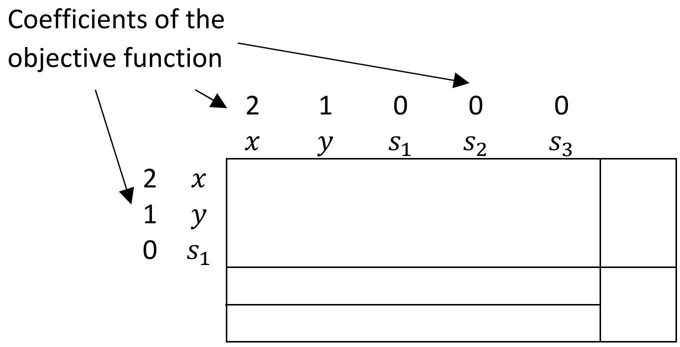
Note that this problem is already in standard form. If our problem is not in standard form, we first need to transform it.

In this example, n=5 and m=3, so a BFS will have 3 basic variables and 2 non-basic variables. Since a simplex tableau is always associated with a BFS, it is also associated with a set of basic variables. So the first thing we need to do is choose some basic variables. For this example, let us select x, y, and s_1 as our basic variables.

The first thing we need to write are the names of the variables above the tableau and the names of the basic variables on the left-hand side of it:



Now we need to copy the coefficients of the variables in the objective function of the problem above the variables at the top and next to the basic variables on the left. In this problem the objective function is $2x + y$, so the variable x has coefficient 2 and the variable y has coefficient 1. The slack variables have a coefficient of 0 since they do not appear in the objective function.



For the next step we need the technical matrix A and the basic matrix B.

Technical matrix A is the matrix that contains the coefficients of the variables in the constraints (each row corresponds to a different constraint and each column corresponds to a variable):

$$A = \begin{pmatrix} 1 & 1 & -1 & 0 & 0 \\ 1 & -2 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 & 1 \end{pmatrix}$$

Basic matrix B contains the columns of A that correspond to the basic variables, which in this case are the first three columns:

$$B = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -2 & 0 \\ 1 & -1 & 0 \end{pmatrix}$$

Now we must calculate $B^{-1}A$, i.e., we need to obtain the inverse matrix of B and multiply the result by A. There are several ways to calculate an inverse matrix. The most common method is to apply the following formula (which you should have seen in Mathematics I):

$$B^{-1} = \frac{1}{|B|}(adj\ B)^t$$

We therefore need the determinant of B:

$$|B| = -1$$

The adjoint matrix of B is:

$$adj\ B = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 2 \\ -2 & -1 & -3 \end{pmatrix}$$

which we need to transpose:

$$(adj\ B)^t = \begin{pmatrix} 0 & 1 & -2 \\ 0 & 1 & -1 \\ 1 & 2 & -3 \end{pmatrix}$$

We then apply the formula for B^{-1}:

$$B^{-1} = \begin{pmatrix} 0 & -1 & 2 \\ 0 & -1 & 1 \\ -1 & -2 & 3 \end{pmatrix}$$

Now we can calculate $B^{-1}A$:

$$B^{-1}A = \begin{pmatrix} 0 & -1 & 2 \\ 0 & -1 & 1 \\ -1 & -2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 & 0 & 0 \\ 1 & -2 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 3 \end{pmatrix}$$

Note that the first three columns of the result are:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which is the identity matrix. This is not by chance as, when calculating $B^{-1}A$, *the result in the columns of the basic variables will always form the identity matrix*. In this example, the basic variables were in the first three positions of the matrix. But if we choose a different combination of basic variables, the column of the identity matrix can appear in other positions (and even in a different order). This means that if we don't see the identity matrix in this step, we must have made a mistake (in calculating the inverse matrix or the product).

Now we need to copy $B^{-1}A$ into the largest rectangle of the simplex tableau:

$B^{-1}A$

		2	1	0	0	0	
		x	y	s_1	s_2	s_3	
2	x	1	0	0	1	2	
1	y	0	1	0	1	1	
0	s_1	0	0	1	2	3	

Our next calculation is $B^{-1}\bar{b}$, where $\bar{b}$ is the vector containing the right-hand side of the constraints. In the example:

$$\bar{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$B^{-1}\bar{b} = \begin{pmatrix} 0 & -1 & 2 \\ 0 & -1 & 1 \\ -1 & -2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

We insert the result of this product into the right-hand column of the tableau:

		2	1	0	0	0	
		x	y	s_1	s_2	s_3	$B^{-1}\bar{b}$
2	x	1	0	0	1	2	2
1	y	0	1	0	1	1	1
0	s_1	0	0	1	2	3	2

The values we obtain for this last column must always be ≥ 0. If we get a negative value there, this may be due to one of two reasons: either we made a mistake or the basic variables we have chosen do not provide a BFS. In the latter case, we must change the basic variables and start again.

Now we fill in the two rows at the bottom of the tableau. For the first row, we need to multiply the coefficients on the left-hand side of the tableau by each column inside it. We do the same with the last column to calculate the value that goes into the box in the bottom right-hand corner of the tableau:

		2	1	0	0	0	
		x	y	s_1	s_2	s_3	
2	x	1	0	0	1	2	2
1	y	0	1	0	1	1	1
0	s_1	0	0	1	2	3	2
		2	1	0	3	5	5

$$(2 \quad 1 \quad 0)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 2 \qquad (2 \quad 1 \quad 0)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 1$$

$$(2 \quad 1 \quad 0)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0 \qquad (2 \quad 1 \quad 0)\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 3$$

$$(2 \quad 1 \quad 0)\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = 5 \qquad (2 \quad 1 \quad 0)\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = 5$$

To obtain the numbers in the last row, we subtract the values we calculated in the previous step from the coefficients at the top of the tableau:

		2	1	0	0	0	
		x	y	s_1	s_2	s_3	
2	x	1	0	0	1	2	2
1	y	0	1	0	1	1	1
0	s_1	0	0	1	2	3	2
		2	1	0	3	5	5
		0	0	0	-3	-5	

$$\begin{array}{r} (2 \quad 1 \quad 0 \qquad 0 \qquad 0) \\ - \quad (2 \quad 1 \quad 0 \qquad 3 \qquad 5) \\ \hline (0 \quad 0 \quad 0 \quad -3 \quad -5) \end{array}$$

As we can see, the first three values in this last row are 0. This is because they correspond to the basic variables we have chosen: *the value of the last row in the columns of the basic variables is always 0*. So, if we get something different from 0 for the basic variables, we must have made a mistake.

The simplex tableau associated with the basis x, y, s_1 is now complete:

		2	1	0	0	0	
		x	y	s_1	s_2	s_3	
2	x	1	0	0	1	2	2
1	y	0	1	0	1	1	1
0	s_1	0	0	1	2	3	2
		2	1	0	3	5	5
		0	0	0	-3	-5	

If we choose a different set of basic variables (a different basis), we will get a different simplex tableau (in some cases, the values in the last column will be negative, which means that this is not a valid simplex tableau).

Examples:

Given the following LP problem:

Max 4x+5y+z

s.t.: 2x+3y+z ≤ 6

x − z ≤ 2

x, y, z ≥0

Calculate the tableau with basic variables x and y.

First we must obtain the standard form of the problem:

$$\begin{array}{ll} Max & 4x + 5y + z \\ s.t.: & 2x + 3y + z + s_1 = 6 \end{array}$$

$$x - z + s_2 = 2$$
$$x, y, z, s_1, s_2 \geq 0$$

We need to obtain technical matrix A, basic matrix B, and vector $\bar{b}$:

$$A = \begin{pmatrix} 2 & 3 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \qquad \bar{b} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

Remember that A is the matrix with the coefficients of the variables in the constraints, B contains the columns of A corresponding to the basic variables (in this case x and y, the first two columns), and $\bar{b}$ is the vector containing the values on the right-hand side of the constraints.

Now we calculate B^{-1}:

$$B^{-1} = \begin{pmatrix} 0 & 1 \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix}$$

And then $B^{-1}A$ and $B^{-1}\bar{b}$:

$$B^{-1}A = \begin{pmatrix} 0 & 1 \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} 3 & 3 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & \frac{1}{3} & -\frac{2}{3} \end{pmatrix}$$

$$B^{-1}\bar{b} = \begin{pmatrix} 0 & 1 \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ \frac{2}{3} \end{pmatrix}$$

We can now start to fill in the simplex tableau:

		4	5	1	0	0	
		x	y	z	s_1	s_2	
4	x	1	0	-1	0	1	2
5	y	0	1	1	1/3	-2/3	2/3

Now we multiply the coefficients on the left-hand side $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$ by each column to obtain the next row:

		4	5	1	0	0	
		x	y	z	s_1	s_2	
4	x	1	0	-1	0	1	2
5	y	0	1	1	1/3	-2/3	2/3
		4	5	1	5/3	2/3	34/3

Finally, we subtract the row we have just calculated from the coefficients at the top to obtain the last row (reduced costs):

		4	5	1	0	0	
		x	y	z	s_1	s_2	
4	x	1	0	-1	0	1	2
5	y	0	1	1	1/3	-2/3	2/3
		4	5	1	5/3	2/3	34/3
		0	0	0	-5/3	-2/3	

Given the following LP problem:

Min 4x+20y

s.t.: 2x - y ≥ 6

4x+2y ≥ 4

x+2y ≥ 4

x, y ≥ 0

Calculate the tableau associated with the solution (x,y)=(4,0).

First we obtain the standard form of the problem:

$$\begin{aligned} Min \quad & 4x + 20y \\ s.t.: \quad & 2x - y - s_1 = 6 \\ & 4x + 2y - s_2 = 4 \\ & x + 2y - s_3 = 4 \\ & x, y, s_1, s_2, s_3 \geq 0 \end{aligned}$$

This problem has n=5 variables and m=3 constraints, so we need 5 basic variables and 3 non-basic variables. However, the solution we are given is (x,y)=(4,0). Since it is a solution of the original form and not of the standard form of the problem, we don't have enough information to know which variables are basic. We must therefore obtain the values of the slack variables to find a solution for the standard form. To obtain the value of the slack variables, we substitute x and y in the constraints:

$$\begin{aligned} 2x - y - s_1 &= 8 - s_1 = 6 \\ 4x + 2y - s_2 &= 16 - s_2 = 4 \\ x + 2y - s_3 &= 4 - s_3 = 4 \end{aligned}$$

From here we get $s_1 = 2, s_2 = 12, s_3 = 0$. The corresponding solution for the standard form is therefore $(x, y, s_1, s_2, s_3) = (4,0,2,12,0)$. Since we know that all variables that are different from 0 must be basic, x, s_1, s_2 are basic variables for this solution, and we already have all the basic variables we need.

We can now start calculating everything we need for the simplex tableau:

$$A = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 \\ 4 & 2 & 0 & -1 & 0 \\ 1 & 2 & 0 & 0 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 2 & -1 & 0 \\ 4 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} \quad \bar{b} = \begin{pmatrix} 6 \\ 4 \\ 4 \end{pmatrix}$$

Now we calculate $B^{-1}, B^{-1}A, B^{-1}\bar{b}$:

$$B^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 2 \\ 0 & -1 & 4 \end{pmatrix} \quad B^{-1}A = \begin{pmatrix} 1 & 2 & 0 & 0 & -1 \\ 0 & 5 & 1 & 0 & -2 \\ 0 & 6 & 0 & 1 & -4 \end{pmatrix} \quad B^{-1}\bar{b} = \begin{pmatrix} 4 \\ 2 \\ 12 \end{pmatrix}$$

We insert these results into the tableau:

		4	20	0	0	0	
		x	y	s_1	s_2	s_3	
4	x	1	2	0	0	-1	4
0	s_1	0	5	1	0	-2	2
0	s_2	0	6	0	1	-4	12

Now we complete the last two rows:

		4	20	0	0	0	
		x	y	s_1	s_2	s_3	
4	x	1	2	0	0	-1	4
0	s_1	0	5	1	0	-2	2
0	s_2	0	6	0	1	-4	12
		4	8	0	0	-4	16
		0	12	0	0	4	

Interpreting the simplex tableau

A simplex tableau is associated with a BFS. We can obtain the values of the basic variables from the last column of the tableau:

		2	1	0	0	0	
		x	y	s_1	s_2	s_3	
2	x	1	0	0	1	2	2
1	y	0	1	0	1	1	1
0	s_1	0	0	1	2	3	2
		2	1	0	3	5	5
		0	0	0	-3	-5	

In this case $x = 2, y = 1, s_1 = 2$. The remaining variables are 0 because they are non-basic: $s_2 = 0, s_3 = 0$. So the BFS corresponding to this tableau is $(2,1,2,0,0)$

The value of the objective function for this BFS can be found in the square in the bottom right-hand corner, which in this case is 5. We could also find this by substituting the values of the variables in the objective function of the problem.

The values in the last row of the tableau are called the reduced costs. They will be used later to check whether the BFS we have obtained is the optimal solution of the problem. These values are usually denoted by the letter w:

$$(w_x, w_y, w_{s_1}, w_{s_2}, w_{s_3}) = (0,0,0,-3,-5)$$

Checking to see if our simplex tableau is right

Let us review all the things we can check while calculating a simplex tableau to make sure we are doing it right:

- When calculating $B^{-1}A$, the columns corresponding to the basic variables must form the identity matrix.
- The values in the last column of the tableau, obtained from calculating $B^{-1}\bar{b}$, must be ≥ 0. If some values are negative, it is either because we have made a mistake or because of our choice of basic variables. In the latter case, we must choose a different combination of basic variables and begin again.
- The reduced costs (last row of the tableau) of the basic variables must be 0.

Even if these three conditions are satisfied, however, it does not necessarily mean that the tableau is right.

Example:

Which values in the following tableau must be wrong?

	x	y	s_1	s_2	s_3	
s_1	1/3	0	1	-1	1	-2
y	0	2	0	-1/3	2/3	2
x	1	0	0	2/3	-1/3	2
	1	2	0	0	1	6
W_j	0	1	0	0	-1	

1/3 : x is the third basic variable. Since the columns of the basic variables must form the identity matrix, this column should be $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. So instead of 1/3, this value should be 0.

2 x is the second basic variable. Since the columns of the basic variables must form the identity matrix, this column should be $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. So instead of 2, this value should be 1.

-2 : the values of all the basic variables in the last column must be ≥ 0, so this cannot be negative.

1 : the reduced cost (last row) of a basic variable must always be 0.

Interpreting the reduced costs – when is the BFS optimal?

We will now learn how to check whether the BFS associated with a simplex tableau is optimal or, in other words, we will learn how to check when the simplex tableau is optimal.

We first need to know the meaning of the reduced costs, which are the numbers in the last row of the simplex tableau and are denoted by w.

		2	1	0	0	0	
		x	y	s_1	s_2	s_3	
2	x	1	0	0	1	2	2
1	y	0	1	0	1	1	1
0	s_1	0	0	1	2	3	2
		2	1	0	3	5	5
Reduced costs w		0	0	0	-3	-5	

In this simplex tableau, the BFS is (2,1,2,0,0) and the reduced costs are $(w_x, w_y, w_{s_1}, w_{s_2}, w_{s_3}) = (0,0,0,-3,-5)$.

The reduced cost of the basic variables is always 0. Interpretation of the reduced costs of non-basic variables is similar to that of the marginals or multipliers:

Let w_x be the reduced cost of variable x. Then, if we increase the value of variable x by 1 unit, the value of the objective function will increase by w_x units.

In the tableau shown above, the non-basic variables are s_2 and s_3, and their reduced costs are $w_{s_2} = -3$ and $w_{s_3} = -5$. This means that, if we increase s_2 by 1 unit (currently its value is 0, so we increase it to 1), the value of the objective function will decrease by 3 (currently its value is 5, so it will decrease to 2). And, if we increase s_3 by 1 unit (from 0 to 1), the value of the objective function will decrease by 5 (from 5 to 0).

Given that in this LP problem we want to maximize the value of the objective function (see part I of this unit), it is not helpful to increase s_2 or s_3 because the value of the objective function will decrease. As for basic variables x, y, s_1, their value cannot be increased further because they are limited by the constraints of the problem.

In this simplex tableau, therefore, there is nothing we can do to improve the value of the objective function. This means that this simplex tableau is optimal and its BFS (2,1,3,0,0) is the optimal solution of the problem.

In general, a simplex tableau is optimal if:

- *it is a maximization problem and $w \leq 0$ for all variables*

or

- *it is a minimization problem and $w \geq 0$ for all variables*

What happens when we are maximizing and there is a variable with $w > 0$ (or when we are minimizing and there is a variable with $w < 0$)?

Moving from one BFS to the next

Let us consider the following LP problem:

$$
\begin{aligned}
Max \quad & x - y \\
s.t. \quad & x + 2y \leq 4 \\
& x + y \leq 3 \\
& x, y \geq 0
\end{aligned}
$$

Its standard form is:

$$
\begin{aligned}
Max \quad & x - y \\
s.t. \quad & x + 2y + s_1 = 4 \\
& x + y + s_2 = 3 \\
& x, y, s_1, s_2 \geq 0
\end{aligned}
$$

This has n=4 variables and m=2 equations, so we need 2 basic variables and 2 non-basic variables. If we choose y, s_2 as basic variables, we will obtain the following simplex tableau:

		1	-1	0	0	
		x	y	s_1	s_2	
-1	y	1/2	1	1/2	0	2
0	s_2	1/2	0	-1/2	1	1
		-1/2	1	-1/2	0	-2
		3/2	0	1/2	0	

Here, the reduced costs are $(w_x, w_y, w_{s_1}, w_{s_2}) = (\frac{3}{2}, 0, \frac{1}{2}, 0)$. Since we are maximizing and there are reduced costs greater than 0 ($w_x = 3/2$ and $w_{s_1} = 1/2$), this tableau is not optimal. According to our interpretation of the reduced costs, if we increase the value of x by 1, the value of the objective function will increase by 3/2, while if we increase s_1 by 1, the objective function will increase by 1/2. The value of the objective function can therefore still be improved by increasing the values of some non-basic variables.

If we could increase only one variable, which one should we increase? Obviously, it seems better to increase x since this will produce a greater increase in the value of the objective function (3/2 vs 1/2).

However, remember that x is a non-basic variable in this BFS, and non-basic variables must take the value 0. If we increase the value of x, it can no longer be non-basic. However, the number of basic and non-basic variables is fixed (2 basic variables and 2 non-basic variables in this example). So we would then have a problem since we would have 3 basic variables and only 1 non-basic variable, which is not possible for a BFS.

To keep the number of basic and non-basic variables correct, if we want to increase the value of x and thus make it basic, we must choose one of the basic variables (y or s_2) and make it 0 so that it can become non-basic.

Basically, the procedure for obtaining a new BFS that is better than the current one involves increasing the value of a non-basic variable as much as possible (to make it basic) and decreasing the value of a basic variable to 0 (to make it non-basic).

When a non-basic variable becomes basic, we say that it enters the basis.

When a basic variable becomes non-basic, we say that it leaves the basis.

Rules for entering and leaving

Now we know that to obtain a new and better BFS, we have to choose one non-basic variable that will enter the basis and one basic variable that will leave the basis. How do we decide which variable enters and which one leaves?

If we are maximizing, only variables with a negative reduced cost can enter the basis. If we are minimizing, only variables with a positive reduced cost can enter. The rule for choosing which variable enters is simple.

Entering rule:

- If it is a maximization problem, the non-basic variable with the greatest positive reduced cost must enter the basis.
- If it is a minimization problem, the non-basic variable with the lowest negative reduced cost must enter the basis.

If there is a tie between several non-basic variables (because they have the same reduced cost), we can choose whichever one we want.

The rule for choosing which variable leaves the basis is a bit more complicated.

Suppose we have applied this rule and the variable that must enter the basis is x_i. Let y_j be the value inside the simplex tableau (those that were obtained from $B^{-1}A$) corresponding to the basic variable number j, and let x_j be its value in the solution (the value in the last column of the simplex tableau that we obtained from $B^{-1}\bar{b}$). First we discard all the basic variables with $y_j \leq 0$ (only those variables with $y_j > 0$ are permitted to leave the basis). The rule for choosing the variable that leaves the basis is then:

Leaving rule:

- The basic variable with the lowest value of $\frac{x_j}{y_j}$ must leave the basis.

Again, if several variables are tied, we can choose the one we prefer. Note that, unlike what occurs in the entering rule, the leaving rule does not depend on whether we are maximizing or minimizing.

Let us apply these rules to the previous example:

x enters the basis

		1	-1	0	0	
		x	y	s_1	s_2	
-1	y	1/2	1	1/2	0	2
0	s_2	1/2	0	-1/2	1	1
		-1/2	1	-1/2	0	-2
		3/2	0	1/2	0	

The LP problem was one of maximization, so we must find the non-basic variable with the greatest positive reduced cost, which in this case is variable x. So x must enter the basis.

To decide which variable leaves the basis, we must first examine the column of the variable that enters, which is x.

		1	-1	0	0	
		x	y	s_1	s_2	
-1	y	1/2	1	1/2	0	2
0	s_2	1/2	0	-1/2	1	1
		-1/2	1	-1/2	0	-2
		3/2	0	1/2	0	

Only the basic variables that have a positive value here can leave the basis. In this case, both variables have a positive value in this column, so both can leave. To decide which one leaves, we must calculate $\frac{x_j}{y_j}$ for each basic variable that can leave.

		1	-1	0	0	
		x	y	s_1	s_2	
-1	y	1/2	1	1/2	0	2
0	s_2	1/2	0	-1/2	1	1
		-1/2	1	-1/2	0	-2
		3/2	0	1/2	0	

s_2 leaves the basis

$$for\ y:\ \frac{x_j}{y_j} = \frac{2}{\frac{1}{2}} = 4$$

$$for\ s_2:\ \frac{x_j}{y_j} = \frac{1}{\frac{1}{2}} = 2$$

We choose the variable with the lowest value of $\frac{x_j}{y_j}$, which in this case is s_2, so s_2 leaves the basis.

For this simplex tableau, therefore, x enters the basis and s_2 leaves the basis. This means that, for the next BFS, the basic variables will be y and x. To continue applying the simplex method, we need now to obtain the simplex tableau with the new basic variables.

The pivot

Once we have determined which variable enters and which variable leaves, the position in the tableau corresponding to the row of the variable that leaves and the column of the variable that enters is called the pivot of the tableau. This number will be used later to obtain the tableau corresponding to a new BFS.

x enters the basis

		1	-1	0	0	
		x	y	s_1	s_2	
-1	y	1/2	1	1/2	0	2
0	s_2	1/2	0	-1/2	1	1
		-1/2	1	-1/2	0	-2
		3/2	0	1/2	0	

s_2 leaves the basis

pivot of the tableau

Example:

Given the following simplex tableau corresponding to a maximization problem:

		1	6	2	0	0	0	
		x1	x2	x3	s1	s2	s3	
1	x1	1	1	0	0	4	1	2
2	x3	0	1	1	0	2	2	1
0	s1	0	-2	0	1	5	0	1
		1	3	2	0	8	5	4
		0	3	0	0	-8	-5	

a) What effect would it have on the objective function if any of the non-basic variables entered the basis?

b) Justify why the tableau is not optimal. Indicate which variable must enter, which variable must leave, and which is the pivot element.

a) The reduced costs of the non-basic variables are:

$$\left(w_{x_2}, w_{s_2}, w_{s_3}\right) = (3, -8, -5)$$

If x_2 enters the basis, the objective function will increase; if s_2 or s_2 enters, the objective function will decrease.

e) Since this is a maximization problem and there is a variable with positive reduced cost (x_2), the tableau is not optimal and x_2 must enter the basis. To know which variable leave, we must calculate $\frac{x_j}{y_j}$ for all basic variables with $y_j > 0$:

$$y_j = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \quad x_j = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \quad x_j/y_j = \begin{pmatrix} 2 \\ 1 \\ - \end{pmatrix}$$

The variable that leaves the basis will be the one with the lowest value of $\frac{x_j}{y_j}$, which in this case is the second basic variable, x_3.

The pivot of the tableau will be the value in the position that corresponds to the row of x_3 and the column of x_2.

enters

		1	6	2	0	0	0	
		x_1	x_2	x_3	s_1	s_2	s_3	
1	x_1	1	1	0	0	4	1	2
2	x_3	0	1	1	0	2	2	1
0	s_1	0	-2	0	1	5	0	1
		1	3	2	0	8	5	4
		0	3	0	0	-8	-5	

leaves

pivot

Given the following LP problem:

$$\begin{array}{ll} \text{Min} & 4x+20y \\ \text{s.t.:} & 2x - y \geq 6 \\ & 4x+2y \geq 4 \\ & x+2y \geq 4 \\ & x, y \geq 0 \end{array}$$

a) Calculate the tableau associated with the solution (x,y)=(4,0).
b) Say whether it is the optimal solution of the problem.

a) This was solved earlier:

		4	20	0	0	0	
		x	y	s_1	s_2	s_3	
4	x	1	2	0	0	-1	4
0	s_1	0	5	1	0	-2	2
0	s_2	0	6	0	1	-4	12
		4	8	0	0	-4	16
		0	12	0	0	4	

b) To know if we have found the optimal solution, we must look at the reduced costs, which in this case are:

$$\left(w_x, w_y, w_{s_1}, w_{s_2}, w_{s_3}\right) = (0{,}12{,}0{,}0{,}4)$$

Since all the reduced costs are ≥ 0 and we have a minimization problem, this tableau is optimal (if any non-basic variable y or s_3 enters the basis, the objective function will increase, which is bad for our solution). We have therefore found the optimal solution of the problem: (4,0,2,12,0).

<u>Different types of solutions in the optimal tableau</u>

Once we obtain the optimal simplex tableau, we can find different situations corresponding to different types of solutions. For simplicity, from now on we will assume that we are considering a maximization problem.

<u>Case 1</u>

		2	1	0	0	0	
		x	y	s_1	s_2	s_3	
2	x	1	0	0	1	2	2
1	y	0	1	0	1	1	1
0	s_1	0	0	1	2	3	2
		2	1	0	3	5	5
		0	0	0	-3	-5	

The reduced costs of all the non-basic variables are $w < 0$

If the reduced costs of all the basic variables are $w < 0$ (and so no variable can enter the basis), the associated BFS is the only optimal solution of the problem. This type of solution is called a vertex solution.

<u>Case 2</u>

		2	1	0	0	0	
		x	y	s_1	s_2	s_3	
2	x	1	0	0	1	2	2
1	y	0	1	0	-2	1	1
0	s_1	0	0	1	2	3	2
		2	1	0	0	5	5
		0	0	0	0	-5	

The reduced costs of all the non-basic variables are $w \leq 0$, but there is a non-basic variable with $w = 0$

If all the reduced costs are $w \leq 0$, but there is one (or several) non-basic variable with $w = 0$, then this BFS is optimal, but not the only one. We have multiple optimal solutions. This type of solution is called an edge solution. Here we can differentiate between two situations:

		2	1	0	0	0	
		x	y	s_1	s_2	s_3	
2	x	1	0	0	1	2	2
1	y	0	1	0	-2	1	1
0	s_1	0	0	1	2	3	2
		2	1	0	0	5	5
		0	0	0	0	-5	

Case 2a: In the column of the variable with $w = 0$, there is at least one positive y_j value.

		2	1	0	0	0	
		x	y	s_1	s_2	s_3	
2	x	1	0	0	0	2	2
1	y	0	1	0	0	1	1
0	s_1	0	0	1	-1	3	2
		2	1	0	0	5	5
		0	0	0	0	-5	

Case 2b: In the column of the variable with $w = 0$, all the y_j values are ≤ 0.

In case 2a, the solution is called a finite edge solution. In this situation, *we can obtain another optimal solution by choosing the variable with $w = 0$ as the variable that enters the basis and then applying the leaving rule to choose the variable that leaves the basis*. In case 2b, we have an infinite edge solution. Here, although there is an infinite number of optimal solutions, no variable can leave the basis, so it is not possible to obtain a new optimal solution.

Case 3

		2	1	0	0	0	
		x	y	s_1	s_2	s_3	
2	x	1	0	0	0	2	2
1	y	0	1	0	-1	1	1
0	s_1	0	0	1	-1	3	2
		2	1	3	-1	5	11
		0	0	0	1	-5	

There is a variable with $w > 0$ that can enter the basis, but all the y_j are ≤ 0, so no basic variable can leave.

If we have a non-basic variable with a positive reduced cost (so it can enter the basis) but all the y_j of this variable are 0 or negative (so no basic variable can leave the basis), we cannot find another BFS. This means that the problem is unbounded (there is no optimal solution because, since we can increase the value of the objective function as much as we want, there is no global maximum).

All the cases explained in this section can also be applied to minimization problems. We just need to change "positive" to "negative", and vice versa, in all references to reduced costs w.

Examples:

- Given the following LP problem:

$$\begin{aligned} Max.\quad & 2x_1 + x_2 \\ s.t.\quad & -x_1 + x_2 \leq 2 \\ & x_1 + 2x_2 \leq 6 \\ & 2x_1 + x_2 \leq 6 \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

and knowing that the optimal solution is $x_1=2$ and $x_2=2$, obtain the optimal simplex tableau without doing any iteration. What type of solution is it?

First, we obtain the standard form:

$$\begin{aligned} Max.\quad & 2x_1 + x_2 \\ s.t.\quad & -x_1 + x_2 + s_1 = 2 \\ & x_1 + 2x_2 + s_2 = 6 \\ & 2x_1 + x_2 + s_3 = 6 \\ & x_1, x_2, s_1, s_2, s_3 \geq 0 \end{aligned}$$

By substituting x_1 and x_1 in the constraints and isolating s_1, s_2 and s_3 we get $s_1 = 2, s_2 = 0, s_3 = 0$. Then, the optimal solution of the standard problem is:

$$(x_1, x_2, s_1, s_2, s_3) = (2,2,2,0,0)$$

Since m=3, we need 3 basic variables, that will be x_1, x_2, s_1. Then we write A, B, and $\bar{b}$:

$$A = \begin{pmatrix} -1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 2 & 0 \\ 2 & 1 & 0 \end{pmatrix} \quad \bar{b} = \begin{pmatrix} 2 \\ 6 \\ 6 \end{pmatrix}$$

We calculate $B^{-1}, B^{-1}\bar{b}, B^{-1}A$

$$B^{-1} = \begin{pmatrix} 0 & -\frac{1}{3} & \frac{2}{3} \\ 0 & \frac{2}{3} & -\frac{1}{3} \\ 1 & -1 & 1 \end{pmatrix} \quad B^{-1}\bar{b} = \begin{pmatrix} 0 & -\frac{1}{3} & \frac{2}{3} \\ 0 & \frac{2}{3} & -\frac{1}{3} \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 6 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

$$B^{-1}A = \begin{pmatrix} 0 & -\frac{1}{3} & \frac{2}{3} \\ 0 & \frac{2}{3} & -\frac{1}{3} \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 1 & 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & -1 & 1 \end{pmatrix}$$

And now we write the simplex tableau:

		2	1	0	0	0	
		x_1	x_2	s_1	s_2	s_3	
2	x_1	1	0	0	-1/3	2/3	2
1	x_2	0	1	0	2/3	-1/3	2
0	s_1	0	0	1	-1	1	2
		2	1	0	0	1	6
		0	0	0	0	-1	

We have a maximization problem and all the reduced costs are ≤ 0, so this tableau is optimal. There is a non-basic variable with reduced cost 0, s_2, so there are multiple solutions. In the column of s_2 there is at least one positive value, the one corresponding to x_2. This means that we have a finite edge solution.

- Given the following linear programming problem:

$$Min. \quad x + 4y$$

$$s.t. \quad x + 3y = 4$$

$$-1 \leq y \leq 8$$

$$x \geq 0$$

Show that the point (x, y^+, y^-, s_1, s_2) = (7,0,1,9,0) is a basic feasible solution of the problem in standard form and obtain the associated simplex tableau. Study if this tableau is optimal. If it is, write the solution of the original linear problem, if not, say which variable should enter, which one should leave and the pivot element of the next iteration.

Note: line $-1 \leq y \leq 8$ represents two constraints, $y \leq 8$ and $y \geq -1$.

First, we transform the problem into standard form. We must add two slack variables and substitute the free variable $y = y^+ - y^-$.

$$Min. \quad x + 4y^+ - 4y^-$$

$$s.t. \qquad x + 3y^+ - 3y^- = 4$$

$$y^+ - y^- + s_1 = 8$$
$$y^+ - y^- - s_2 = -1$$

$$x, y^+, y^-, s_1, s_2 \geq 0$$

Now we are going to check that (7,0,1,9,0) is a BFS:

- There are n=5 variables and m=3 equalities, so we need 3 basic variables ant two non-basic variables. y^+ and s_2 have value 0, so we have enough non-basic variables.
- If we put the values of the variables into the constraints, we can see that they are all satisfied.
- Since x, y^- and s_1 are basic, the basic matrix B is:
$$B = \begin{pmatrix} 1 & -3 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$
and $|B| = 1 \neq 0$.
- Finally, the value of all the variables is ≥ 0.

Therefore, this solution is a BFS. Let us now obtain its simplex tableau. First, we calculate B^{-1}:

$$B^{-1} = \frac{1}{|B|}(adj\ B)^t = \frac{1}{1}\begin{pmatrix} 1 & 0 & -3 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

And now $B^{-1}\overline{b}$ and $B^{-1}A$:

$$B^{-1}\overline{b} = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} 4 \\ 8 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 9 \end{pmatrix}$$

$$B^{-1}A = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 3 & -3 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Note: Actually, we did not need to calculate $B^{-1}\overline{b}$, since the result is the vector containing the values of the basic variables in the BFS, and we already know these values.

Now we start completing the simplex tableau

		1	4	-4	0	0	
		x	y^+	y^-	s_1	s_2	
1	x	1	0	0	0	3	7
-4	y^-	0	-1	1	0	1	1
0	s_1	0	0	0	1	1	9

And we calculate the last two rows:

		1	4	-4	0	0	
		x	y^+	y^-	s_1	s_2	
1	x	1	0	0	0	3	7
-4	y^-	0	-1	1	0	1	1
0	s_1	0	0	0	1	1	9
		1	4	-4	0	-1	3
		0	0	0	0	1	

Since we are minimizing and all the reduced costs are ≥ 0, the tableau is optimal, and (7,0,1,9,0) is an optimal solution of the standard form of the problem.

However, there is a non-basic variable with $w = 0$, namely y^+. This means that this is not the only optimal solution of the problem. If we look at the column of this variable, there is no basic variable that can leave the basis, because all the values of y_j are ≤ 0. Therefore, this is an infinite edge solution.

Note: If we want an optimal solution of the original form of the problem, we must calculate $y = y^+ - y^- = 0 - 1 = -1$, and so the corresponding optimal solution in the original form will be (7,-1).

What we know so far

We already know how to transform an LP problem to its standard form, obtain a simplex tableau, and check whether the tableau is optimal. If the simplex tableau is optimal, we can determine which type of solution we have. If it is not, we decide which variable enters the basis and which variable leaves.

Taking all the above into account, we can now apply the following method for solving an LP problem:

1. Transform the problem into standard form.
2. Choose some basic variables.
3. Obtain the simplex tableau corresponding to our basic variables.
4. Check whether the simplex tableau is optimal:
    a. If it is optimal, determine which type of solution we have. The method is finished.
    b. If it is not optimal, determine which variable enters the basis and which one leaves. With the new set of basic variables, go back to step 3.

This is basically how the Simplex algorithm works. However, if we have to go back to step 3 several times before reaching the optimal solution, it can be quite time-consuming as we will need to calculate several simplex tableaus. To speed up the process, we can obtain a new simplex tableau from the previous one without having to redo all the calculations.

Moving from one simplex tableau to the next

Let us continue with an example we saw earlier:

$$
\begin{aligned}
&Max \quad x-y \\
&s.t. \quad x+2y+s_1=4 \\
&\qquad\ \ x+y+s_2=3 \\
&\qquad\ \ x,y,s_1,s_2\geq 0
\end{aligned}
$$

Suppose we have chosen y and s_2 as basic variables. The corresponding simplex tableau is then:

		1	-1	0	0	
		x	y	s_1	s_2	
-1	y	1/2	1	1/2	0	2
0	s_2	1/2	0	-1/2	1	1
		-1/2	1	-1/2	0	-2
		3/2	0	1/2	0	

As we have seen, the variable that must enter the basis is x and the one that must leave the basis is s_2. We now need to calculate a new simplex tableau with y and x as basic variables. Let's see how we obtain this new tableau.

First we need to determine the pivot of the tableau:

variable that enters

		1	-1	0	0	
		x	y	s_1	s_2	
-1	y	1/2	1	1/2	0	2
0	s_2	1/2	0	-1/2	1	1
		-1/2	1	-1/2	0	-2
		3/2	0	1/2	0	

variable that leaves

pivot

Now we draw a new tableau in which we replace the variable that leaves with the variable that enters (remember to update the coefficient on the left). The inside part of the tableau will be empty:

		1	-1	0	0	
		x	y	s_1	s_2	
-1	y					
1	x					

Now we need to take the row of the variable that has left the basis, s_2, and multiply it or divide it by the number that makes the pivot equal to 1. Since the value of the pivot in this case is 1/2, we must multiply this row by 2:

Row of s_2: 1/2 0 -1/2 1 1

Row of s_2 x 2: 1 0 -1 2 2

We can now write this new row in its original place in the tableau:

		1	-1	0	0	
		x	y	s_1	s_2	
-1	y					
1	x	1	0	-1	2	2

Now we need to calculate the remaining rows. Our goal is to modify these rows so that, in the column of the variable that has entered (x in this example), all values are 0 (except in the position of the pivot, which is now 1). To achieve this, we must take each row and add or subtract to this row the row of the pivot multiplied or divided by some number.

In this example, there is only one row we need to recalculate, which is the one that corresponds to variable y. We must obtain a value of 0 in the first position in this row (x column). To do so, we can use the row of the pivot (from either the first tableau or the new one, whichever we prefer). All we need to do in this case is subtract the row of the pivot from the row of y:

Row of y:	–	1/2	1	1/2	0	2
Row of the pivot:		1/2	0	-1/2	1	1
Row of y – row of pivot:		0	1	1	-1	1

This is the position that we must make 0

Now we can write the resulting row in the simplex tableau:

		1	-1	0	0	
		x	y	s_1	s_2	
-1	y	0	1	1	-1	1
1	x	1	0	-1	2	2

If we had more rows, we would have to proceed in the same way to complete the simplex tableau. Finally, the last two rows of the tableau are computed as usual, i.e. first multiplying the coefficient on the left by each column and then subtracting the new line from the line at the top:

		1	-1	0	0	
		x	y	s_1	s_2	
-1	y	0	1	1	-1	1
1	x	1	0	-1	2	2
		1	-1	-2	3	1
		0	0	2	-3	

Now we have completed the new simplex tableau. Remember all the steps you can take to check that your tableau is correct:

- The values of the basic variables (in the final column) must be ≥ 0.
- The columns of the basic variables (y and x in this example) must form the identity matrix.
- The reduced cost (final row) of the basic variables must be 0.

There is a further condition we can check when we move from one tableau to the next:

- The value of the objective function (in the lower right-hand corner of the tableau) must be equal to or better than the value in the previous tableau.

In this tableau, the value of the objective function has gone from -2 to 1. Since we are maximizing, this is an improvement, which is correct.

The new tableau we have obtained is not yet optimal because the reduced cost of variable s_1 is positive. s_1 should therefore now enter the basis to obtain a new simplex tableau.

The process of choosing which variables enter and leave the basis and calculating the new tableau is called a simplex iteration. The simplex algorithm consists of performing several simplex iterations until we reach the optimal tableau.

Advice for obtaining the first simplex tableau

Usually, the exercises tell you which basic variables you should choose to obtain the first simplex tableau, or at least provide you with enough information to deduce which variables to use. However, if no information is given, there is no way to know which are the best variables for obtaining the first tableau. Some variables will provide you with the optimal tableau right from the start, some will provide you with an intermediate tableau and you will need to perform several iterations before you obtain the optimal one, and some will not even provide you with a valid simplex tableau and you will need to change your choice.

There is no way to identify which choice of basic variables is best, though sometimes there is a way to choose them that will provide a first simplex tableau that will be very easy to calculate.

If the right-hand side of all the constraints is not negative and the technical matrix contains the identity matrix, you can choose as basic variables those whose columns contain this identity matrix. In this way, basic matrix B will be the identity matrix, B^{-1} will also be the identity matrix. Therefore $B^{-1}\bar{b} = \bar{b}$ and $B^{-1}A = A$ and you will not need to perform any matrix calculations. All you need to do is copy A and $\bar{b}$ into the simplex tableau.

For example:

$$\begin{aligned} Max \quad & 3x + 7y \\ s.t.: \quad & x + y \geq 5 \\ & y \leq 3 \\ & x, y \geq 0 \end{aligned}$$

The standard form of the problem is:

$$
\begin{aligned}
&Max \quad 3x + 7y \\
&s.t.: \; x + y - s_1 = 5 \\
&\qquad\quad y + s_2 = 3 \\
&\qquad\quad x, y, s_1, s_2 \geq 0
\end{aligned}
$$

columns of the identity matrix

$$
A = \begin{array}{c} \begin{array}{cccc} x & y & s_1 & s_2 \end{array} \\ \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \end{array} \qquad \bar{b} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}
$$

There are no negative values in $\bar{b}$, and the columns of variables x and s_2 form the identity matrix, so we can choose these as basic variables. To obtain the simplex tableau, we just copy A and $\bar{b}$ and then complete the last two rows as usual.

		3	7	0	0		
		x	y	s_1	s_2		
3	x	1	1	-1	0	5	
0	s_2	0	1	0	1	3	$\bar{b}$
	A	3	3	-3	0	15	
		0	4	3	0		

This situation occurs, for example, with problems in which all constraints are $\leq$. In such cases, the slack variables always form the identity matrix.

Examples:

- Given the following simplex tableau corresponding to a maximization problem:

		1	6	2	0	0	0	
		x1	x2	x3	s1	s2	s3	
1	x1	1	1	0	0	4	1	2
2	x3	0	1	1	0	2	2	1
0	s1	0	-2	0	1	5	0	1
		1	3	2	0	8	5	4
		0	3	0	0	-8	-5	

f) Calculate the next tableau. Check that the new tableau satisfies all the conditions a simplex tableau must satisfy.

In question e) we saw that variable x_2 must enter the basis and variable x_3 must leave.

enters

		1	6	2	0	0	0	
		x_1	x_2	x_3	s_1	s_2	s_3	
1	x_1	1	1	0	0	4	1	2
2	x_3	0	1	1	0	2	2	1
0	s_1	0	-2	0	1	5	0	1
		1	3	2	0	8	5	4
		0	3	0	0	-8	-5	

leaves

pivot

Now we will calculate the next simplex tableau. First we replace variable x_3 with variable x_2 on the left-hand side of the tableau:

		1	6	2	0	0	0	
		x_1	x_2	x_3	s_1	s_2	s_3	
1	x_1							
6	x_2							
0	s_1							

Now we divide the row of the variable that has left the basis (the second row here) by the value of the pivot but, since the pivot of this tableau is 1, the row will stay the same:

		1	6	2	0	0	0	
		x_1	x_2	x_3	s_1	s_2	s_3	
1	x_1							
6	x_2	0	1	1	0	2	2	1
0	s_1							

To calculate the first row, we must use row 1 and the row of the pivot to obtain the value 0 in the position above the pivot (the second column here). The operations we can perform are of the type $Row\ 1 \pm \propto * \ Row\ of\ the\ pivot$, where $\propto$ can be any value we choose. In this case, all we need to do is $Row\ 1 - Row\ of\ the\ pivot$

Row 1:	−	1	1	0	0	4	1	2
Row of the pivot:		0	1	1	0	2	2	1
		1	0	-1	0	2	-1	1

must be 0

		1	6	2	0	0	0	
		x_1	x_2	x_3	s_1	s_2	s_3	
1	x_1	1	0	-1	0	2	-1	1
6	x_2	0	1	1	0	2	2	1
0	s_1							

We need to proceed in the same way for row 3. In this case, we will need to perform the following operation:

$$Row\ 3 + 2 * Row\ of\ the\ pivot$$

Row 3:	0	-2	0	1	5	0	1
+ 2*Row of the pivot:	0	2	2	0	4	4	2
	0	0	2	1	9	4	3

must be 0

		1	6	2	0	0	0	
		x_1	x_2	x_3	s_1	s_2	s_3	
1	x_1	1	0	-1	0	2	-1	1
6	x_2	0	1	1	0	2	2	1
0	s_1	0	0	2	1	9	4	3

Finally, we complete the last two rows as usual:

		1	6	2	0	0	0	
		x_1	x_2	x_3	s_1	s_2	s_3	
1	x_1	1	0	-1	0	2	-1	1
6	x_2	0	1	1	0	2	2	1
0	s_1	0	0	2	1	9	4	3
		1	6	5	0	14	11	7
		0	0	-3	0	-14	-11	

Now we can check that the basic variables form the identity matrix, that all values of the basic variables are ≥ 0, that the reduced costs of the basic variables are 0, and that the value of the objective function has increased (from 4 to 7).

This new tableau is optimal because all the reduced costs are ≤ 0 (and this is a maximization problem). Since there are no non-basic variables with reduced cost 0, this is a vertex solution (i.e., with only one optimal solution). The optimal solution is then (1,1,0,3,0,0) and the value of the objective function is 7.

- Solve the problem using the Simplex method:

$$\begin{aligned} Min \quad & x_1 - 2x_2 + 4x_3 \\ s.t.: \quad & x_1 - x_2 \le 10 \\ & x_1 - 2x_2 + x_3 \ \le 14 \\ & x_1 \ge 0, x_2 \le 0, x_3 \ \ free \end{aligned}$$

First we transform the problem into the standard form by adding slack variables and turning all variables into non-negative ones:

$$\begin{aligned} Min \quad & x_1 - 2x_2 + 4x_3^+ - 4x_3^- \\ s.t.: \quad & x_1 - x_2 + s_1 = 10 \\ & x_1 - 2x_2 + x_3^+ - x_3^- + s_2 = 14 \\ & x_1, x_2, x_3^+, x_3^-, s_1, s_2 \ge 0 \end{aligned}$$

This problem has n=6 variables and m=2 equations, so we need 2 basic variables. The technical matrix $A$and the vector $\bar{b}$ are:

$$A = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3^+ & x_3^- & s_1 & s_2 \end{matrix} \\ \begin{pmatrix} 1 & -1 & 0 & 0 & 1 & 0 \\ 1 & -2 & 1 & -1 & 0 & 1 \end{pmatrix} \end{array} \qquad \bar{b} = \begin{pmatrix} 10 \\ 14 \end{pmatrix}$$

Since we are not told which variables to use as basic variables for the first tableau, we can look for the identity matrix in A. Here, we find it in the columns of s_1 and s_2, so we can try using these variables as basic (we could also try s_1 and x_3^+, which also form the identity matrix). Now all we have to do is copy A and $\bar{b}$ into the simplex tableau:

		1	-2	4	-4	0	0	
		x_1	x_2	x_3^+	x_3^-	s_1	s_2	
0	s_1	1	-1	0	0	1	0	10
0	s_2	1	-2	1	-1	0	1	14

Now we complete the tableau:

		1	-2	4	-4	0	0	
		x_1	x_2	x_3^+	x_3^-	s_1	s_2	
0	s_1	1	-1	0	0	1	0	10
0	s_2	1	-2	1	-1	0	1	14
		0	0	0	0	0	0	0
		1	-2	4	-4	0	0	

It seems that this tableau is not optimal because we are minimizing and there are negative reduced costs. The variable with the lowest negative reduced cost must therefore enter the basis; in this case it is variable x_3^-, whose reduced cost is -4. Now we need to choose the variable that leaves the basis. However, only variables with a positive value in the column can leave and since both basic variables have a value ≤ 0 in the column of x_3^-, no variables can leave.

must enter

≤ 0, no variable can leave

		1	-2	4	-4	0	0	
		x_1	x_2	x_3^+	x_3^-	s_1	s_2	
0	s_1	1	-1	0	0	1	0	10
0	s_2	1	-2	1	-1	0	1	14
		0	0	0	0	0	0	0
		1	-2	4	-4	0	0	

If a variable must enter but no variable can leave, the tableau is optimal but it means that the problem is unbounded.

Finding the inverse matrix of B in the tableau

As we will see in unit 5, some situations will require the inverse of the basic matrix, B^{-1}, which corresponds to the basic variables of the simplex tableau. If this is the first tableau we calculated, we will probably already have B^{-1}, since B^{-1} is needed to calculate this tableau. However, if it is a tableau we obtained in a subsequent iteration using the procedure described above, we will not already have B^{-1}. In this case, if we want to obtain B^{-1}, we can extract B from technical matrix A and calculate B^{-1}in the usual way.

However, there are certain situations in which obtaining B^{-1} may be easier. *If technical matrix A contains the identity matrix in some of its columns, we can always find B^{-1} in the same columns of the simplex tableau.*

Consider the example from the previous page:

$$\begin{aligned} Max \quad & 3x + 7y \\ s.t.: \quad & x + y - s_1 = 5 \\ & \qquad y + s_2 = 3 \\ & x, y, s_1, s_2 \geq 0 \end{aligned}$$

$$A = \begin{array}{cccc} x & y & s_1 & s_2 \end{array} \atop \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Here, the columns of x and s_2 form the identity matrix. Suppose that we have the following simplex tableau, which corresponds to the basic variables x and y:

		3	7	0	0	
		x	y	s_1	s_2	
3	x	1	0	-1	-1	2
7	y	0	1	0	1	3
		3	7	-3	4	27
A		0	0	3	-4	

To determine the inverse matrix B^{-1}, we just need to take the columns of the tableau that correspond to the variables that formed the identity matrix in A, i.e. x and s_2:

$$B^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

Of course, we could have found B in technical matrix A and then calculated B^{-1}, but this way is easier and faster.

If there is no identity matrix in A, we will need to calculate B^{-1} in the usual way.

Unit 5. Post-optimization and sensitivity analysis

Post-optimization vs. sensitivity analysis

In this unit we will study what happens to the optimal solution of an LP when some of the data change. We can consider two possible situations:

- We want to know whether the solution we have remains optimal if something changes. If it doesn't, we want to know what the new optimal solution is (for example, what happens to the optimal solution if the price of a certain ingredient increases by €1?). This is called post-optimization.
- We want to study by how much we can change a specific datum of the problem without the change affecting the optimal solution (for example, by how much can we increase the price of a certain product without this affecting the optimal production?). This is called sensitivity analysis.

In post-optimization we will study how the following changes affect the problem:

- changing the coefficient of a variable in the objective function
- changing the right-hand side of a constraint
- changing the coefficient of a non-basic variable in a constraint
- adding a new variable to the problem
- adding a new constraint to the problem

In sensitivity analysis we will study by how much we can alter the following elements of the problem:

- the coefficients of the variables in the objective function
- the right-hand side of the constraints
- the coefficients of non-basic variables in the constraints

Optimality and feasibility conditions

When we modify something in a maximization problem after having solved it with the simplex method, we must check two conditions:

- $x_B \geq 0$: the value of the basic variables (the last column of the tableau) must not be negative. This is the feasibility condition. If this is satisfied, the solution will still be feasible.
- $w_j \leq 0$: the reduced cost (last row of the tableau) must be ≤ 0 for the non-basic variables (for the basic variables it is always 0). This is the optimality condition. If this condition is satisfied, the solution will still be optimal. For a minimization problem, the condition would be $w_j \geq 0$.

Coefficients of the objective function: post-optimization

We will study what happens to the optimal solution if we modify the coefficient of a variable in the objective function. Since slack variables do not appear in the objective function, we can only modify the coefficients of the original variables.

Since this change does not affect the feasibility of our solution, we only need to check the optimality condition.

Let us see first what happens if we modify the coefficient of a **non-basic variable**. Consider the following LP problem:

$$\begin{aligned} Max \quad & x + 2y \\ s.t.: \quad & x + y \leq 4 \\ & 2x + y \leq 6 \\ & x, y \geq 0 \end{aligned}$$

Its standard form is:

$$\begin{aligned} Max \quad & x + 2y \\ s.t.: \quad & x + y + s_1 = 4 \\ & 2x + y + s_2 = 6 \\ & x, y, s_1, s_2 \geq 0 \end{aligned}$$

Suppose we have solved this problem using the simplex method and obtained the following optimal simplex tableau:

		1	2	0	0	
		x	y	s_1	s_2	
2	y	1	1	1	0	4
0	s_2	1	0	-1	1	2
		2	2	2	0	8
		-1	0	-2	0	

The optimal solution is then (0,4,0,2), and the optimal value of the objective function is 8.

Now suppose we want to change the coefficient of variable x in the objective function, denoted c_1, from 1 to 3/2, so that the objective function is now $\frac{3}{2}x + 2y$.

If we look at the tableau, the coefficient of x appears only once at the top of the tableau:

		1	2	0	0	
		x	y	s_1	s_2	
2	y	1	1	1	0	4
0	s_2	1	0	-1	1	2
		2	2	2	0	8
		-1	0	-2	0	

If we change this coefficient to 3/2, this will only affect the reduced cost of x, which we can now easily recalculate:

		3/2	2	0	0	
		x	y	s_1	s_2	
2	y	1	1	1	0	4
0	s_2	1	0	-1	1	2
		2	2	2	0	8
		-1/2	0	-2	0	

Since the new reduced cost we have calculated is still negative, the optimality condition is satisfied and the optimal solution is the same.

Now let's see what happens if we modify the same coefficient but, instead of 1 or 3/2, it now takes the value 3. We need to recalculate the reduced cost of x as before:

		3	2	0	0	
		x	y	s_1	s_2	
2	y	1	1	1	0	4
0	s_2	1	0	-1	1	2
		2	2	2	0	8
		1	0	-2	0	

However, the new reduced cost is positive, which does not satisfy the optimality condition. This means that this solution is no longer optimal. To obtain the new optimal solution we need to do one more iteration of the simplex algorithm. The variable that must enter the basis is x, and the variable that must leave is s_2. The new (and optimal) simplex tableau will be:

		3	2	0	0	
		x	y	s_1	s_2	
2	y	0	1	2	-1	2
3	x	1	0	-1	1	2
		3	2	1	1	10
		0	0	-1	-1	

The new optimal solution will be (2,2,0,0), and the optimal value of the objective function will be 10.

Now let's study what happens when the coefficient we modify corresponds to a **basic variable**. Consider the same problem with its original coefficients and suppose that we want to change the coefficient of variable y to 4. The coefficient of y appears not only at the top of the tableau but also on the left-hand side:

		1	2	0	0	
		x	y	s_1	s_2	
2	y	1	1	1	0	4
0	s_2	1	0	-1	1	2
		2	2	2	0	8
		-1	0	-2	0	

This implies that modifying this coefficient will affect all the reduced costs. We must therefore erase the last two rows and recalculate them using the new coefficient:

		1	4	0	0	
		x	y	s_1	s_2	
4	y	1	1	1	0	4
0	s_2	1	0	-1	1	2

		1	4	0	0	
		x	y	s_1	s_2	
4	y	1	1	1	0	4
0	s_2	1	0	-1	1	2
		4	4	4	0	16
		-3	0	-4	0	

The new reduced costs are all still ≤ 0, so the optimality condition is satisfied and the optimal solution is the same, (0,4,0,2), though the optimal value of the objective function has now changed to 16.

Now suppose that we change the coefficient of y to 1/2. By proceeding as before, we obtain the following simplex tableau:

		1	1/2	0	0	
		x	y	s_1	s_2	
1/2	y	1	1	1	0	4
0	s_2	1	0	-1	1	2
		1/2	1/2	1/2	0	2
		1/2	0	-1/2	0	

This tableau is no longer optimal. Variable x must enter the basis and variable y must leave, so we obtain the following optimal simplex tableau:

		1	1/2	0	0	
		x	y	s_1	s_2	
1/2	y	0	1	2	-1	2
1	x	1	0	-1	1	2
		1	1/2	0	1/2	3
		0	0	0	-1/2	

In this case we have a new optimal solution, (2,2,0,0), with an optimal value of 3 for the objective function. Moreover, this is not the only solution (it is a finite edge solution) since the reduced cost of s_1 is 0 and variable y can leave the basis (if we wish to obtain another optimal solution, we can do one more iteration with s_1 entering and y leaving).

Examples:

- We have the following linear programming problem:

$$Max.\ \ F(x) = 2x_1 + 4x_2 + 8x_3$$

$$s.t. \qquad 2x_1 + 2x_2 + 4x_3 \le 8$$

$$2x_1 + x_2 + 2x_3 \le 6$$

$$x_1 \ge 0, x_2 \ge 0, x_3 \ge 0$$

The optimal solution for this problem is given in the following tableau:

		2	4	8	0	0	
		x_1	x_2	x_3	s_1	s_2	
4	x_2	1	1	2	1/2	0	4
0	s_2	1	0	0	-1/2	1	2
		4	4	8	2	0	16
		-2	0	0	-2	0	

Obtain the new optimal solution when c1=5.

We need to modify the coefficient of x_1 in the simplex tableau and recalculate its reduced cost:

		5	4	8	0	0	
		x_1	x_2	x_3	s_1	s_2	
4	x_2	1	1	2	1/2	0	4
0	s_2	1	0	0	-1/2	1	2
		4	4	8	2	0	16
		1	0	0	-2	0	

The reduced cost of x_1 is now positive and, since this is a maximization problem, this tableau is not optimal. x_1 must enter the basis and s_2 must leave. The new simplex tableau will be:

		5	4	8	0	0	
		x_1	x_2	x_3	s_1	s_2	
4	x_2	0	1	2	1	-1	2
5	x_1	1	0	0	-1/2	1	2
		5	4	8	3/2	1	18
		0	0	0	-3/2	-1	

This tableau is optimal. The new optimal solution is (2,2,0,0,0) and the optimal value of the objective function is 18. There is a non-basic variable, x_3, with reduced cost 0, so there are multiple optimal solutions. If x_3 entered the basis, x_2 would leave, so this is a finite edge solution.

Although this was not required in the exercise, let us obtain another optimal solution by doing one more iteration in which x_3 enters the basis and x_2 leaves:

		5	4	8	0	0	
		x_1	x_2	x_3	s_1	s_2	
8	x_3	0	1/2	1	1/2	-1/2	1
5	x_1	1	0	0	-1/2	1	2
		5	4	8	3/2	1	18
		0	0	0	-3/2	-1	

This tableau gives us another optimal solution, (2,0,1,0,0), while the value of the objective function is the same.

- We have the following problem:

$$Max. \quad 2x + 6y$$

$$s.t. \qquad x + 3y \leq 9$$

$$2x + y \leq 8$$

$$x, y \geq 0$$

The optimal tableau for this problem is:

		2	6	0	0	
		x	y	s_1	s_2	
2	x	1	0	-1/5	3/5	3
6	y	0	1	2/5	-1/5	2
		2	6	2	0	18
		0	0	-2	0	

If we set c1= 10, is the tableau still optimal? What is the optimal value of the objective function in this case?

Since we are modifying the coefficient of x, which is basic, we must modify it in two places and recalculate the last two rows of the tableau:

		10	6	0	0	
		x	y	s_1	s_2	
10	x	1	0	-1/5	3/5	3
6	y	0	1	2/5	-1/5	2
		10	6	2/5	24/5	42
		0	0	-2/5	-24/5	

The simplex tableau is still optimal because all reduced costs are ≤ 0. The optimal solution is the same but the optimal value of the objective function is now 42.

Now we will see the post-optimization for the remaining elements of the problem and the sensitivity analysis for some of them.

Coefficient of a variable in the objective function: sensitivity interval

In the first part of this unit we saw how to perform post-optimization for these coefficients. Now we will obtain the sensitivity interval, which will tell us by how much we can modify the coefficient without affecting the optimal solution.

Let us consider the same LP problem as in the first part of this unit. The standard form and optimal simplex tableau for this problem are:

$$
\begin{aligned}
Max \quad & x + 2y \\
s.t.: \quad & x + y + s_1 = 4 \\
& 2x + y + s_2 = 6 \\
& x, y, s_1, s_2 \geq 0
\end{aligned}
$$

		1	2	0	0	
		x	y	s_1	s_2	
2	y	1	1	1	0	4
0	s_2	1	0	-1	1	2
		2	2	2	0	8
		-1	0	-2	0	

The optimal solution is (0,4,0,2), and the optimal value of the objective function is 8.

We will obtain the sensitivity interval for the coefficient of a **non-basic** variable. Let us consider, for example, variable x and its coefficient in the objective function, c_1. We need to replace the coefficient of x in the simplex tableau by parameter c_1 and recalculate the reduced cost using this parameter:

		c_1	2	0	0	
		x	y	s_1	s_2	
2	y	1	1	1	0	4
0	s_2	1	0	-1	1	2
		2	2	2	0	8
		$c_1 - 2$	0	-2	0	

The new reduced cost of x is $c_1 - 2$. For the tableau to be optimal, since this is a maximization problem, all the reduced costs must be ≤ 0. So we need $c_1 - 2 \leq 0$, which implies $c_1 \leq 2$. Then, for any value of c_1 less than or equal to 2, the optimal solution will be the same. The sensitivity interval of c_1 is therefore $]-\infty, 2]$. For any value of the coefficient inside this interval, the optimal solution will not change.

Now let us see how to obtain this interval for a **basic** variable (for example, y) and its coefficient c_2. As before, we replace the coefficient of y by parameter c_2 but, since the variable is now basic, we need to do this in two places in the tableau. This implies that we have to recalculate the last two rows of the tableau for all variables:

		1	c_2	0	0	
		x	y	s_1	s_2	
c_2	y	1	1	1	0	4
0	s_2	1	0	-1	1	2
		c_2	c_2	c_2	0	$4c_2$
		$1 - c_2$	0	$-c_2$	0	

For this tableau to be optimal, all the reduced costs must be negative and so the following two inequalities must be satisfied:

$$1 - c_2 \leq 0 \text{ and } -c_2 \leq 0$$

This implies that:

$$c_2 \geq 1 \text{ and } c_2 \geq 0$$

The second inequality is redundant (if $c_2 \geq 1$, then $c_2 \geq 0$ will necessarily be true), so we only need the first condition, $c_2 \geq 1$. The sensitivity interval of c_2 will therefore be $[1, +\infty[$. This means that the optimal solution will be the same for any coefficient of y that belongs to this interval.

Example:

- We have the following linear programming problem:

$$Max.\ \ F(x) = 2x_1 + 4x_2 + 8x_3$$

$$s.t. \quad 2x_1 + 2x_2 + 4x_3 \leq 8$$

$$2x_1 + x_2 + 2x_3 \leq 6$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

The optimal solution for this problem is given in the following tableau:

		2	*4*	*8*	*0*	*0*	
		x_1	x_2	x_3	s_1	s_2	
4	x_2	*1*	*1*	*2*	*1/2*	*0*	*4*
0	s_2	*1*	*0*	*0*	*-1/2*	*1*	*2*
		4	*4*	*8*	*2*	*0*	*16*
		-2	*0*	*0*	*-2*	*0*	

Obtain the sensitivity interval of c1 from the optimal tableau.

We need to replace the coefficient of x_1 in the simplex tableau with c_1 and recalculate its reduced cost:

		c_1	*4*	*8*	*0*	*0*	
		x_1	x_2	x_3	s_1	s_2	
4	x_2	*1*	*1*	*2*	*1/2*	*0*	*4*
0	s_2	*1*	*0*	*0*	*-1/2*	*1*	*2*
		4	*4*	*8*	*2*	*0*	*16*
		$c_1 - 4$	*0*	*0*	*-2*	*0*	

Since this is a maximization problem, the reduced cost in the optimal tableau must be ≤ 0. Therefore:

$$c_1 - 4 \leq 0 \Rightarrow c_1 \leq 4$$

and the sensitivity interval of c_1 is $]-\infty, 4]$.

- We have the following problem:

$$Max. \quad 2x + 6y$$
$$s.t. \quad x + 3y \leq 9$$
$$2x + y \leq 8$$
$$x, y \geq 0$$

The optimal tableau for this problem is:

		2	6	0	0	
		x	y	s_1	s_2	
2	x	1	0	-1/5	3/5	3
6	y	0	1	2/5	-1/5	2
		2	6	2	0	18
		0	0	-2	0	

- Obtain the sensitivity interval of the coefficient of variable x in the objective function and explain its meaning.

Again we replace the coefficient of x with c_1 but this time in two places of the tableau because the variable is basic. We then calculate the last two rows of the tableau using c_1:

		c_1	6	0	0	
		x	y	s_1	s_2	
c_1	x	1	0	-1/5	3/5	3
6	y	0	1	2/5	-1/5	2
		c_1	6	$-\frac{1}{5}c_1 + \frac{12}{5}$	$\frac{3}{5}c_1 - \frac{6}{5}$	$3c_1 + 12$
		0	0	$\frac{1}{5}c_1 - \frac{12}{5}$	$-\frac{3}{5}c_1 + \frac{6}{5}$	

For the tableau to be optimal, we need that:

$$\frac{1}{5}c_1 - \frac{12}{5} \leq 0 \text{ and } -\frac{3}{5}c_1 + \frac{6}{5} \leq 0$$
$$\Downarrow$$
$$c_1 \leq 12 \text{ and } c_1 \geq 2$$

The sensitivity interval of c_1 is therefore $[2,12]$. This means that if the coefficient of variable x changes but remains between 2 and 12, the optimal solution will be the same but the optimal value will change to $3c_1 + 12$.

The right-hand side of the constraints: post-optimization

We will now study what happens to the optimal solution if we modify the right-hand side of a constraint.

Unlike what happened with the coefficients of the objective function, this change does not affect the optimality of our solution but the feasibility. We must therefore check the feasibility condition and, if this condition is still satisfied, the solution will also be optimal (though the values of the basic variables and the optimal value of the objective function may have changed!).

Now suppose that we wish to change the right-hand side of the first constraint, which is denoted b_1, from 4 to 5 so that the first constraint is now $x + y + s_1 = 5$.

This affects vector $\bar{b}$. This vector is used to calculate $B^{-1}\bar{b}$, which then goes in the last column of the tableau, so we must again recalculate $B^{-1}\bar{b}$. We already know the new vector $\bar{b}$:

$$\bar{b} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

Now we must obtain B^{-1}. If this simplex tableau were the first one we calculated, we would probably have B^{-1} already. However, if we did not calculate it or it is the result of some simplex iterations, we'll need to calculate it in some other way. The technical matrix A of this problem is:

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}$$

In this case there are two ways to obtain B^{-1}:

- Since we know that the basic variables of the current simplex tableau are y and s_2, we can obtain B from A and then use the usual formula to calculate B^{-1}:

$$B = \begin{array}{c} \begin{array}{cc} y & s_2 \end{array} \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \end{array}$$

$$B^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

- In this case technical matrix A contains the identity matrix in the columns of s_1 and s_2, so we can find B^{-1} in the same columns of the simplex tableau (we saw this at the end of unit 4):

Identity matrix

$$A = \begin{array}{c} \begin{array}{cccc} x & y & s_1 & s_2 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

B^{-1}

		1	2	0	0	
		x	y	s_1	s_2	
2	y	1	1	1	0	4
0	s_2	1	0	-1	1	2
		2	2	2	0	8
		-1	0	-2	0	

The second way to obtain B^{-1} is easier and helps avoid calculation mistakes because we don't need to calculate anything. However, it can be applied only when A contains the identity matrix.

Once we have obtained B^{-1}, we calculate $B^{-1}\bar{b}$, replace the last column of the tableau with the result, and recalculate the value of the objective function.

$$B^{-1}\bar{b} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

		1	2	0	0	
		x	y	s_1	s_2	
2	y	1	1	1	0	5
0	s_2	1	0	-1	1	1
		2	2	2	0	10
		-1	0	-2	0	

If all values obtained in $B^{-1}\bar{b}$ are ≥ 0, the feasibility condition is satisfied and the simplex tableau is optimal. In this example, the optimal solution is (0,5,0,1) and the optimal value of the objective function is 10. As you can see, the values of the solution have changed but the tableau is still optimal, and so the basic and non-basic variables of the optimal solution are the same.

Now let's suppose that we want to know what happens if the right-hand side of the first constraint changes to 8. We already have B^{-1}, so we can directly calculate $B^{-1}\bar{b}$:

$$B^{-1}\bar{b} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 8 \\ 6 \end{pmatrix} = \begin{pmatrix} 8 \\ -2 \end{pmatrix}$$

In this case the value of the second basic variable is negative, which means that the solution is no longer feasible. In this case we cannot obtain the new optimal solution easily because the basic and non-basic variables will change and, to obtain the optimal solution, we would need to solve the problem again from the beginning by choosing another combination of basic variables.

The right-hand side of the constraints: the sensitivity interval

To obtain the sensitivity interval of the right-hand side of a constraint, we do something similar to what we do with the coefficients of the objective function. Here, we must replace the value on the right-hand side with a parameter.

Suppose that we want to obtain the sensitivity interval of the right-hand side of the first constraint, b_1. We must then calculate $B^{-1}\bar{b}$, but by using b_1 in vector $\bar{b}$:

$$B^{-1}\bar{b} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ 6 \end{pmatrix} = \begin{pmatrix} b_1 \\ -b_1 + 6 \end{pmatrix}$$

For the solution to be feasible, we need all values of this vector to be ≥ 0 and we obtain these two inequalities:

$$b_1 \geq 0 \text{ and } -b_1 + 6 \geq 0$$

which is the same as:

$$b_1 \geq 0 \text{ and } b_1 \leq 6$$

The sensitivity interval of b_1 is then $[0,6]$. This means that, for any value on the right-hand side of the first constraint that lies in this interval, the simplex tableau will still be optimal (i.e., the basic and non-basic variables of the optimal solution will be the same), though the values of the basic variables of the optimal solution will change ($y = b_1$ and $s_2 = -b_1 + 6$).

Example:

- For the following LP problem:

$$\begin{aligned} Max.\ & 2x_1 + x_2 + 3x_3 \\ s.t.\ & x_1 + 2x_3 \geq 12 \\ & 3x_1 + 2x_2 + x_3 \leq 18 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

a) Obtain the optimal simplex tableau from the following one, which you need to complete:

		2	1	3	0	0	
		x_1	x_2	x_3	s_1	s_2	x_B
1	x_2	5/4	1	0	1/4	1/2	
3	x_3	1/2	0	1	-1/2	0	
	z_j						
	w_j						

b) Perform the sensitivity analysis of b2 and explain the meaning of the interval.

a) The standard form is:

$$\begin{aligned} Max.\ & 2x_1 + x_2 + 3x_3 \\ s.t.\ & x_1 + 2x_3 - s_1 = 12 \\ & 3x_1 + 2x_2 + x_3 + s_2 = 18 \\ & x_1, x_2, x_3, s_1, s_2 \geq 0 \end{aligned}$$

The technical matrix is:

$$A = \begin{pmatrix} 1 & 0 & 2 & -1 & 0 \\ 3 & 2 & 1 & 0 & 1 \end{pmatrix}$$

Since the basic variables of the tableau are x_2 and x_3, the basic matrix and its inverse matrix are:

$$B = \begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix} \qquad B^{-1} = \begin{pmatrix} -\frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$$

To obtain the values of the basic variables, we perform this operation:

$$B^{-1}\bar{b} = \begin{pmatrix} -\frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 12 \\ 18 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \end{pmatrix}$$

Now we can complete the tableau:

		2	1	3	0	0	
		x_1	x_2	x_3	s_1	s_2	x_B
1	x_2	5/4	1	0	1/4	1/2	6
3	x_3	1/2	0	1	-1/2	0	6
	z_j	11/4	1	3	-5/4	1/2	24
	w_j	-3/4	0	0	5/4	-1/2	

Since this is a maximization problem, variable s_1 must enter the basis and variable x_2 must leave. After one simplex iteration the new tableau is:

		2	1	3	0	0	
		x_1	x_2	x_3	s_1	s_2	x_B
0	s_1	5	4	0	1	2	24
3	x_3	3	2	1	0	1	18
	z_j	9	6	3	0	3	54
	w_j	-7	-5	0	0	-3	

This is the optimal simplex tableau because all reduced costs are ≤ 0.

b) We must first obtain B^{-1}. Since A does not contain the identity matrix, we cannot find B^{-1} in the simplex tableau and we need to calculate it in the usual way by considering the basic variables s_1 and x_3 (note that matrix B^{-1}, which we calculated in a), corresponds to the first simplex tableau with basic variables x_2, x_3; it is therefore not valid for the optimal tableau and we cannot use it here).

$$B = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \qquad B^{-1} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$$

Now we multiply $B^{-1}\overline{b}$ while replacing the second value in $\overline{b}$ with b_2:

$$B^{-1}\overline{b} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 12 \\ b_2 \end{pmatrix} = \begin{pmatrix} -12 + 2b_2 \\ b_2 \end{pmatrix}$$

All values here must always be ≥ 0, so we have:

$$-12 + 2b_2 \geq 0 \text{ and } b_2 \geq 0$$
$$\Downarrow$$
$$b_2 \geq 6 \text{ and } b_2 \geq 0$$

The second condition is redundant, so the sensitivity interval of b_2 is $[6, +\infty[$. For any value of b_2 that is greater than or equal to 6, the optimal simplex tableau will be the same and the optimal solution will have the same basic and non-basic variables.

- We have the following linear programming problem:

$$\begin{aligned} Min.\quad & -x + 5y - 3z \\ s.t.\quad & x - 2y + 3z \leq 6 \\ & 4x - y - 6z \leq 12 \\ & x, y, z \geq 0 \end{aligned}$$

We also have the following optimal tableau corresponding to the equivalent maximization problem:

		1	-5	3	0	0	
		x	y	z	s	t	
3	z	1/3	-2/3	1	1/3	0	2
0	t	6	-5	0	2	1	24
	Z_j	1	-2	3	1	0	6
	W_j	0	-3	0	-1	0	

If the right-hand side of the first constraint changed to *12* ($b_1=12$), what would the optimal solution of the problem be? Write the value of the variables and the objective function and explain what type of solution it is.

The standard form of the problem is:

$$\begin{aligned} Min.\quad & -x+5y-3z \\ s.t.\quad & x-2y+3z+s_1 = 6 \\ & 4x-y-6z+s_2 = 12 \\ & x, y, z, s_1, s_2 \geq 0 \end{aligned}$$

The technical matrix A is then:

$$A = \begin{pmatrix} 1 & -2 & 3 & 1 & 0 \\ 4 & -1 & -6 & 0 & 1 \end{pmatrix}$$

In this case, we can find the identity matrix in the last two columns, which correspond to variables s_1 and s_2. To obtain the inverse matrix B^{-1}, therefore, we can observe these two columns in the simplex tableau:

		1	-5	3	0	0	
		x	y	z	s	t	
3	z	1/3	-2/3	1	1/3	0	2
0	t	6	-5	0	2	1	24
	Z_j	1	-2	3	1	0	6
	W_j	0	-3	0	-1	0	

$$B^{-1} = \begin{pmatrix} 1/3 & 0 \\ 2 & 1 \end{pmatrix}$$

Now we calculate $B^{-1}\bar{b}$ with the new value of b_1:

$$B^{-1}\bar{b} = \begin{pmatrix} 1/3 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 12 \\ 12 \end{pmatrix} = \begin{pmatrix} 4 \\ 36 \end{pmatrix}$$

Then we update the simplex tableau with the new values:

		1	*-5*	*3*	*0*	*0*	
		x	*y*	*z*	*s*	*t*	
3	*z*	*1/3*	*-2/3*	*1*	*1/3*	*0*	*4*
0	*t*	*6*	*-5*	*0*	*2*	*1*	*36*
	Z_j	*1*	*-2*	*3*	*1*	*0*	*12*
	W_j	*0*	*-3*	*0*	*-1*	*0*	

Since all the values obtained for the basic variables are ≥ 0, the tableau is still feasible (and optimal). The new optimal solution is (0,0,4,0,36) and the optimal value of the objective function is 12.

This is an edge solution because variable x is non-basic and its reduced cost is 0. If x entered the basis, variable t would leave, so we have a finite edge solution.

Coefficient of a variable in a constraint: post-optimization

Now we will see what happens if we modify the coefficient of a variable in one of the constraints. We will consider only changes to the coefficients of non-basic variables because if we changed the coefficient of a basic variable, too many things would be affected and it would be better to start the problem from the beginning.

If we modify the coefficient of a non-basic variable, only the reduced cost of this variable will be affected and we will only need to check the optimality condition, since the feasibility will always be satisfied.

Suppose that we want to change the coefficient of the first variable, x, in the first constraint from 1 to 2 (this coefficient is denoted by a_{11}). This means that the first constraint will now be $2x + y + s_1 = 4$. Since this will affect technical matrix A, we will need to recalculate $B^{-1}A$. If we do not yet have B^{-1}, we will need to obtain it. In this case, however, we already have it from the previous examples:

$$B^{-1}A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

In fact, the only part of $B^{-1}A$ that will change is the column of the variable whose coefficient we are modifying, which in this case is x. We therefore do not need to recalculate all the other columns. Now we replace $B^{-1}A$ in the tableau and recalculate the last two rows (but only for variable x):

		1	2	0	0	
		x	y	s_1	s_2	
2	y	2	1	1	0	4
0	s_2	0	0	-1	1	2
		4	2	2	0	8
		-3	0	-2	0	

Since the reduced cost of x is still negative and this is a maximization problem, the tableau is still optimal and the optimal solution does not change.

Suppose now that we change this same coefficient to -1. By proceeding as before, we obtain:

$$B^{-1}A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 & 0 \\ 3 & 0 & -1 & 1 \end{pmatrix}$$

		1	2	0	0	
		x	y	s_1	s_2	
2	y	-1	1	1	0	4
0	s_2	3	0	-1	1	2
		-2	2	2	0	8
		3	0	-2	0	

The reduced cost of variable x is now positive, so this tableau is no longer optimal. To obtain the new optimal solution, therefore, we need to do one more simplex iteration in which x enters the basis and s_2 leaves (we will not do this here for the sake of brevity).

Coefficient of a variable in a constraint: the sensitivity interval

As earlier, we will consider only the coefficients of non-basic variables.

Suppose that we want to obtain the sensitivity interval of the coefficient of the first variable, x, in the first constraint (a_{11}). We now replace the coefficient in A with a_{11} and proceed as before (we need only to recalculate the column of x):

$$B^{-1}A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & 1 & 1 & 0 \\ -a_{11}+2 & 0 & -1 & 1 \end{pmatrix}$$

		1	2	0	0	
		x	y	s_1	s_2	
2	y	a_{11}	1	1	0	4
0	s_2	$-a_{11}+2$	0	-1	1	2
		$2a_{11}$	2	2	0	8
		$1-2a_{11}$	0	-2	0	

Since the reduced cost of x must be negative, we get:

$$1 - 2a_{11} \leq 0$$

And then:

$$a_{11} \geq \frac{1}{2}$$

The sensitivity interval of a_{11} is therefore $[\frac{1}{2}, +\infty[$. For any value inside this interval, the optimal solution will be the same.

Example:

- We have the following linear problem:

$$Max. \qquad 3x_1 + x_2 + 3x_3 + 4x_4$$

$$s.t. \qquad 2x_1 + 2x_2 + 3x_3 + 8x_4 \leq 20$$

$$3x_1 + 5x_2 + 2x_3 + 2x_4 \leq 15$$

$$x_i \geq 0, \qquad i = 1, \ldots, 4$$

We also have the following optimal tableau:

		3	1	3	4	0	0	
		x_1	x_2	x_3	x_4	s_1	s_2	
3	x_1	1	11/5	0	-2	-2/5	3/5	1
3	x_3	0	-4/5	1	4	3/5	-2/5	6
	Z_j	3	21/5	3	6	3/5	3/5	21
	W_j	0	-16/5	0	-2	-3/5	-3/5	

a) If the coefficient of x4 in the first constraint changes to a14=3, calculate the optimal solution.

b) Obtain the sensitivity interval and explain its a14 meaning.

a) The standard form is:

$$Max. \qquad 3x_1 + x_2 + 3x_3 + 4x_4$$

$$s.t. \qquad 2x_1 + 2x_2 + 3x_3 + 8x_4 + s_1 = 20$$

$$3x_1 + 5x_2 + 2x_3 + 2x_4 + s_2 = 15$$

$$x_i, \geq 0, \qquad i = 1, \ldots, 4$$

$$s_1, s_2 \geq 0$$

The technical matrix is:

$$A = \begin{pmatrix} 2 & 2 & 3 & 8 & 1 & 0 \\ 3 & 5 & 2 & 2 & 0 & 1 \end{pmatrix}$$

This contains the identity matrix in the columns of s_1 and s_2, so B^{-1} can be found in the columns of these variables in the simplex tableau:

$$B^{-1} = \begin{pmatrix} -2/5 & 3/5 \\ 3/5 & -2/5 \end{pmatrix}$$

Now we want to change the technical coefficient a_{14} to 3. This coefficient is in row 1, column 4 of the technical matrix:

$$A = \begin{pmatrix} 2 & 2 & 3 & 8 & 1 & 0 \\ 3 & 5 & 2 & 2 & 0 & 1 \end{pmatrix} \Longrightarrow A = \begin{pmatrix} 2 & 2 & 3 & 3 & 1 & 0 \\ 3 & 5 & 2 & 2 & 0 & 1 \end{pmatrix}$$

So we need to re-calculate $B^{-1}A$ and replace the corresponding part of the simplex tableau.

$$B^{-1}A = \begin{pmatrix} -\frac{2}{5} & \frac{3}{5} \\ \frac{3}{5} & -\frac{2}{5} \end{pmatrix} \begin{pmatrix} 2 & 2 & 3 & 3 & 1 & 0 \\ 3 & 5 & 2 & 2 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 11/5 & 0 & 0 & -2/5 & 3/5 \\ 0 & -4/5 & 1 & 1 & 3/5 & -2/5 \end{pmatrix}$$

		3	1	3	4	0	0	
		x_1	x_2	x_3	x_4	s_1	s_2	
3	x_1	1	11/5	0	0	-2/5	3/5	1
3	x_3	0	-4/5	1	1	3/5	-2/5	6
	Z_j	3	21/5	3		3/5	3/5	21
	W_j	0	-16/5	0		-3/5	-3/5	

In fact, since only the column of x_4 changes, it would be sufficient to calculate

$$B^{-1}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and replace this column in the simplex tableau.

Now we calculate the last two rows of the x_4 column:

		3	1	3	4	0	0	
		x_1	x_2	x_3	x_4	s_1	s_2	
3	x_1	1	11/5	0	0	-2/5	3/5	1
3	x_3	0	-4/5	1	1	3/5	-2/5	6
	Z_j	3	21/5	3	3	3/5	3/5	21
	W_j	0	-16/5	0	1	-3/5	-3/5	

This tableau is no longer optimal because the reduced cost of x_4 is positive. We must therefore do one more simplex iteration to obtain the optimal solution. Variable x_4 enters the basis and x_3 leaves. Since the column of x_4 already has the value of 1 in the position of the pivot and the value of 0 in the other row, in the tableau we only need to replace the basic variables and calculate the last two rows accordingly:

		3	1	3	4	0	0	
		x_1	x_2	x_3	x_4	s_1	s_2	
3	x_1	1	11/5	0	0	-2/5	3/5	1
4	x_4	0	-4/5	1	1	3/5	-2/5	6
	Z_j	3	17/5	4	4	6/5	1/5	27
	W_j	0	-12/5	-1	0	-6/5	-1/5	

This is the optimal tableau and the new optimal solution is (1,0,0,6,0,0). The optimal value of the objective function is 27.

b) Here we need to do the same but use a_{14} instead of 3:

$$B^{-1}\begin{pmatrix} a_{14} \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{2}{5} & \frac{3}{5} \\ \frac{3}{5} & -\frac{2}{5} \end{pmatrix}\begin{pmatrix} a_{14} \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{2}{5}a_{14} + \frac{6}{5} \\ \frac{3}{5}a_{14} - \frac{4}{5} \end{pmatrix}$$

		3	1	3	4	0	0	
		x_1	x_2	x_3	x_4	s_1	s_2	
3	x_1	1	11/5	0	$-\frac{2}{5}a_{14}+\frac{6}{5}$	-2/5	3/5	1
3	x_3	0	-4/5	1	$\frac{3}{5}a_{14}-\frac{4}{5}$	3/5	-2/5	6
	Z_j	3	21/5	3	$\frac{3}{5}a_{14}+\frac{6}{5}$	3/5	3/5	21
	W_j	0	-16/5	0	$-\frac{3}{5}a_{14}+\frac{14}{5}$	-3/5	-3/5	

For the tableau to be optimal, we need:

$$-\frac{3}{5}a_{14} + \frac{14}{5} \leq 0$$

And then:

$$a_{14} \geq \frac{14}{3}$$

The sensitivity interval of a_{14} is $[\frac{14}{3}, +\infty[$. If the coefficient of x_4 in the first constraint takes any value greater than or equal to 14/3, the optimal solution will be the same.

Adding a new variable: post-optimization

If we add a new variable to the problem (in a practical case we could consider, for example, incorporating the manufacture of a new product), we need to add one more column to the simplex tableau and fill in the values. This can affect the optimality of the tableau but not the feasibility.

Suppose, for example, that we want to add a new variable z to the problem and that this variable has coefficient $c_z = 1$ in the objective function and technical coefficients $A_z = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ in the constraints (this means that it has a coefficient of 1 in the first constraint and a coefficient of -1 in the second one. The new tableau will have one new column:

		1	2	1	0	0	
		x	y	z	s_1	s_2	
2	y	1	1		1	0	4
0	s_2	1	0		-1	1	2
		2	2		2	0	8
		-1	0		-2	0	

To complete this column, we multiply $B^{-1}A_z$ and calculate the last two rows as usual:

$$B^{-1}A_z = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

		1	2	1	0	0	
		x	y	z	s_1	s_2	
2	y	1	1	1	1	0	4
0	s_2	1	0	-2	-1	1	2
		2	2	2	2	0	8
		-1	0	-1	-2	0	

Since the reduced cost of the new variable is negative, the tableau is still optimal and the optimal solution is (0,4,0,0,2) (the new variable z has the value of 0 because it has not entered the basis and is therefore non-basic).

If the reduced cost of the new variable were positive, it would have to enter the basis and we would need to do one more simplex iteration.

Example:

- For the following LP problem:

$$\begin{aligned} Max.\ \ & x_1 + 4x_2 + x_3 \\ s.t.\ \ & x_2 + 2x_3 \geq 4 \\ & x_1 + 2x_2 + 4x_3 = 12 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \end{aligned}$$

a) Obtain the optimal simplex tableau without doing any iteration but knowing that the optimal solution is (x1, x2, x3)=(0,6,0).

b) If a new variable x4 with coefficient c4=3 in the objective function and technical coefficients $A_4 = \binom{2}{1}$ is added, explain whether the optimal solution in a) is still optimal and, if it is not, obtain the new optimal solution.

a) Standard form:

$$\begin{aligned} Max.\ \ & x_1 + 4x_2 + x_3 \\ s.t.\ \ & x_2 + 2x_3 - s_1 = 4 \\ & x_1 + 2x_2 + 4x_3 = 12 \\ & x_1, x_2, x_3, s_1 \geq 0 \end{aligned}$$

Complete solution (0,6,0,2). Basic variables: x_2, s_1.

Technical matrix:

$$A = \begin{pmatrix} 0 & 1 & 2 & -1 \\ 1 & 2 & 4 & 0 \end{pmatrix}$$

Basic matrix B and B^{-1}:

$$B = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix} \quad B^{-1} = \begin{pmatrix} 0 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{pmatrix}$$

$$B^{-1}\bar{b} = \begin{pmatrix} 0 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 4 \\ 12 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

$$B^{-1}A = \begin{pmatrix} 0 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & -1 \\ 1 & 2 & 4 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 1 & 2 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \end{pmatrix}$$

Simplex tableau:

		1	*4*	*1*	*0*	
		x_1	x_2	x_3	s_1	
4	x_2	*1/2*	*1*	*2*	*0*	*6*
0	s_1	*1/2*	*0*	*0*	*1*	*2*
		2	*4*	*8*	*0*	*24*
		-1	*0*	*-7*	*0*	

b) We need to make some space for variable x_4 in the simplex tableau:

		1	*4*	*1*		*0*	
		x_1	x_2	x_3	x_4	s_1	
4	x_2	*1/2*	*1*	*2*		*0*	*6*
0	s_1	*1/2*	*0*	*0*		*1*	*2*
		2	*4*	*8*		*0*	*24*
		-1	*0*	*-7*		*0*	

Now we multiply $B^{-1}A_4$ and insert the result into the tableau:

$$B^{-1}A_4 = \begin{pmatrix} 0 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix}$$

		1	*4*	*1*	*3*	*0*	
		x_1	x_2	x_3	x_4	s_1	
4	x_2	*1/2*	*1*	*2*	*1/2*	*0*	*6*
0	s_1	*1/2*	*0*	*0*	*-3/2*	*1*	*2*
		2	*4*	*8*	*2*	*0*	*24*
		-1	*0*	*-7*	*1*	*0*	

The tableau is no longer optimal: variable x_4 must enter the basis and variable x_2 must leave. By doing one simplex iteration, we obtain the following optimal tableau:

		1	4	1	3	0	
		x_1	x_2	x_3	x_4	s_1	
3	x_4	1	2	4	1	0	12
0	s_1	2	3	6	-3/2	1	20
		3	6	12	3	0	36
		-2	-2	-11	0	0	

The new optimal solution is (0,0,0,12,0), and the optimal value of the objective function is 36.

Adding a new constraint: post-optimization

We can also consider incorporating a new constraint to the problem. In this case, all we have to do is check whether the current optimal solution satisfies the new constraint. If it does, the solution is still optimal and nothing changes. If it does not, all we can do is begin the problem again from the beginning.

For example, suppose that we add the constraint $x + y \leq 5$ to the problem. We then check it with the optimal solution, which is (0,4,0,2), and we have $x + y = 0 + 4 = 4 \leq 5$. The constraint is therefore satisfied and the solution is still optimal.

Example:

- We have the following linear programming problem:

$$Min.\ 8x + 3y + 2z$$

$$s.t.\ \ y + 2z \geq 6$$

$$2x + y + 4z \geq 8$$

$$x, y, z \geq 0$$

We also have the following optimal tableau corresponding to the equivalent maximization problem:

		-8	-3	-2	0	0	
		x	y	z	s	t	
-2	z	0	1/2	1	-1/2	0	3
0	t	-2	1	0	-2	1	4
	Z_j	0	-1	-2	1	0	-6
	W_j	-8	-2	0	-1	0	

Calculate the optimal solution if we add the constraint $x + y + z \geq 2$. Which variables are basic?

The optimal solution of the problem is (0,0,3,0,4) and the value of the objective function is -6. If we add the constraint $x + y + z \geq 2$, we must check whether it is satisfied by the current optimal solution:

$$x + y + z = 0 + 0 + 3 = 3 \geq 2$$

The constraint is satisfied and the optimal solution is the same.

Unit 6. Integer Linear Programming

In some contexts, using fractional values for the variables is not acceptable. For example, if the variable represents the number of cars produced or the number of people assigned to a certain task, the variable should take only integer values (0, 1, 2, etc.). In other cases we may want to use integer variables for other purposes. For example, binary variables (variables that can take only values of 0 or 1) can be used to represent decisions such as whether to construct a facility or to buy a machine, etc.

The methods we have seen for solving LP problems do not ensure that the variables in the optimal solution will take integer variables. We therefore need to find other ways of solving problems with such conditions.

Types of problems with integer variables

When all variables in the problem must take integer values, we have an Integer Programming (IP) problem. When, in addition, all functions of the model (the objective function and the constraints) are linear, we have what is called an Integer Linear Programming (ILP) problem.

Sometimes we have integer variables (those that can have only integer values) and continuous variables (those that can also take fractional values). These problems are called Mixed Integer Programming (MIP) problems or Mixed Integer Linear Programming (MILP) problems.

Suppose we have the following LP problem:

$$\begin{array}{ll} Max & 8x + 10y \\ s.t.: & 4x + 6y \leq 24 \\ & 8x + 3y \leq 24 \\ & x, y \geq 0 \end{array}$$

Suppose also that we want to impose the condition that both variables must be integers. We can just write:

$$\begin{array}{ll} Max & 8x + 10y \\ s.t.: & 4x + 6y \leq 24 \\ & 8x + 3y \leq 24 \\ & x, y \geq 0 \quad integer \end{array}$$

or

$$\begin{array}{ll} Max & 8x + 10y \\ s.t.: & 4x + 6y \leq 24 \\ & 8x + 3y \leq 24 \\ & x, y \in \mathbb{Z}^+ \end{array}$$

Both $x, y \geq 0 \quad integer$ and $x, y \in \mathbb{Z}^+$ mean that the variables must be integers and ≥ 0 (we will not consider problems with integer variables that are not ≥ 0).

Linear relaxation

If we take an ILP problem and remove the condition that the variables must be integers, the resulting problem is called the linear relaxation (LR) of the ILP problem.

We can try to solve the LR using simplex. If we are lucky, the variables will take integer values in the optimal solution and we will have the optimal solution of the ILP problem.

Consider, for example, the following ILP problem:

$$\begin{aligned} &Max \quad y \\ &s.t.: \quad x + y \leq 2 \\ &\qquad\quad x, y \geq 0 \; integer \end{aligned}$$

The LR of this problem is:

$$\begin{aligned} &Max \quad y \\ &s.t.: \quad x + y \leq 2 \\ &\qquad\quad x, y \geq 0 \end{aligned}$$

The optimal solution of this LR is $x = 0, y = 2$ (the solution can be obtained with simplex or by solving it graphically since it has only two variables). Solutions in which all variables take integer values are called integer solutions. Since the optimal solution of the LR is integer, it is also the optimal solution of the ILP problem.

However, this will not occur very often. Consider this other ILP problem:

$$\begin{aligned} &Max \quad 3x + 12y \\ &s.t.: \quad 3x + 16y \leq 68 \\ &\qquad\quad 2x + 2y \leq 15 \\ &\qquad\quad x, y \geq 0 \; integer \end{aligned}$$

If we solve the LR, the optimal solution is $x = 4, y = 3.5$. In this case, since variable y is not an integer, this solution is not feasible for the ILP problem.

We could try rounding the fractional value of y to obtain an integer solution. If we round the value of y to the nearest higher integer value (i.e., 4), we obtain the integer solution $x = 4, y = 4$. However, this solution is not feasible for the ILP problem because it does not satisfy the first constraint:

$$3x + 16y = 12 + 64 = 76 \nleq 68$$

If we round the value of y to the nearest lower integer value (i.e., 3), we get $x = 4, y = 3$. This integer solution is feasible for the ILP but it is not the optimal one. The value of the objective function for this solution is

$$3x + 12y = 12 + 36 = 48$$

while the optimal solution of the ILP problem is $x = 1, y = 4$, which has a value of

$$3x + 12y = 3 + 48 = 51$$

As we can see, rounding the fractional values of the optimal solution of the LR is not a good idea since we can get solutions that are not optimal or are even infeasible.

The branch-and-bound method

The branch-and-bound method is one of many algorithms for solving ILP problems. The basic idea is to iteratively split the solution set of the LR into smaller subsets and solve an LP problem in each subset until we find the optimal solution of the ILP.

Consider the following ILP problem:

$$\begin{array}{ll} Max & 8x+10y \\ s.t.: & 4x+6y \leq 24 \\ & 8x+3y \leq 24 \\ & x,y \geq 0 \quad integer \end{array}$$

The first step in the branch-and-bound method is to solve the LR of the problem:

$$\begin{array}{ll} Max & 8x+10y \\ s.t.: & 4x+6y \leq 24 \\ & 8x+3y \leq 24 \\ & x,y \geq 0 \end{array}$$

We will call this LP problem LP-0. To solve LP-0, we can use the methods for LP problems (for example, simplex). The optimal solution of this LP-0 is $x=2, y=2.66, O.F.=42.66$ (O.F. is the value of the objective function in the optimal solution). Unfortunately, since this solution is not integer, it is not the optimal solution of the ILP. In this solution, variable y is not an integer. We know that no feasible solution of the ILP problem can have $y=2.66$, so any solution of the ILP will satisfy either $y \leq 2$ or $y \geq 3$. We will now consider these two new LP problems:

LP-1
$$\begin{array}{ll} Max & 8x+10y \\ s.t.: & 4x+6y \leq 24 \\ & 8x+3y \leq 24 \\ & y \leq 2 \\ & x,y \geq 0 \end{array}$$

LP-2
$$\begin{array}{ll} Max & 8x+10y \\ s.t.: & 4x+6y \leq 24 \\ & 8x+3y \leq 24 \\ & y \geq 3 \\ & x,y \geq 0 \end{array}$$

LP-1 is the same problem as LP-0 but with a new constraint $(y \leq 2)$, while LP-2 is LP-0 plus constraint $y \geq 3$. We know that all feasible solutions of the original ILP will be in either LP-1 or LP-2. The only solutions that have been left out are those in which $2 < y < 3$, but these are not feasible solutions of the ILP because y has a fractional value. Specifically, the solution of LP-0 is neither in LP-1 nor LP-2. We have therefore removed the optimal solution of LP-0, but we know that the optimal solution of the ILP must be in one of these two new problems.

Splitting a problem into two new ones by adding a constraint to each is called branching. Now we have to solve LP-1 and LP-2. If the optimal solutions are not integers, we will need to branch again, thus creating two new problems from each one.

If we solve LP-1, we obtain the optimal solution $x=2.25, y=2, O.F.=38$. The optimal solution of LP-2 is $x=1.5, y=3, O.F.=42$.

Since neither of these solutions is integer, we need to split both problems into two new ones. Let us begin, for example, with LP-2. In this case, the fractional variable is $x=1.5$. To remove this solution from the solution set, therefore, we need to add two new constraints: $x \leq 1$ and $x \geq 2$. We will thus create two new problems (LP-3 and LP-4) by adding these new inequalities to LP-2:

LP-3

$$
\begin{aligned}
Max \quad & 8x + 10y \\
s.t.: \quad & 4x + 6y \leq 24 \\
& 8x + 3y \leq 24 \\
& y \geq 3 \\
& x \leq 1 \\
& x, y \geq 0
\end{aligned}
$$

LP-4

$$
\begin{aligned}
Max \quad & 8x + 10y \\
s.t.: \quad & 4x + 6y \leq 24 \\
& 8x + 3y \leq 24 \\
& y \geq 3 \\
& x \geq 2 \\
& x, y \geq 0
\end{aligned}
$$

Now we solve LP-3 and LP-4. The optimal solution of LP-3 is $x = 1, y = 3.33, O.F. = 41.33$, while LP-4 is infeasible and so has no solution.

If we look at the value of the objective function, we will see that whenever we have added new constraints, this value has decreased: the solution of LP-0 had a value of 42.66, while the solutions of LP-1 and LP-2 had values of 38 and 42, respectively. The solution of LP-2 is 42, while the value of LP-3 is 41.33. This occurs because when we add new inequalities, we have fewer solutions, so the optimal solution is always worse. If this were a minimization problem, the value of the objective function would increase instead of decrease.

At this point we have still not found any integer solution. We need to choose an LP problem and branch by adding two new inequalities. We could choose LP-1 or LP-3 (LP-2 has already been studied and LP-4 is infeasible). Let us choose LP-3, for example, whose optimal solution was $x = 1, y = 3.33, O.F. = 41.33$. We therefore need to add the inequalities $y \leq 3$ and $y \geq 4$ to remove the solutions with $y = 3.33$. With these inequalities, we generate two new problems (LP-5 and LP-6):

LP-5

$$
\begin{aligned}
Max \quad & 8x + 10y \\
s.t.: \quad & 4x + 6y \leq 24 \\
& 8x + 3y \leq 24 \\
& y \geq 3 \\
& x \leq 1 \\
& y \leq 3 \\
& x, y \geq 0
\end{aligned}
$$

LP-6

$$
\begin{aligned}
Max \quad & 8x + 10y \\
s.t.: \quad & 4x + 6y \leq 24 \\
& 8x + 3y \leq 24 \\
& y \geq 3 \\
& x \geq 2 \\
& y \geq 4 \\
& x, y \geq 0
\end{aligned}
$$

When we solve these new problems, we obtain the optimal solutions $x = 1, y = 3, O.F. = 38$ in LP-5 and $x = 0, y = 4, O.F. = 40$ in LP-6. At last we have found two integer solutions. Since we are maximizing, the solution of LP-6 is better than the solution of LP-5, so we can discard the latter. Now our candidate for the optimal solution of the ILP is the solution of LP-6, $x = 0, y = 4, O.F. = 40$. To be sure that this is the optimal solution, however, we need to study all the problems completely. Problems LP-5 and LP-6 can no longer be studied because they already have integer optimal solutions. We still need to consider problem LP-1, whose solution was $x = 2.25, y = 2, O.F. = 38$. We could add two more inequalities from here but the value of the objective function (38) is already worse than the value of the integer solution we found in LP-6, which was 40. Since this value is not going to improve when we add new inequalities, we can be sure that we will not find any solution that is better than the one we already have. This means that $x = 0, y = 4, O.F. = 40$ is the optimal solution of the ILP.

Branch-and-bound step-by-step

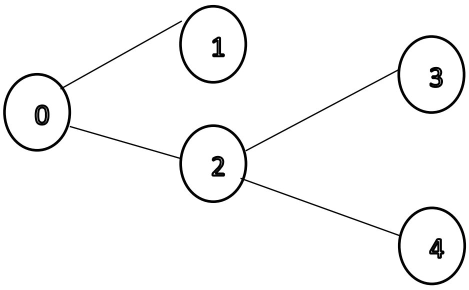
Follow these steps to apply the branch-and-bound method:

- Start by solving the LR of the ILP problem. If the optimal solution of the LR is an integer, it is also the optimal solution of the ILP. Otherwise, choose a variable with a fractional value.
- Let x be this variable and x^* its fractional value. Then, $\lfloor x^* \rfloor$ is the nearest integer value lower than x^* and $\lceil x^* \rceil$ is the nearest integer value higher than x^*.
- Create two new problems by adding the inequalities $x \leq \lfloor x^* \rfloor$ and $x \geq \lceil x^* \rceil$, respectively.
- Choose a problem we have not yet solved and solve it. Whenever we solve one of the problems, one of these four things may happen:
    - the problem is infeasible: we do not need to study this problem further.
    - the optimal solution of the problem is an integer: if this solution is better than the best integer solution we have found so far (or if it is the first integer solution we have found), we keep it and we don't need to study this problem any further.
    - the optimal solution is not an integer but the value of the objective function is already equal to or worse than the value of an integer solution we have found in another problem: in this case, we don't need to study this problem any further.
    - the optimal solution is not an integer and the value of the objective function is better than the value of all the integer solutions we have found so far: we must then create two new problems as we saw in steps 2 and 3.

When there are no more problems to study, the best integer solution found will be the optimal solution of the ILP.

Branch-and-bound tree

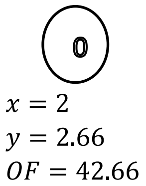
Solving an ILP problem with the branch-and-bound method requires solving several LP problems and can be a long process that generates a lot of information. To keep track of all the data, we can represent all this information in a graphical structure called a branch-and-bound tree:



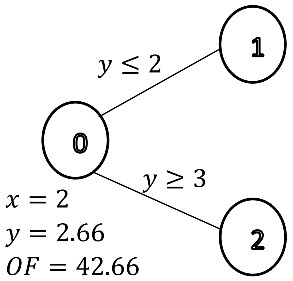
The circles in the branch-and-bound tree are called nodes. Each node represents an LP problem. Next to each node we write the value of the variables and the objective function of the optimal solution of the corresponding problem. Two lines, called branches, emerge from each node. These represent the inequalities we need to add to create two new problems. We write these inequalities next to the branches.

Let's see how the branch-and-bound tree of the problem solved above would be obtained.

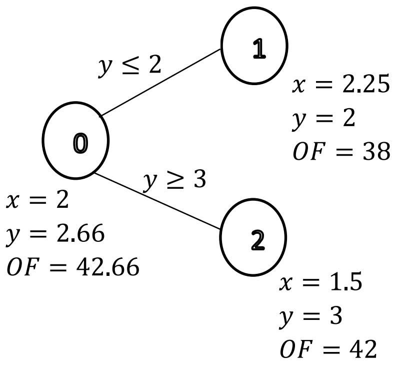
First we solved the LR relaxation, which we called LP-0. The optimal solution was $x = 2, y = 2.66, O.F. = 42.66$. We draw a node with number 0 and write this optimal solution next to it:



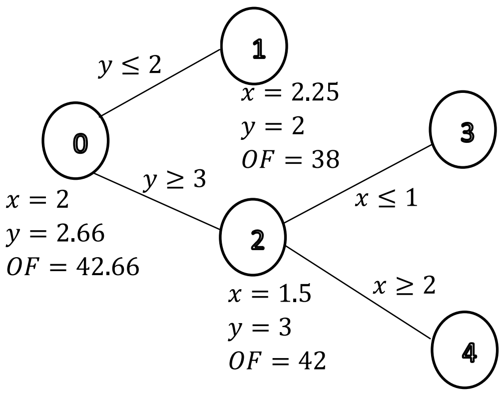
Then we added inequalities $y \leq 2$ and $y \geq 3$ to generate problems LP-1 and LP-2, so we draw two new branches from node 0 leading to two new nodes 1 and 2, which represent these new problems:



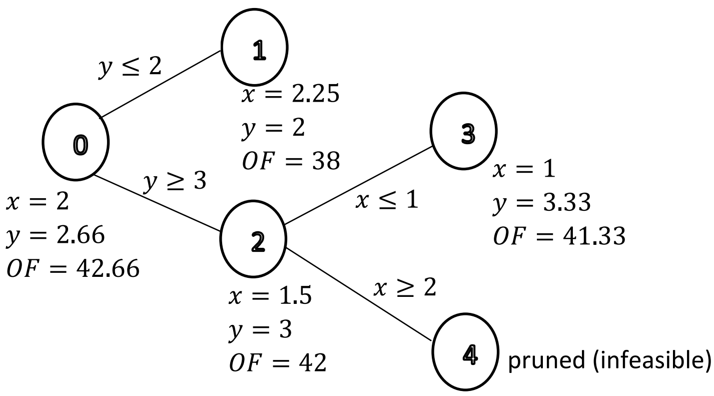
In the next step we solved problems LP-1 and LP-2 to obtain the solutions $x = 2.25, y = 2, O.F. = 38$ and $x = 1.5, y = 3, O.F. = 42$, respectively:



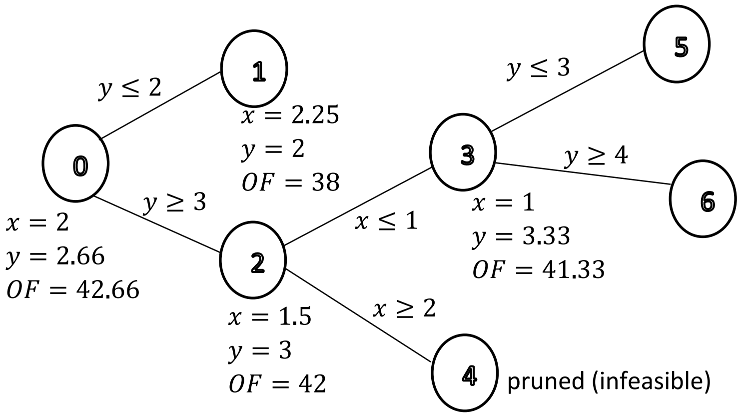
We chose to continue by studying problem LP-2 (we could have chosen LP-1 since all options are acceptable), so we created two new problems using inequalities $x \leq 1$ and $x \geq 2$:



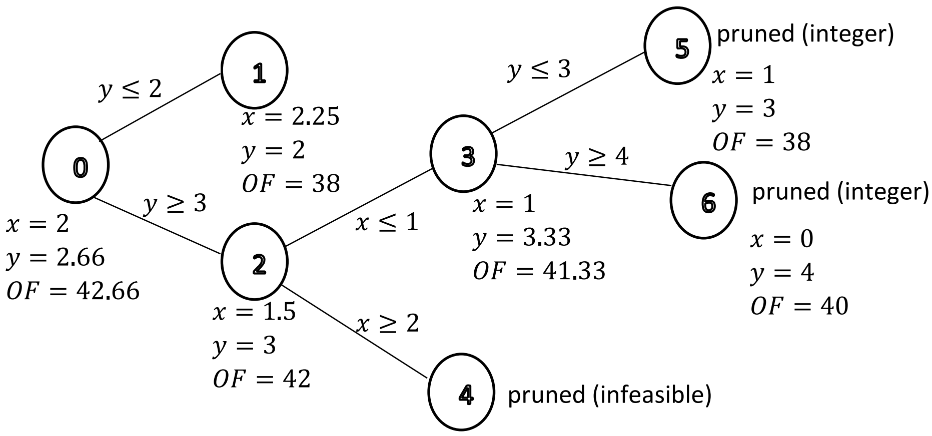
Problem LP-4 was infeasible, so we did not need to study it any further. When we stop studying a problem, we say that its node is pruned. The optimal solution of LP-3 was $x = 1, y = 3.33, O.F. = 41.33$:



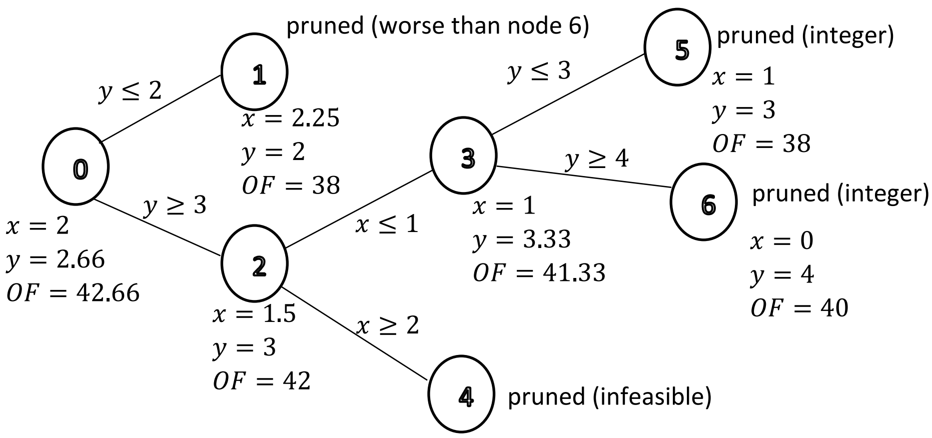
We continued by studying problem LP-3 (though we could have chosen LP-1 instead) by adding two new branches (inequalities), $y \leq 3$ and $y \geq 4$:



The optimal solutions of problems LP-5 and LP-6 were $x = 1, y = 3, O.F. = 38$ and $x = 0, y = 4, O.F. = 40$, respectively. Since these solutions are integers, we consider nodes 5 and 6 to be pruned and they need not be studied any further:



We still need to study LP-1. However, the value of the objective function is worse than the value of the integer solution we found in node 6, so we consider node 1 to also be pruned:



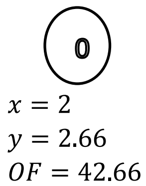
Now all nodes have been studied or pruned, so we have completed the branch-and-bound method. The optimal solution of the ILP is the best integer solution found in the tree, which is the one that corresponds to node 6.

Solving MILP problems with branch-and-bound

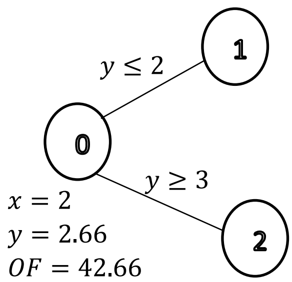
The branch-and-bound method can also be used to solve MILP problems, i.e., those in which some variables are integers and some are continuous. Suppose we have the same problem as before but now only variable y is an integer:

$$
\begin{aligned}
Max \quad & 8x + 10y \\
s.t.: \quad & 4x + 6y \leq 24 \\
& 8x + 3y \leq 24 \\
& x \geq 0 \\
& y \geq 0 \quad integer
\end{aligned}
$$

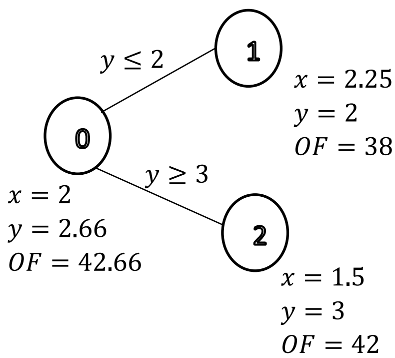
As in the previous case we begin by solving the LR of the problem, which is exactly the same as before and so has the same optimal solution $x = 2, y = 2.66, O.F. = 42.66$:



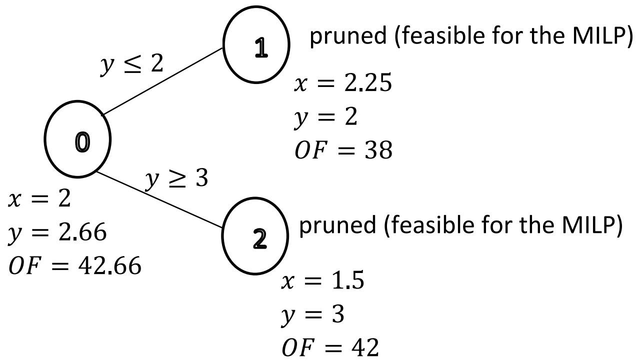
In this solution, variable y takes a fractional value but in the MILP it should be an integer. We therefore need to branch using the constraints $y \leq 2$ and $y \geq 3$ (exactly as before):



Now we solve problems LP-1 and LP-2 to obtain the solutions $x = 2.25, y = 2, O.F. = 38$ and $x = 1.5, y = 3, O.F. = 42$, respectively (since they are the same problems as in the previous example, we have the same solutions):



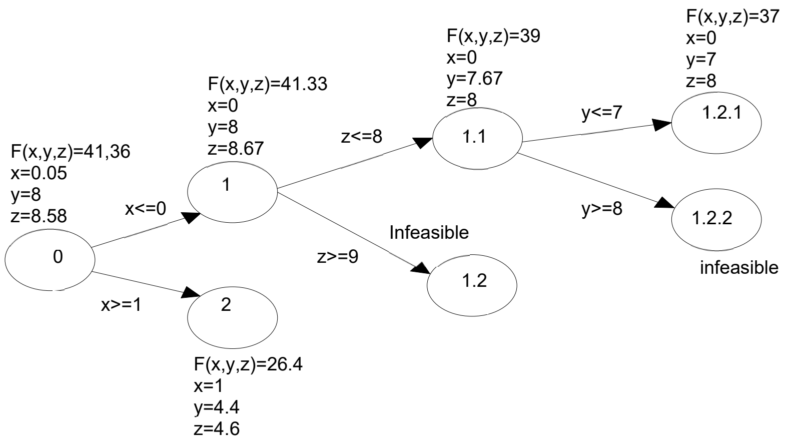
Now variable x is fractional in node 1. Remember, however, that this variable does not need to be an integer, so we cannot branch using this variable. This solution satisfies the condition that y must be an integer, so it is a feasible solution of the MILP. We can therefore prune this node. The same rationale can be applied to node 2:



All nodes have now been studied or pruned. We have therefore finished and the optimal solution is the best solution that is feasible for the MILP, which is the solution in node 2, i.e. $x = 1.5, y = 3, O.F. = 42$.

Examples:

- Given a certain integer liner programming problem with the following branch-and-bound tree:



a) Explain whether the objective of the problem is to maximize or minimize.
b) Have we reached the optimal solution? If we have, explain why. Otherwise, say from which node we should branch and what the constraints should be.

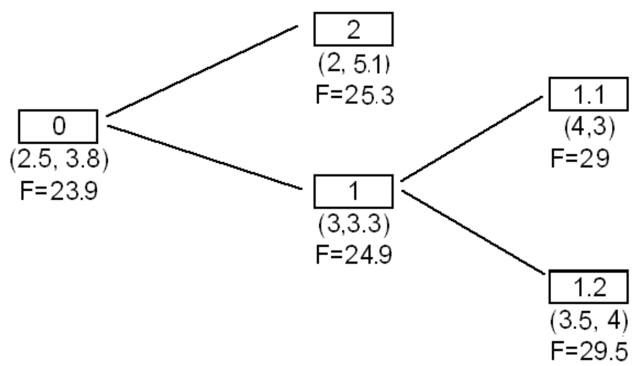
a) To determine whether this is a minimization or a maximization problem, we just need to study what happens to the value of the objective function F(x,y,z) when we move from one node to the next. If this value increases, it must be a minimization problem (remember that the value of the objective function always gets worse when we proceed further along the tree). If the value decreases, it must be a maximization problem.

In this case we can look, for example, at nodes 0 and 2 (though any pair of consecutive nodes will do). The value of F in node 0 is 41.36, while the value of F in node 2 is 26.4, which is lower. This means that this is a maximization problem.

b) To know if we have reached the optimal solution, we need to check whether all nodes that have not been studied (i.e., those that have not been branched) are pruned. In this case, they are nodes 1.2.1, 1.2.2, 1.2, and 2. Remember that there may be three reasons for pruning a node, i.e., if it is infeasible, if it has an integer solution, or if the value of the objective function is equal to or worse than the value of an integer solution we have found.
   - In node 1.2.1 we have an integer solution, so this node is pruned.
   - Node 1.2.2 is infeasible, so it is also pruned.
   - Node 1.2 is infeasible, so it is pruned.
   - Node 2 has a fractional solution but its value (26.4) is worse than that of the integer solution we have in node 1.2.1 (37), so this node is also pruned.

   Since all nodes are pruned, we have found the optimal solution, which will be the best integer solution found in the tree; in this case it is the solution of node 1.2.1, $x = 0, y = 7, z = 8, O.F. = 37$.

- Consider an ILP problem with two variables, x and y. Suppose we have the following associated branch-and-bound tree:



a) Explain whether this is a maximization or a minimization problem.

b) Write the constraints that have been added at each branch.

c) Explain whether the optimal solution has already been found. If it has not, branch from the appropriate node.

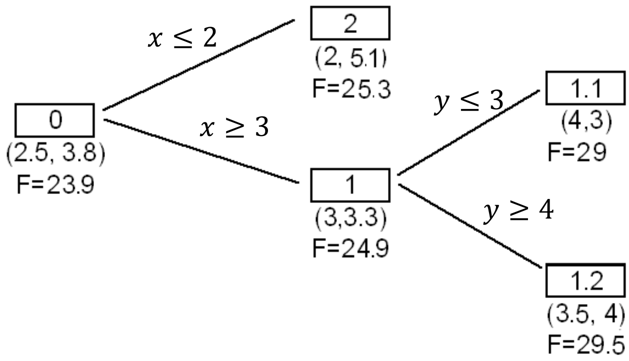
d) If the original problem was a mixed ILP where only variable x had to be integer, what would the answer to question (c) be?

a) The value of the objective function from node 0 to node 1 increases from 23.9 to 25.3, so it is a minimization problem

b) Branches from node 0:

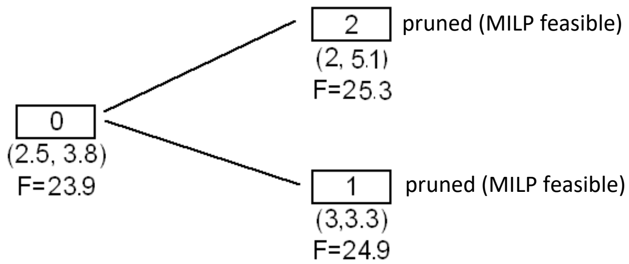
   Since both variables have a fractional value at node 0, either could have been used to branch. If we had chosen variable y, the added constraints would have been $y \leq 3$ and $y \geq 4$. However, the solution of node 1 does not satisfy either of these inequalities, so y cannot be the chosen variable. This must be variable x, with inequalities $x \leq 2$ and $x \geq 3$. The inequality in the upper branch must be $x \leq 2$ because, in the solution of node 2, $x = 2$, and the inequality in the lower branch must be $x \geq 3$.

The branching variable in node 1 can only be y, since it is the only variable with a fractional value. The added inequalities will be $y \leq 3$ and $y \geq 4$. According to the solutions obtained in nodes 1.1 and 1.2, $y \leq 3$ must go in the upper branch and $y \geq 4$ must go in the lower branch. So we finally have:



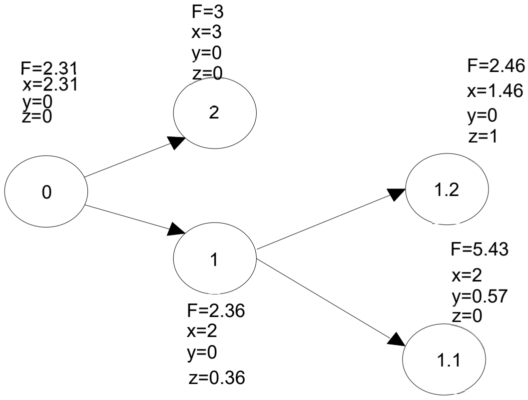
c) Node 1.1 is pruned because it has an integer solution. Node 1.2 is also pruned because its value (29.5) is worse than the value of the integer solution in node 1.1 (29). However, node 2 has a fractional solution whose value (25.3) is better than the only integer solution found (29), so this node is not pruned. This means that we must branch node 2 using the inequalities $y \leq 5$ and $y \geq 6$.

d) If only variable x must be an integer, the solutions in nodes 1 and 2 are feasible (so node 1 would not have been branched and nodes 1.1 and 1.2 would not exist). In this case nodes 1 and 2 are pruned and both have feasible solutions The optimal solution will be the best of the two, which in this case is the solution of node 1, i.e., $x = 3, y = 3.3, O.F. = 24.9$:



- Solving the following integer linear programming problem we obtain the following branch-and-bound tree:

$$\begin{aligned} ? \quad & x + 6y + z \\ s.t. \quad & 13x + 7y + 11z \geq 30 \\ & x, y, z \in \mathbb{Z}^{+} \end{aligned}$$

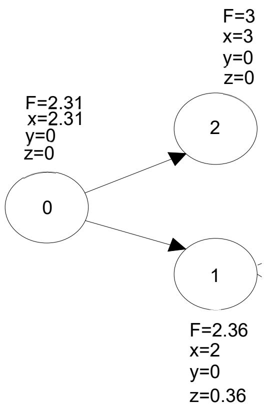


a) Explain whether this is a maximization or a minimization problem and write the constraint associated with each branch.

b) Have we reached the optimal solution? If we have, explain why and indicate which one it is. If we haven't, write the next branches and their associated constraints.

c) Consider the same branch-and-bound tree for the mixed integer problem with x, y $\in Z^{+}$, z ≥ 0. Have we reached the optimal solution? If we have, explain why and indicate which one it is. If we haven't, write the next branches and their associated constraints.

a) Since the value of the objective function increases, this is a minimization problem.
b) Node 2 is pruned because it has an integer solution.
Node 1.1 is pruned because its objective function has a value which is worse than the integer solution of node 2.
Since node 1.2 is a fractional solution and the value of the objective function is better than the only integer solution found, we need to branch this node with inequalities $x \leq 1$ and $x \geq 2$
c) If z did not need to be integer, node 1 would have been pruned and the tree would have been:



Both nodes 1 and 2 would have been pruned because both solutions are feasible for the new problem. The optimal solution would be the best of these two, which is the one in node 1, i.e., $x = 2, y = 0, z = 0.36, O.F. = 2.36$.