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Research paper

Partial regularity for manifold constrained quasilinear elliptic systems

Esther Cabezas-Rivas [a], Salvador Moll [b,*], Vicent Pallardó-Julià [b,c]

[a] Departament de Matemàtiques, Universitat de València, Dr. Moliner, 50, Burjassot, 46100, Spain
[b] Departament d'Anàlisi Matemàtica, Universitat de València, Dr. Moliner, 50, Burjassot, 46100, Spain
[c] Kimera Technologies, Lanzadera, C/ del Moll de la Duana, s/n, València, Spain

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ABSTRACT

We consider manifold constrained weak solutions of quasilinear uniformly elliptic systems of divergence type with a source term that grows at most quadratically with respect to the gradient of the solution. As we impose that the solution lies on a Riemannian manifold, the classical smallness condition for regularity can be relaxed to an inequality relating strict convexity of the squared distance and growth of the leading order term in the tangent component of the source. As a key tool for the proof of a partial regularity result, we derive a fully intrinsic Caccioppoli inequality which may be of independent interest. Finally we show how the systems under consideration have a variational nature and arise in the context of F- or V-harmonic maps.

1. Introduction and statement of main results

Let $(\mathcal{N}, g)$ be a complete (without boundary, non necessarily compact) connected smooth n-dimensional Riemannian manifold, which by Nash theorem [22] can be regarded as isometrically embedded $\mathcal{N} \hookrightarrow \mathbb{R}^N$ in some Euclidean space. Given $\Omega \subseteq \mathbb{R}^m$ $(m \geq 2)$ an open bounded Lipschitz domain, we study regularity of weak solutions $u : \Omega \to \mathcal{N}$ to the following system of PDEs:

$$-\operatorname{div}\left(b(x,u,\nabla u)\nabla u\right) = f(x,u,\nabla u) \qquad \text{in } \Omega. \tag{1}$$

Setting $B(x,v,\mathbf{z}) := b(x,v,\mathbf{z})\mathbf{z}$, we have two measurable vector fields B, f defined on $\Omega \times \mathbb{R}^N \times \mathbb{R}^{mN}$ and taking values in $\mathbb{R}^{mN}$ and $\mathbb{R}^N$, respectively, while b is a scalar function on the same domain. The interest in these systems of diagonal type comes from their ubiquitous appearance in differential geometry and physics: isothermal coordinates, Plateau and capillarity problems, minimal surfaces and harmonic maps on Riemannian manifolds, or elliptic Monge–Ampère equations (see [14] for details).

It is well-known, since De Giorgi's counterexample [6] in the 60s, that there is no hope to get regularity of weak solutions in general. Accordingly, the goal is to focus on partial regularity, including estimates on the size of the singular set. The main point of the present note is to prove that a theorem in this spirit still holds in a suitable manifold constrained setting, which in fact allows us to relax the so-called classical smallness assumption. Furthermore, as a key tool of independent interest, we obtain an intrinsic Caccioppoli type inequality (see Proposition 1).

In the Euclidean framework (cf. [7,11,17]), the following structural and regularity conditions are often assumed:

* Corresponding author.
E-mail addresses: Esther.Cabezas-Rivas@uv.es (E. Cabezas-Rivas), j.salvador.moll@uv.es (S. Moll), vicent.pallardo@uv.es (V. Pallardó-Julià).

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H1. $0 < \ell \le b(x, v, \mathbf{z}) \le L$.

H2. B is differentiable with respect to $\mathbf{z}$ with continuous derivatives: $\lambda|\xi|^2 \le D_{\mathbf{z}} B(x, v, \mathbf{z})\xi \otimes \xi \le \Lambda|\xi|^2$.

H3. $\left|B(x+y, v+w, \mathbf{z}) - B(x, v, \mathbf{z})\right| \le L(|y| + |w|)^\alpha(1 + |\mathbf{z}|)$ for some $\alpha \in (0, 1)$.

H4. $|f(x, v, \mathbf{z})| \le C_1|\mathbf{z}|^2 + D_1$. We further require that $f(x, u, \nabla u)$ is measurable for all $u \in H^1(\Omega; \mathbb{R}^N)$.

The above conditions hold for some $\ell, L, \lambda, \Lambda, C_1, D_1 > 0$ and all $x, y \in \Omega$, $v, w \in \mathbb{R}^N$, $\mathbf{z}, \xi \in \mathbb{R}^{mN}$. Then it is well-known that any weak solution $u \in H^1(\Omega; \mathbb{R}^N) \cap L^\infty(\Omega; \mathbb{R}^N)$, under the smallness condition

$$|u|_{L^\infty} < \lambda/2C_1, \tag{2}$$

is partially regular in the sense that ∇u is Hölder continuous on an open subset $\Omega_0 \subset \Omega$ of full measure. We stress that condition (2) is crucial to get a Caccioppoli inequality, which usually is a starting point for the regularity study.

In our manifold constrained setting, f is decomposed into two terms; the tangent and normal components to $\mathcal{N}$:

$$f(x, v, \mathbf{z}) = \mathbf{t}(x, v, \mathbf{z}) + \mathbf{n}(x, v, \mathbf{z}) \in T_v\mathcal{N} \oplus T_v^\perp\mathcal{N} \qquad \text{for all } (x, v, \mathbf{z}) \in \Omega \times \mathcal{N} \times \mathbb{R}^{mN}.$$

By using intrinsic methods, we show that assumption H4 can be relaxed (see Section 2 for details) by asking instead that there exist some constants $C_{\mathbf{t}}, D \ge 0$ so that

H4*. $|\mathbf{t}(x, v, \mathbf{z})| \le C_{\mathbf{t}}|\mathbf{z}|^2 + D(1 + |\mathbf{z}|)$, for all $(x, v, \mathbf{z}) \in \Omega \times \mathcal{N} \times \mathbb{R}^{mN}$.

and we just require a smallness condition for $C_{\mathbf{t}}$, i.e., about the growth of the leading order term in the tangent part, cf. (3). We point out that the use of intrinsic methods allows to impose no restriction on the size of the extrinsic $\mathbf{n}$. As before, $\mathbf{t}(x, u, \nabla u)$ and $\mathbf{n}(x, u, \nabla u)$ are assumed to be measurable for all $u \in H^1(\Omega; \mathbb{R}^N)$.

To state our main result, we first introduce some standard notations:

$$R_\kappa := \frac{1}{2}\min\left\{\operatorname{inj}(\mathcal{N}), \frac{\pi}{\sqrt{\kappa}}\right\}, \quad \operatorname{co}_\kappa(t) = \begin{cases} \sqrt{\kappa}\cot(\sqrt{\kappa}\, t), & \kappa > 0, \\ 1/t, & \kappa = 0, \\ \sqrt{|\kappa|}\coth(\sqrt{|\kappa|}\, t), & \kappa < 0, \end{cases}$$

where $\operatorname{inj}(\mathcal{N})$ and κ denote the injectivity radius and an upper bound for all sectional curvatures of $\mathcal{N}$, respectively, with $R_\kappa = \frac{1}{2}\operatorname{inj}(\mathcal{N})$ for $\kappa \le 0$. Recall that the squared distance on the manifold to a point $p \in \mathcal{N}$, denoted by $d_g^2(p, \cdot)$ is a convex function in geodesic balls $B_g(p, R)$ in the case that $R \le R_\kappa$, strictly convex if $R < R_\kappa$ (cf. [3, p. 404]). In this setting, we prove

Theorem 1. *Let $u \in H^1(\Omega; \mathbb{R}^N)$ be a weak solution of (1) such that $u(\Omega) \subset B_g(p, R)$ for some $p \in \mathcal{N}$ and $R > 0$ with $R < \frac{R_\kappa}{2}$, and let assumptions* H1-H3 *and* H4* *hold, as well as the following smallness condition*

$$C_{\mathbf{t}} < \ell \min\left\{\frac{1}{2R}, \operatorname{co}_\kappa(2R)\right\}. \tag{3}$$

Then the first derivatives of u are Hölder continuous on an open set Ω_0 with $\mathcal{L}^m(\Omega \setminus \Omega_0) = 0$. Moreover, $\Omega_0 = \Omega$ for $m = 2$.

The above partial regularity result may be regarded as a natural extension of the already classical regularity statements by Eells and Sampson [9], Hildebrandt, Kaul and Widman [15] or Schoen and Uhlenbeck [25, Corollary, p. 310] for harmonic mappings into Riemannian manifolds to the wider framework of F or V-harmonic maps (see Section 4 for details about the interpretation of (1) as the Euler–Lagrange equation associated to an F-energy).

About the assumptions on $u(\Omega)$, we point out that for F- and V-harmonic maps, there holds a comparison principle, for instance, when coupled with Dirichlet boundary conditions (see [18] and [4, Theorem 3]). In this case, if the Dirichlet constraint lies in a geodesic ball with smaller radius than a threshold, then the solution does so too. Moreover, we highlight that in case that $B(x, v, \mathbf{z}) = b(x, u)\mathbf{z}$ our smallness condition (3) recovers that in [16] for the case $\kappa = 0$. Despite the condition $u(\Omega) \subset B_g(p, R)$ imposes certainly a restriction, as it amounts to remove any (local) topological defects of the map u, such a requirement is nevertheless necessary in light of e.g. the construction by T. Rivière [23] of everywhere discontinuous weakly harmonic maps into $\mathbb{S}^2$.

Concerning the strategy of the proof, under condition (3) and within half of the range of strict convexity, we prove an intrinsic Caccioppoli inequality in Proposition 1. This result is, to the best of our knowledge, the first inequality of this type where the so-called excess function (that is, the right hand side) is fully intrinsic. This is novel even for the previous literature working on manifold constrained settings (compare with e.g. [5, Lemma 8] or [10, Theorem 5.2]).

The next step to obtain regularity of solutions is to prove a higher integrability result. In Proposition 2, we prove that weak solutions of the system belong to a Sobolev space $W^{1,q}(\Omega; \mathbb{R}^N)$ with a $q > 2$. As pointed out in [11, Remark 2.4], the smallness condition (2) is only needed to provide the integrability result. Therefore, applying [11, Theorem 2 and Corollary 4.4] (in which no smallness condition is used), we immediately get the proof of Theorem 1.

The paper is organized as follows: Section 2 shows how to get the hypothesis H4 from the milder requirement H4*. Section 3 is devoted to prove the Caccioppoli inequality and a higher integrability result for weak solutions to (1). In Section 4 we give details of how the variational problem which gives rise to F- and V-harmonic maps can be interpreted in the framework of the elliptic systems under consideration.

2. How to get hypothesis H4 from the weaker H4*

In this section, we aim to show that the assumption H4 follows from the weaker requirement H4*. After proving this, as a by-product of Theorem 1, we recover the classical regularity result, but with a milder smallness condition.

Let us thus assume hereafter that H4* holds.

First, if we split the left hand side of (1) into a tangent and a normal part, the extrinsic part $\mathbf{n}$ is completely characterized in terms of the second fundamental form $\mathcal{A}_v$ of $\mathcal{N} \hookrightarrow \mathbb{R}^N$ at $v \in \mathcal{N}$ as

$$\mathbf{n}(x,u,\nabla u) = b(x,u,\nabla u) \sum_{\alpha=1}^{m} \mathcal{A}_{u(x)}(\partial_\alpha u, \partial_\alpha u), \qquad \text{in} \quad T^{\perp}_{u(x)}\mathcal{N}.$$

Next, we extend this expression to the whole of $\Omega \times \mathbb{R}^N \times \mathbb{R}^{mN}$ in a way that H4 holds. For this, we follow the ideas in [13, p. 108]. For a sufficiently small $\delta > 0$, we consider a tubular neighborhood of $\overline{B_g(p,R)}$: $T_\delta(\overline{B_g(p,R)}) := \{x \in \mathbb{R}^{N+1} : \operatorname{dist}(x, \overline{B_g(p,R)}) < \delta\}$, where dist denotes the Euclidean distance. Let $0 \le \psi \le 1$ be a test function such that $\psi(\overline{B_g(p,R)}) = 1$ and $\operatorname{supp}(\psi) \subset T_\delta(\overline{B_g(p,R)})$. Observe that any point in $T_\delta(\overline{B_g(p,R)})$ can be projected onto $\overline{B_g(p,R)}$ uniquely by a submersion π_δ (e.g. see [26]).

In this setting, we can define the extensions

$$\widetilde{\mathbf{n}}(x,v,\mathbf{z}) := \begin{cases} \psi(v)\, b(x,v,\mathbf{z}) \displaystyle\sum_{\alpha=1}^{m} \mathcal{A}_{\pi_\delta(p)}(z_\alpha, z_\alpha), & (x,v,\mathbf{z}) \in \Omega \times T_\delta(\overline{B_g(p,R)}) \times \mathbb{R}^{mN} \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

$$\tilde{\mathbf{t}}(x,v,\mathbf{z}) := \begin{cases} \psi(v)\, \mathbf{t}(x,v,\mathbf{z}), & (x,v,\mathbf{z}) \in \Omega \times T_\delta(\overline{B_g(p,R)}) \times \mathbb{R}^{mN}, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Hence, by imposing just H4*, it follows that H4 holds for $\tilde{f} = \tilde{\mathbf{t}} + \widetilde{\mathbf{n}}$ with $C_1 := C_{\mathfrak{t}} + \frac{D}{2} + L\,\|\mathcal{A}\|_{L^\infty(B_g(p,R))}$ and $D_1 := 3D/2$. Here we have used the notation

$$\|\mathcal{A}\|_{L^\infty(B_g(p,R))} := \inf\left\{C \in \mathbb{R}^+ \;:\; \mathcal{A}_q(X,X) \le C|X|^2 \text{ for all } q \in B_g(p,R) \text{ and } X \in T_q\mathcal{N}\right\}.$$

Finally, let us remark that u also solves (1) with $\tilde{f}$ on the right hand side, since f and $\tilde{f}$ coincide on $\Omega \times \mathcal{N} \times \mathbb{R}^{mN}$. Therefore, without loss of generality, we will work with the extension $\tilde{f}$ in the sequel (but we keep the notation f for clarity).

3. Caccioppoli inequality and integrability of solutions

We say that $\mathfrak{b} \in \overline{B_g(p,R)}$ is a barycenter of a Radon measure μ on $\overline{B_g(p,R)}$ if $\mathfrak{b}$ is a minimizer of the function

$$\overline{B_g(p,R)} \ni q \longmapsto \frac{1}{2}\int_{\mathcal{N}} d_g^2(\cdot, q)\, d\mu.$$

In [1, Theorem 2.1], it is proved that, under the conditions above, a unique barycenter exists. Notice that this is true for simply connected nonpositively curved manifolds without any restriction on R (cf. [8]). Recall that in the literature the barycenter is also known as the Riemannian center of mass or Karcher mean [20]. Given $x_0 \in \Omega$ and $r < \operatorname{dist}(x_0, \partial\Omega)$ we define the barycenter of u in the ball $B_r(x_0)$ as the barycenter of the particular measure $\mu := u\#\mathcal{L}^m|_{\overline{B_r(x_0)}}$, whose support by the assumption about $u(\Omega)$ lies on $\overline{B_g(p,R)}$. Namely, the barycenter can be regarded simply as the minimizer of

$$\mathfrak{b} \mapsto \int_{B_r(x_0)} d_b^2(u(x), \mathfrak{b})\, dx.$$

Proposition 1. *Under the assumptions of Theorem 1, the following intrinsic Caccioppoli inequality holds:*

$$\int_{B_r(x_0)} |\nabla u|^2 \le \frac{C}{r^2}\int_{B_{2r}(x_0)} (1 + d_g^2(u, \mathfrak{b}_{2r})), \quad \text{for some } C > 0, \text{ for all } x_0 \in \Omega \text{ and } r < \min\left\{\tfrac{\operatorname{dist}(x_0,\partial\Omega)}{2}, 1\right\}, \tag{6}$$

where $\mathfrak{b}_{2r}$ denotes the barycenter of u in the ball $B_{2r}(x_0)$.

Proof. First of all, we observe that the condition $u(\Omega) \subseteq B_g(p,R)$ permits to say that $\mathfrak{b}_{2r}$ exists, and that $\mathfrak{r}_{\mathfrak{b}} := d_g(u, \mathfrak{b}_{2r}) < R_\kappa$ for all $x_0 \in \Omega$ and $r < \frac{\operatorname{dist}(x_0,\partial\Omega)}{2}$. Unless otherwise stated, B_r denotes hereafter a ball centered in x_0 with radius r.

Let us consider x_0 and r as above, and let $\eta \in C_0^\infty(\Omega)$ be such that $0 \le \eta \le 1$, $\eta_{|B_r} \equiv 1$, $\operatorname{supp}(\eta) \subset B_{2r}$ and $|\nabla\eta| < c/r$ for some $c > 0$. We take $\varphi = -\eta^2 \exp_u^{-1}\mathfrak{b}_{2r} =: \eta^2 X_{\mathfrak{b}} \in T_u\mathcal{N}$. Note that $X_{\mathfrak{b}}$ inherits the regularity of the map u, and by [19, Theorem 6.6.1] it holds

$$\operatorname{Hess}\left(\frac{\mathfrak{r}_{\mathfrak{b}}^2}{2}\right)(\nabla u, \cdot)^\sharp = \nabla_{\nabla u}\left(\frac{\mathfrak{r}_{\mathfrak{b}}^2}{2}\right) = -\nabla_{\nabla u}\left(\exp_x^{-1}\mathfrak{b}_{2r}\right) = -\sum_{\alpha=1}^{m} \nabla_{e_\alpha}\left(\exp_{u(x)}^{-1}\mathfrak{b}_{2r}\right) = \nabla X_{\mathfrak{b}}, \tag{7}$$

where $\{e_\alpha\}_{\alpha=1}^m$ is the canonical orthonormal basis and $\sharp$ is the musical isomorphism i.e. $\omega^\sharp$ is the vector field given by $g(\omega^\sharp, \cdot) = \omega(\cdot)$ for any 1-form ω.

Now, if we multiply (1) by φ, we can write

$$-\int_{B_{2r}} \eta^2 \operatorname{div}(B(x,u,\nabla u)) \cdot X_{\mathfrak{b}} = \int_{B_{2r}} \eta^2\, \mathbf{t}(x,u,\nabla u) \cdot X_{\mathfrak{b}}.$$

Integrating by parts and using (7), we get

$$\int_{B_{2r}} \eta^2\, b(x,u,\nabla u)\, \mathrm{Hess}\left(\frac{\mathfrak{r}_{\mathfrak{b}}^2}{2}\right)(\nabla u, \nabla u) = \int_{B_{2r}} \eta^2\, \mathbf{t}(x,u,\nabla u)\cdot X_{\mathfrak{b}} - 2\int_{B_{2r}} \eta\, b(x,u,\nabla u)\nabla\eta\ \nabla u\ X_{\mathfrak{b}} =: \mathcal{J}. \tag{8}$$

Concerning the left hand side of (8), by means of the comparison theorem (see [19, Theorem 6.6.1] or [24, p. 154]), we get

$$\int_{B_{2r}} \eta^2\, b(x,u,\nabla u)\, \mathrm{Hess}\left(\frac{\mathfrak{r}_{\mathfrak{b}}^2}{2}\right)(\nabla u, \nabla u) \geq \ell \int_{B_{2r}} \eta^2\, \mathfrak{r}_{\mathfrak{b}}\, \mathrm{co}_\kappa(\mathfrak{r}_{\mathfrak{b}})|\nabla u|^2 \geq \ell\, \min\{1, 2R\,\mathrm{co}_\kappa(2R)\}\int_{B_{2r}} \eta^2|\nabla u|^2. \tag{9}$$

On the other hand, using Young's inequality, we have, for any $\varepsilon > 0$,

$$\begin{aligned}\mathcal{J} &\leq 2RC_{\mathbf{t}} \int_{B_{2r}} \eta^2|\nabla u|^2 + D\int_{B_{2r}} \eta^2(1+|\nabla u|)\,\mathfrak{r}_{\mathfrak{b}} + \varepsilon\int_{B_{2r}} \eta^2|\nabla u|^2 + \frac{L^2c^2}{r^2\varepsilon}\int_{B_{2r}} \mathfrak{r}_{\mathfrak{b}}^2 \\ &\leq 2RC_{\mathbf{t}} \int_{B_{2r}} \eta^2|\nabla u|^2 + \frac{D}{2}\int_{B_{2r}} (1+\mathfrak{r}_{\mathfrak{b}}^2) + \varepsilon(1+D)\int_{B_{2r}} \eta^2|\nabla u|^2 + \frac{1}{\varepsilon}\left(\frac{L^2c^2}{r^2} + \frac{D}{4}\right)\int_{B_{2r}} \mathfrak{r}_{\mathfrak{b}}^2. \end{aligned} \tag{10}$$

Under the smallness condition, combining the expressions (8), (9), (10), and having in mind that $r < 1$, we easily get (6). $\square$

Proposition 2. *Let $u \in H^1(\Omega;\mathcal{N})$ be such that $u(\Omega) \subseteq B_g(p,R)$ for some $p \in \mathcal{N}$ and $R < \frac{\mathrm{inj}(\mathcal{N})}{2}$ and satisfying (6). Then, there exists $q > 2$ and $C > 0$ such that $u \in W^{1,q}(\Omega;\mathbb{R}^N)$ and*

$$\left(\fint_{B_r(x_0)} (1+|\nabla u|)^q\, dx\right)^{2/q} \leq C\fint_{B_{2r}(x_0)} (1+|\nabla u|^2)\, dx \tag{11}$$

for every $x_0 \in \Omega$ and $0 < r < \frac{\mathrm{dist}(x_0,\partial\Omega)}{2}$.

Proof. First, observe that the range condition $u(\Omega) \subseteq B_g(p,R)$, with $R < \frac{\mathrm{inj}(\mathcal{N})}{2}$ yields the existence of $\mathfrak{b}_{2r}$ and the injectivity of the exponential map centered at $\mathfrak{b}_{2r}$ in the range of u. Therefore, we can choose polar coordinates centered at $\mathfrak{b}_{2r}$, and we find that

$$d_g(u,\mathfrak{b}_{2r})^2 = |u^\rho|^2 = \left|u^\rho - \fint_{B_{2r}} u^\rho\right|^2$$

where u^ρ is the radial component of u. Consequently, using Sobolev–Poincaré inequality we note that

$$\frac{1}{r^2}\int_{B_{2r}} d_g^2(u,\mathfrak{b}_{2r}) \leq Cr^{m-2}\fint_{B_{2r}} |u^\rho|^2 \leq Cr^{m-2}r^2\left(\fint_{B_{2r}} |\nabla u^\rho|^{\frac{2m}{m+2}}\right)^{\frac{m+2}{m}} \leq Cr^m\left(\fint_{B_{2r}} |\nabla u|^{\frac{2m}{m+2}}\right)^{\frac{m+2}{m}}$$

where, in the last inequality we use the fact that $|\nabla u^\rho| \leq |\nabla u|$. Therefore, from (6), we find

$$\fint_{B_r} |\nabla u|^2 \leq C\left(\fint_{B_{2r}} (1+|\nabla u|)^{\frac{2m}{m+2}}\right)^{\frac{m+2}{m}}.$$

Finally, the result is obtained by [12, Proposition 5.1]. $\square$

4. Application to recover several generalizations of harmonic maps

Quasilinear elliptic systems of constrained PDEs of type (1) naturally arise when one minimizes the following energy functionals in the class of H^1 functions with range in $\mathcal{N}$, denoted by $H^1(\Omega;\mathcal{N})$:

$$\mathcal{F}(u) := \begin{cases} \displaystyle\int_\Omega F\left(x, \frac{|du|^2}{2}\right)dx, & \text{if } u \in H^1(\Omega;\mathcal{N}) \\ +\infty, & \text{otherwise.}\end{cases} \tag{12}$$

In fact, the Euler–Lagrange equation associated to $\mathcal{F}$ can be written in terms of the tension field τ as

$$0 = \tau_F(u) = F'\left(x, \frac{|du|^2}{2}\right)\tau(u) + \nabla u\cdot\nabla\left(F'\left(x,\frac{|du|^2}{2}\right)\right), \qquad \text{where} \quad \tau(u) = P_u(\Delta u), \tag{13}$$

$P_u : \mathbb{R}^N \to T_u\mathcal{N}$ is the orthogonal projection and F' the partial derivative of $F(x,s)$ with respect to s. Notice that (13) includes the well studied cases of F-harmonic maps [2], if there is no x-dependence, as well as f-harmonic maps (introduced by Lichnerowicz in [21]) with $F(x,s) = f(x)s$. We point out that (13) can be written as

$$-\mathrm{div}\left(F'\left(x,\frac{|du|^2}{2}\right)\nabla u\right) = F'\left(x,\frac{|du|^2}{2}\right)\sum_{\alpha=1}^m \mathcal{A}_u(\partial_\alpha u, \partial_\alpha u).$$

Therefore, as explained in Section 2, (13) can be put into the form (1) with right hand side satisfying H4.

Another example in which our study can be applied is the case of $V-$harmonic maps, with V a given smooth vector field on Ω [4]. Then the system of PDEs reads as

$$\tau(u) - \left\langle V, \nabla u \right\rangle = 0.$$

In this case, $\mathbf{t}^i(x, \mathbf{z}) := V^\alpha(x)\, \mathbf{z}^i_\alpha$, satisfies H4* with $C_\mathbf{t} = 0$.

Notice that, in the case that $F \in C^2(\Omega \times \mathbb{R})$, such that $0 < \ell \leq F'(x,s) \leq L$ and the function $G(x, \mathbf{z}) := F\left(x, \frac{|\mathbf{z}|^2}{2}\right)$ is uniformly convex in $\mathbf{z}$, assumptions H1-H3 are satisfied.

Data availability

No data was used for the research described in the article.

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