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ORIGINAL ARTICLE

Ergodic properties of multiplication and weighted composition operators on spaces of holomorphic functions

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Abstract
We obtain different results about the mean ergodicity of weighted composition operators when acting on the spaces $H(B)$, $H_b(B)$, and $H^\infty(B)$, where B is the open unit ball of a Banach space, as well as about the compactness and the mean ergodicity of the multiplication operator. This study relates these properties of the operators with properties of the symbol and the weight defining such operators.

KEYWORDS
bounded type, holomorphic function on a Banach space, mean ergodic, power bounded, weighted composition operator

1 INTRODUCTION

If $\mathbb{D}$ is the open unit disk of $\mathbb{C}$ and $\psi : \mathbb{D} \to \mathbb{C}$ and $\varphi : \mathbb{D} \to \mathbb{D}$ are holomorphic mappings, then $f \mapsto \psi \cdot (f \circ \varphi)$ defines an operator on the space holomorphic functions $H(\mathbb{D})$, called *weighted composition operator*. When restricted to some subspace of $H(\mathbb{D})$, the properties of such an operator depend on the maps ψ and φ, as well as, on the subspace on which the operator is defined.

Linear dynamics, in general, try to answer a natural question: if $T : E \to E$ is a continuous linear operator acting on some topological space, how do the compositions of T with itself behave? This leads to a variety of different situations, that have been extensively studied and described in [3, 12]. Some dynamical properties of weighted composition operators on spaces of holomorphic functions on $\mathbb{D}$ have been studied in [4, 14].

The authors in [6] study composition operators (which are weighted composition operators taking the mapping ψ equal to 1) acting on spaces of holomorphic functions defined on a subset of a finite-dimensional space. Following this idea, we find studies on (weighted) composition operators defined on spaces of holomorphic functions defined on the open unit ball of a Banach space from two different points of view; the study of compactness and related properties in [2, 7, 11] and the study of dynamical properties of the composition operator in [16]. The aim in this note is to extend some of these results, in two senses, from finite-dimensional domains to domains in arbitrary Banach spaces and from composition operators to weighted composition operators. We are mainly interested in power boundedness and uniform mean ergodicity (precise definitions are given below). As we mentioned before, the properties of the operator depend not only on the mappings ψ and φ, but also on the space on which the operator is defined. We consider three cases of functions defined on the ball of a Banach space: the space of all holomorphic functions, the space of holomorphic functions of bounded type and the space of bounded holomorphic functions (again, definitions to be found below). We seek for conditions on the mappings ψ and φ so that in each case the corresponding weighted composition operator has the properties we are interested in.

1.1 | Preliminaries

We begin by fixing some notation and basic notions that will be used throughout the paper. From now on, E will always be assumed to be a locally convex Hausdorff space (lcHs for short). The family of continuous seminorms that induces the locally convex topology on E is denoted by $cs(E)$. We denote by $\mathcal{L}(E)$ the space of continuous linear operators acting on E. Given $T \in \mathcal{L}(E)$, the operator

$$T_{[n]} = \frac{1}{n}\sum_{m=0}^{n-1} T^m \tag{1}$$

is the nth Cesáro mean of T. We say that $T \in \mathcal{L}(E)$ is:

- *Power bounded* if the family $(T^n)_{n\in\mathbb{N}}$ is an equicontinuous set on $\mathcal{L}(E)$.
- *Cesáro bounded* if the family $(T_{[n]})_{n\in\mathbb{N}}$ is an equicontinuous set on $\mathcal{L}(E)$.
- *Mean ergodic* if there is $P \in \mathcal{L}(E)$ such that $\lim_{n\to\infty} T_{[n]}x = Px$ for each $x \in E$.
- *Uniformly mean ergodic* if the sequence of operators $(T_{[n]})_{n\in\mathbb{N}}$ converges uniformly on the bounded sets of E to some $P \in \mathcal{L}(E)$.
- *Topologizable* if for every continuous seminorm $p \in cs(E)$ there is $q \in cs(E)$ such that for every $n \in \mathbb{N}$ there is $A_n > 0$ such that
$$p(T^n(x)) \leq A_n q(x) \tag{2}$$
for every $x \in E$.

The following result is a straightforward consequence of Eberleins's mean ergodic theorem (see [8, Corollary 5.24] or [18, Theorem 2.1]). Similar results in same spirit can be found in [1, 5].

Proposition 1.1. *Let E be an lcHs and $T \in \mathcal{L}(E)$.*

(a) Assume T is Cesáro bounded. Then, T is mean ergodic if and only if $\frac{T^n}{n} \to 0$ pointwise and $(T_{[n]}x)_{n\in\mathbb{N}}$ is $\sigma(E,E')$-relatively compact for each $x \in E$.
(b) Assume E is barreled. Then, T is mean ergodic if and only if $\frac{T^n}{n} \to 0$ pointwise and $(T_{[n]}x)_{n\in\mathbb{N}}$ is $\sigma(E,E')$-relatively compact for each $x \in E$.

We follow [10] and [20] for notation and theory of homogeneous polynomials and holomorphic functions.

Let X and Y be complex Banach spaces. We denote by B the open unit ball of X. A mapping $p : X \to Y$ is an m-homogeneous polynomial if there is a continuous linear form $L : X \times \cdots \times X \to Y$ so that $p(x) = L(x, \ldots, x)$. We denote by $\mathcal{P}(^m X)$ the space of m-homogeneous polynomials with $Y = \mathbb{C}$. A function $f : B \to Y$ is holomorphic if for each $x_0 \in B$ there exists a (unique) sequence $(p_m)_{m\in\mathbb{N}}$, where each $p_m : X \to Y$ is an m-homogeneous polynomials and $p_0 \in Y$ so that

$$f(x) = \sum_{m=0}^{\infty} p_m(x)$$

converges uniformly on some open ball centered at x_0. The space of holomorphic functions on B with $Y = \mathbb{C}$ is denoted by $H(B)$. This space becomes a locally convex Haussdorff space when endowed with the topology of uniform convergence on the compact sets of B, or shortly τ_0.

We say that a set $V \subset B$ is B-bounded if its distance to the boundary of B is strictly positive. A function $f : B \to \mathbb{C}$ is said to be of *bounded type* if it maps B-bounded sets into bounded sets on $\mathbb{C}$. Similarly, we also say that a mapping $\varphi : B \to B$ is of bounded type when it maps B-bounded sets into B-bounded sets. As usual, we denote by $H_b(B)$ the space of functions in $H(B)$ which are of bounded type. With the topology of uniform convergence on the B-bounded sets the space $H_b(B)$ is Fréchet. Note that this topology is given by the seminorms

$$\sup_{\|x\|<1-\frac{1}{n+1}} |f(x)|,$$

for $f \in H_b(B)$ and $n \in \mathbb{N}$. Finally, $H^\infty(B)$ is the space of holomorphic functions $f : B \to \mathbb{C}$ which are bounded on B. Endowed with the norm

$$\|f\|_\infty := \sup_{x\in B} |f(x)|$$

it becomes a Banach space.

For standard theory of functional analysis, we refer to [19].

Now, let $\psi : B \to \mathbb{C}$ and $\varphi : B \to B$ be holomorphic functions. We denote by $C_{\psi,\varphi} : H(B) \to H(B)$ the weighted composition operator defined by the action $C_{\psi,\varphi}(f) = \psi \cdot (f \circ \varphi)$. As usual, we say that ψ is the *weight* and φ is the *symbol* of the weighted composition operator.

The operator $C_{\psi,\varphi}$ can be written as the composition of the multiplication operator given by $M_\psi(f) = \psi \cdot f$ with the composition operator $C_\varphi(f) = f \circ \varphi$. That is, $C_{\psi,\varphi} = M_\psi \circ C_\varphi$, and taking φ as the identity on B or ψ as the constant function 1 we recover the operators M_ψ and C_φ, respectively.

The paper is organized as follows. In Section 2, we investigate the power boundedness and mean ergodicity of $C_{\psi,\varphi}$ acting on the spaces $H(B)$, $H_b(B)$, and $H^\infty(B)$. We focus on properties of the weight and the symbol as well as their composition. In Section 3, we characterize compactness of the multiplication operator M_ψ on the spaces of holomorphic functions previously mentioned. Additionally, ergodic properties of M_ψ are studied. Finally, in Section 4 we give several examples.

2 | ERGODIC PROPERTIES OF WEIGHTED COMPOSITION OPERATORS

On this section, we study ergodic properties of $C_{\psi,\varphi}$ when acting on the spaces $H(B)$, $H_b(B)$, $H^\infty(B)$, and $\mathcal{P}(^mX)$.

First, we briefly focus on weighted composition operators acting on spaces of homogeneous polynomials, which are natural subspaces of $H(B)$. In some cases, we can rewrite the weighted composition operator as a composition operator whose symbol is a simple modification of the original one.

Proposition 2.1. *Let $\psi : X \to \mathbb{C}$ and $\varphi : X \to X$ be two continuous mappings. Assume there is $m \in \mathbb{N}$ and a continuous function $\widehat{\psi} : X \to \mathbb{C}$ such that*

$$\psi(x) = \left(\widehat{\psi}(x)\right)^m \quad \text{for every } x \in X. \tag{3}$$

Then, $C_{\psi,\varphi} : \mathcal{P}(^mX) \to \mathcal{P}(^mX)$ is well defined if and only if the map $\widehat{\psi} \cdot \varphi : X \to X$ is a continuous linear operator. In addition, the operator $C_{\psi,\varphi}$ coincides with the composition operator $C_{\widehat{\psi}\cdot\varphi}$.

Proof. Fix $p \in \mathcal{P}(^mX)$ and $x \in X$, by Equation (3) we have

$$p\left(\widehat{\psi}(x) \cdot \varphi(x)\right) = \left(\widehat{\psi}(x)\right)^m p(\varphi(x)) = \psi(x)p(\varphi(x)).$$

Since p and x were arbitrary we obtain that the operator $C_{\psi,\varphi}$ coincides with the composition operator $C_{\widehat{\psi}\cdot\varphi}$. Observe that the symbol $\widehat{\psi} \cdot \varphi : X \to X$ is continuous and applying [15, Proposition 2.2] (see also [21, Proposition 2.3]) we obtain the result. □

The constant functions are trivial examples of weights satisfying Equation (3). In this case, if there is $c \in \mathbb{C} \setminus \{0\}$ such that $\psi(x) = c$ for every $x \in X$, we can find $z \in \mathbb{C} \setminus \{0\}$ with $c = z^m$. Thus, the weighted composition operator is well defined if and only if $z \cdot \varphi : X \to X$ is a linear operator, and this forces φ to be a linear operator. We can construct more elaborated examples. For each $u \in X' \setminus \{0\}$ observe that the function $\psi = u^m$ satisfies Equation (3) taking, for example, $\widehat{\psi} = u$. Now if $\varphi(x) = x_0 \in X$ for every $x \in X$, we have that $C_{u^m,x_0} : \mathcal{P}(^mX) \to \mathcal{P}(^mX)$ coincides with the composition operator $C_{u\cdot x_0}$.

Properties of composition operators acting on spaces of m-homogeneous polynomials have been studied in [15]. Then, for weighted composition operators whose symbol and weight satisfy Proposition 2.1, the results of [15] clearly apply.

However, there are combinations of weights and symbols that do not satisfy the assumption on Proposition 2.1, but the associated weighted composition operator is well defined on $\mathcal{P}(^mX)$. Indeed, consider $\psi \in \mathcal{P}(^2\mathbb{C}^2)$ given by

$\psi(z_1, z_2) = z_1 z_2$. Clearly, ψ does not satisfy Equation (3) since $\widehat{\psi}(1, z)$ would give a continuous determination of the square root on $\mathbb{C}$. Now, taking the constant symbol $\varphi(z_1, z_2) = (1, 1)$ for every $z_1, z_2 \in \mathbb{C}$, we have

$$\psi(z_1, z_2) \cdot p(\varphi(z_1, z_2)) = \psi(z_1, z_2) \cdot p(1, 1),$$

for any $p \in \mathcal{P}(^2\mathbb{C}^2)$ and every $(z_1, z_2) \in \mathbb{C}^2$. This is again a 2-homogeneous polynomial and then $C_{\psi,\varphi}$ is well defined.

In general, observe that the iterates of the operators $C_{\psi,\varphi}$ can be rewritten as follows:

$$(C_{\psi,\varphi})^n f = \psi \cdots (\psi \circ \varphi^{n-1}) \cdot (f \circ \varphi^n) = \left(\prod_{m=0}^{n-1} (\psi \circ \varphi^m) \right) \cdot (f \circ \varphi^n)$$

for any function f, $n \in \mathbb{N}$, where φ^0 denotes the identity. Now taking $\psi_{[n]} := \prod_{m=0}^{n-1} (\psi \circ \varphi^m)$ we obtain the following identity:

$$(C_{\psi,\varphi})^n(f) = \psi_{[n]} \cdot (f \circ \varphi^n). \tag{4}$$

Then, it seems natural to assume that there are infinitely many $n \in \mathbb{N}$ for which $\psi_{[n]}$ is not zero. If this is not the case, the iterates $(C_{\psi,\varphi})^n$ will be the zero operator from some n_0 big enough on.

Remark 2.2. If $\psi : B \to \mathbb{C}$ and $\varphi : B \to B$ are such that $\psi \circ \varphi^{n_0} = 0$ for some $n_0 \in \mathbb{N}$, then clearly $\psi_{[n]} = 0$ for every $n \geq n_0$. This, in view of (4), gives that $(C_{\psi,\varphi})^n = 0$ for every $n \geq n_0$ (whenever the operator is well defined).

So, in order to avoid trivial cases, from now on we will always assume that $\psi \circ \varphi^n \neq 0$ for every $n \in \mathbb{N}$. Let us point out that if φ is open, then this condition is satisfied for every $\psi \neq 0$. Indeed, if $\psi \circ \varphi^{n_0} = 0$ for some $n_0 \in \mathbb{N}$, then $\psi \circ \varphi^{n_0 - 1}(\varphi(B)) = \{0\}$. Since $\varphi(B)$ is an open nonempty set, the identity principle gives $\psi \circ \varphi^{n_0 - 1} = 0$. Repeating this argument a finite number of times we finally get that $\psi = 0$, as we wanted.

Of course we may have $\psi \circ \varphi^n \neq 0$ for every $n \in \mathbb{N}$ without φ being open. On the open unit ball of $\mathbb{C}^2$, consider the mappings $\psi((x_1, x_2)) = (x_1)(x_2 + 1)$ and $\varphi((x_1, x_2)) = (x_1, 0)$. Clearly, φ is not open and $\psi \circ \varphi^n \neq 0$ is satisfied for all $n \in \mathbb{N}_0$.

The assumption of φ not being open is not interesting on $H(\mathbb{D})$ since the constant symbols are the only holomorphic mappings that are not open. However, on Banach spaces of higher dimension the situation is very different (see Examples 4.2).

2.1 | The space of holomorphic functions

We begin by looking at weighted composition operators acting on $H(B)$. We have that the operator $C_{\psi,\varphi} : H(B) \to H(B)$ is continuous if and only if $\psi \in H(B)$ and $\varphi : B \to B$ is holomorphic (see [7, Section 2]). From now on, we will always assume that $\psi : B \to \mathbb{C}$ and $\varphi : B \to B$ are holomorphic. Analogs of results obtained in this subsection will be given in the following subsections for the spaces $H_b(B)$ and $H^\infty(B)$.

We begin with the following extension of [4, Proposition 3.1] to $H(B)$.

Proposition 2.3. *The following are satisfied.*

(a) If $C_{\psi,\varphi} : H(B) \to H(B)$ is power bounded then $C_{\psi,\varphi}$ is uniformly mean ergodic and $(\psi_{[n]})_{n \in \mathbb{N}}$ is bounded in $H(B)$.
(b) If $C_{\psi,\varphi} : H(B) \to H(B)$ is mean ergodic then $\lim_{n \to \infty} \frac{1}{n} \psi_{[n]} = 0$ in $H(B)$.

Proof.

(a) If $C_{\psi,\varphi}$ is power bounded then the set $\{C^n_{\psi,\varphi}(f) : n \in \mathbb{N}\}$ is bounded in $H(B)$ for every $f \in H(B)$. In particular, taking the constant function $f = 1$ we have (recall Equation (4)) that $\{\psi_{[n]} : n \in \mathbb{N}\}$ is bounded in $H(B)$. Since $H(B)$ is a semi-Montel space, as a consequence of [5, Proposition 3.3] the operator $C_{\psi,\varphi}$ is also uniformly mean ergodic (a complete proof of this fact can be found in [15, Proposition 3.1]).

(b) If $C_{\psi,\varphi}$ is mean ergodic we clearly have

$$\lim_{n\to\infty} \frac{1}{n} C^n_{\psi,\varphi}(f) = \lim_{n\to\infty} \left[\frac{n+1}{n} \left(C_{\psi,\varphi}\right)_{[n+1]}(f) - \left(C_{\psi,\varphi}\right)_{[n]}(f) \right] = 0,$$

for every $f \in H(B)$. Therefore, $\lim_{n\to\infty} \frac{1}{n} C^n_{\psi,\varphi}(1) = 0$, which (again by Equation (4)) yields the assertion. □

Since we have obtained a necessary condition for the mean ergodicity and a sufficient condition for the uniform mean ergodicity now, we pay attention to the properties of being topologizable and power boundedness. As in the case of composition operators (see [16]), φ having stable orbits plays a fundamental role. Several authors have used this property to characterize topologizable, power bounded, and/or mean ergodic weighted composition operators on different function spaces (see [4, 6, 17]).

Following [6] we say that φ has *stable orbits* if for every compact subset K of B there is a compact $L \subset B$ such that $\varphi^n(K) \subseteq L$ for every $n \in \mathbb{N}$.

Proposition 2.4. *Assume φ has stable orbits. Then, $C_{\psi,\varphi} : H(B) \to H(B)$ is topologizable.*

Proof. Fix some compact $K \subset B$ and take $L \subset B$ compact such that $\varphi^n(K) \subseteq L$ for all $n \in \mathbb{N}$. Let $a_n = \sup_{x\in K} |\psi_{[n]}(x)| < \infty$. Then, using Equation (4) we have

$$\sup_{x\in K} |C^n_{\psi,\varphi}(f)(x)| \le \sup_{x\in K} |\psi_{[n]}(x)| \sup_{x\in K} |f(\varphi^n(x))| \le a_n \sup_{x\in L} |f(x)|$$

and $C_{\psi,\varphi}$ is topologizable. □

For $C_{\psi,\varphi}$ being power bounded we obtain a similar result.

Proposition 2.5. *Consider the following conditions.*

(a) φ has stable orbits and $(\psi_{[n]})_n$ is a bounded sequence in $H(B)$.
(b) $C_{\psi,\varphi} : H(B) \to H(B)$ is power bounded.
(c) $(\psi_{[n]})_n$ is a bounded sequence in $H(B)$.

Then, (a) implies (b) and (b) implies (c).

Proof. First, assume (a) holds. Fix a compact set $K \subset B$ and consider

$$A = \sup_{n\in\mathbb{N}} \sup_{x\in K} \left| \prod_{m=0}^{n} (\psi\circ\varphi^m)(x) \right| < \infty.$$

Since φ has stable orbits there exists a compact set $L \subset B$ such that $\varphi^n(K) \subseteq L$ for every $n \in \mathbb{N}$. Then, using Equation (4) we have

$$\sup_{x\in K} |C^n_{\psi,\varphi}(f)(x)| \le A \sup_{x\in K} |f(\varphi^n(x))| \le A \sup_{x\in L} |f(x)|,$$

for each $n \in \mathbb{N}$ and all $f \in H(B)$. Thus, $C_{\psi,\varphi}$ is power bounded.

Now, if (b) holds, it follows from Proposition 2.3 that $(\prod_{m=0}^{n} (\psi\circ\varphi^m))_n$ is a bounded sequence in $H(B)$. □

We discuss the mean ergodicity of $C_{\psi,\varphi}$ assuming that the operator is Cesáro bounded.

Proposition 2.6. *Assume that $C_{\psi,\varphi} : H(B) \to H(B)$ is Cesáro bounded. Consider the following conditions.*

(a) $\left((C_{\psi,\varphi})_{[n]} f\right)_n$ *is* $\sigma(H(B), H(B)')$*-relatively compact for each* $f \in H(B)$*.*
(b) $\lim_{n\to\infty} \frac{\psi_{[n]}}{n} = 0$ *in* $H(B)$*.*
(c) φ *has stable orbits.*

Then, if (a), (b) and (c) hold $C_{\psi,\varphi} : H(B) \to H(B)$ *is mean ergodic. Conversely, if* $C_{\psi,\varphi}$ *is mean ergodic then conditions (a) and (b) are satisfied.*

Proof. Assume that (a), (b), and (c) hold. By Proposition 1.1, it is enough to show that

$$\lim_{n\to\infty} \frac{C^n_{\psi,\varphi}(f)}{n} = 0 \tag{5}$$

holds for every $f \in H(B)$.

Fix a compact set $K \subset B$ and using (c) take $L \subset B$ compact such that $\varphi^n(K) \subseteq L$ for every $n \in \mathbb{N}$. For an arbitrary $f \in H(B)$, we obtain

$$\begin{aligned} \lim_{n\to\infty} \sup_{x\in K} \left| \frac{C^n_{\psi,\varphi}(f)(x)}{n} \right| &= \lim_{n\to\infty} \sup_{x\in K} \left| \frac{\psi_{[n]}(x) \cdot f(\varphi^n(x))}{n} \right| \\ &\leq \lim_{n\to\infty} \sup_{x\in K} \left| \frac{\psi_{[n]}(x)}{n} \right| \sup_{y\in K} |f(\varphi^n(y))| \\ &\leq \lim_{n\to\infty} \sup_{x\in K} \left| \frac{\psi_{[n]}(x)}{n} \right| \sup_{y\in L} |f(y)|. \end{aligned}$$

By (b) the last term tends to 0 and the result follows.

Now, assume that $C_{\psi,\varphi}$ is mean ergodic. Proposition 1.1 gives that (a) and Equation (5) are satisfied for every $f \in H(B)$. Taking f to be the constant function 1 we have

$$\lim_{n\to\infty} \sup_{x\in K} \left| \frac{\psi_{[n]}(x)}{n} \right| = \lim_{n\to\infty} \sup_{x\in K} \left| \frac{C^n_{\psi,\varphi}(1)}{n} \right| = 0,$$

for every compact $K \subset B$. Hence, (b) holds. □

2.2 | The space of holomorphic functions of bounded type

The goal now is to study mean ergodicity, power boundedness, and topologizable for $C_{\psi,\varphi}$ defined on $H_b(B)$. In this case, we have more tools at hand, which allows us to get more precise results than in the previous section.

To this aim, instead of assuming that the symbol $\varphi : B \to B$ has stable orbits we consider a 'bounded type' version of this property. The map φ has *B-stable orbits* if for every B-bounded set $U \subset B$ there is a B-bounded set $V \subset B$ such that $\varphi^n(U) \subseteq V$ for every $n \in \mathbb{N}$. The interested reader can find a study of this property on [16, Section 3].

Assuming ψ is non-zero, we recall that the operator $C_{\psi,\varphi} : H_b(B) \to H_b(B)$ is continuous if and only if both ψ and φ are of bounded type (see [7, Proposition 2.1]).

Remark 2.7. In this case, mean ergodicity has no direct connection with power boundedness for this type of operators. On one hand, since $H_b(B)$ is not a Montel space in general we cannot obtain (as in Proposition 2.3 (a)) that power boundedness implies uniformly mean ergodicity. In fact, the examples in [16, Proposition 5.4] are particular cases of weighted composition operator which are power bounded but not even mean ergodic. On the other hand, since $H_b(B)$ is a barreled space Proposition 1.1 (b) is available (see also [1, Theorem 2.4]). Applying this to the constant function $f = 1$ gives that $\lim_{n\to\infty} \frac{\psi_{[n]}}{n} = 0$ in $H_b(B)$, whenever $C_{\psi,\varphi} : H_b(B) \to H_b(B)$ is mean ergodic.

The following result shows that the converse implication of Proposition 2.4 does hold for $C_{\psi,\varphi}$ acting on $H_b(B)$. However, the argument that we use here does not work on $H(B)$, since compact subsets of B may have empty interior, as it happens on every infinite-dimensional Banach space. This extends [4, Proposition 3.2]. As we mentioned before, in order to avoid trivial situations, we assume that $\psi\circ\varphi^n \neq 0$ for every $n \in \mathbb{N}_0$.

Proposition 2.8. *Let ψ and φ be of bounded type such that $\psi\circ\varphi^n \neq 0$ holds for every $n \in \mathbb{N}_0$. The following assertions are equivalent:*

(a) $C_{\psi,\varphi} : H_b(B) \to H_b(B)$ is topologizable.
(b) φ has B-stable orbits.

Proof. Assume $C_{\psi,\varphi}$ is topologizable. Then, for every $0 < r < 1$ there exists $0 < s < 1$ such that for every $n \in \mathbb{N}$ there is $a_n > 0$ with

$$\sup_{\|x\|<r} |C^n_{\psi,\varphi}(f)(x)| \leq a_n \sup_{\|x\|<s} |f(x)| \tag{6}$$

for all $f \in H_b(B)$. On the other hand, by [16, Lemma 5.1] the $H_b(B)$-hull of sB given by

$$M = \widehat{(sB)}_{H_b(B)} := \{x \in B : |f(x)| \leq \sup_{y\in sB} |f(y)|, \text{ for every } f \in H_b(B)\}$$

is again a B-bounded set.

Since M is the intersection of closed sets, it is closed. Let us see now that $\varphi^n(rB) \subseteq M$ for every $n \in \mathbb{N}$. Suppose that this is not the case and take $n_0 \in \mathbb{N}$ and $z_0 \in rB$ such that $\varphi^{n_0}(z_0) \notin M$. In other words, $rB \bigcap \varphi^{-n_0}(B \setminus M) \neq \emptyset$. By assumption, the function $\psi_{[n_0]}$ is not 0 and, since $rB \bigcap \varphi^{-n_0}(B \setminus M)$ is an open set, by the identity principle we can take x_0 in this set such that $|\psi_{[n_0]}(x_0)| > 0$. The definition of M gives $f_0 \in H_b(B)$ such that $\sup_{y\in sB} |f_0(y)| < |f_0(\varphi^{n_0}(x_0))|$. Then, there is $m \in \mathbb{N}$ with

$$\sup_{y\in sB} \frac{|f_0(y)|^m}{|f_0(\varphi^{n_0}(x_0))|^m} < \frac{|\psi_{[n_0]}(x_0)|}{a_{n_0}}.$$

Observe that the function $f = f_0^m \in H_b(B)$ does not satisfy Equation (6) and this yields a contradiction.

The converse implication follows with exactly the same proof as in Proposition 2.4, replacing “compact” by “B-bounded” and “stable orbits” by “B-stable orbits.” □

Observe that we only use condition $\psi\circ\varphi^n \neq 0$ to show that if $C_{\psi,\varphi}$ is topologizable then φ has stable orbits. As a consequence of this result, we characterize power bounded weighted composition operators on $H_b(B)$ extending [4, Theorem 3.3].

Theorem 2.9. *Let ψ and φ be of bounded type such that $\psi\circ\varphi^n \neq 0$ holds for all $n \in \mathbb{N}_0$. The following assertions are equivalent:*

(a) $C_{\psi,\varphi} : H_b(B) \to H_b(B)$ is power bounded,
(b) φ has B-stable orbits and $(\psi_{[n]})_n$ is a bounded sequence in $H_b(B)$.

Proof. If $C_{\psi,\varphi}$ is power bounded it is in particular topologizable. Applying Proposition 2.8, we obtain φ has B-stable orbits. Additionally, the set $\{C^n_{\psi,\varphi}(f) : n \in \mathbb{N}\}$ is bounded in $H_b(B)$ for every $f \in H_b(B)$. We are done if we take the constant function $f = 1$.

Now, (b) implies (a) follows as in Theorem 2.5 replacing "compact" by "B-bounded" and "stable orbits" by "B-stable orbits." □

In contrast to Proposition 2.6 in the following result, we do not need to assume that $C_{\psi,\varphi}$ is Cesáro bounded.

Theorem 2.10. *Let ψ and φ be of bounded type such that $\psi\circ\varphi^n \neq 0$ holds for all $n \in \mathbb{N}_0$. Then, $C_{\psi,\varphi} : H_b(B) \to H_b(B)$ is mean ergodic if and only if the following conditions are satisfied.*

(a) $\left((C_{\psi,\varphi})_{[n]} f\right)_n$ is $\sigma(H_b(B), H_b(B)')$-relatively compact for each $f \in H_b(B)$,
(b) φ has B-stable orbits, and
(c) $\lim\limits_{n\to\infty} \frac{\psi_{[n]}}{n} = 0$ on $H_b(B)$.

Proof. Since $H_b(B)$ is barreled, Proposition 1.1 is available and it suffices to show that (b) and (c) are satisfied if and only if

$$\lim_{n\to\infty} \frac{C^n_{\psi,\varphi}(f)}{n} = 0 \quad \text{for every } f \in H_b(B). \tag{7}$$

Observe that Equation (7) gives that the sequence $\left(\frac{C^n_{\psi,\varphi}}{n}\right)_n$ is pointwise bounded in $\mathcal{L}(H_b(B))$ and so it is equicontinuous. Then, for every $0 < r < 1$ there are $c > 0$ and $0 < s < 1$ so that

$$\sup_{\|x\|<r} \left|\frac{C^n_{\psi,\varphi}(f)(x)}{n}\right| \leq c \sup_{\|x\|<s} |f(x)|,$$

for every $f \in H_b(B)$ and $n \in \mathbb{N}$. Just taking $A_n = cn$ in Equation (2) we obtain that $C_{\psi,\varphi}$ is topologizable and Proposition 2.8 yields (b). Now, taking the constant function 1 in Equation (7) immediately gives (c).

Assuming (b) and (c) hold, we obtain Equation (7) as in the first part of Proposition 2.6 replacing "compact" by "B-bounded" and "stable orbits" by "B-stable orbits." □

2.3 | The space of bounded holomorphic functions

We ask again the same questions about the ergodic properties of the weighted composition operator, this time acting on $H^\infty(B)$.

We recall that the operator $C_{\psi,\varphi} : H^\infty(B) \to H^\infty(B)$ is continuous if and only if ψ is a bounded holomorphic function on B (see [7, Proposition 2.2]). Note that, since every continuous linear operator acting on a Banach space is clearly topologizable, in particular, the operator $C_{\psi,\varphi}$ is always topologizable in $H^\infty(B)$.

Again, we can characterize power boundedness for weighted composition operators.

Proposition 2.11. *Let $\psi \in H^\infty(B)$. Then, $C_{\psi,\varphi} : H^\infty(B) \to H^\infty(B)$ is power bounded if and only if $(\psi_{[n]})_n$ is a bounded sequence in $H^\infty(B)$.*

Proof. If we assume that $C_{\psi,\varphi}$ is power bounded, then the sequence $\left(C^n_{\psi,\varphi}(f)\right)_n$ is bounded in $H^\infty(B)$ for every $f \in H^\infty(B)$. Taking the constant function 1 we obtain the claim.

Conversely, let

$$c = \sup_{n\in\mathbb{N}} \sup_{x\in B} \left|\psi_{[n]}(x)\right| < \infty.$$

Then, we have

$$\sup_{x\in B} |C^n_{\psi,\varphi}(f)(x)| \leq \sup_{x\in B} |\psi_{[n]}(x)| \sup_{x\in B} |f(\varphi^n(x))| \leq c \sup_{x\in B} |f(x)|,$$

for all $n \in \mathbb{N}$ and $f \in H^\infty(B)$, which gives that $C_{\psi,\varphi}$ is power bounded. □

The following result is a consequence of Proposition 1.1.

Proposition 2.12. *Let $\psi \in H^\infty(B)$. Then, $C_{\psi,\varphi} : H^\infty(B) \to H^\infty(B)$ is mean ergodic if and only if the following conditions are satisfied:*

(a) $\left((C_{\psi,\varphi})_{[n]} f\right)_n$ *is $\sigma(H^\infty(B), H^\infty(B)')$-relatively compact for each $f \in H^\infty(B)$, and*
(b) $\lim_{n\to\infty} \frac{\psi_{[n]}}{n} = 0$ *in $H^\infty(B)$.*

3 | COMPACTNESS AND ERGODIC PROPERTIES OF THE MULTIPLICATION OPERATOR

As we already noted, the multiplication operator can be seen as a particular case of the weighted composition operator. So, we may use [7] and some results from the previous section to obtain some information on compactness and mean ergodic properties of multiplication operators, when acting on $H(B)$, $H_b(B)$, or $H^\infty(B)$.

Let $\mathcal{H}$ be either $H^\infty(B)$, $H_b(B)$ or $H(B)$. We recall that the operator $M_\psi : \mathcal{H} \to \mathcal{H}$ is continuous if and only if the function ψ is in $\mathcal{H}$ (see [7, Proposition 2.3]).

3.1 | Compactness

We begin with an observation on the invertibility of the multiplication operator.

Lemma 3.1. *Let $\mathcal{H}$ be either $H^\infty(B)$, $H_b(B)$, or $H(B)$. Then, $M_\psi : \mathcal{H} \to \mathcal{H}$ is invertible if and only if $\frac{1}{\psi} \in \mathcal{H}$.*

Proof. Let us suppose that M_ψ is invertible. Then

$$f(x) = M_\psi(M_\psi)^{-1}(f)(x) = \psi(x) \cdot (M_\psi)^{-1}(f)(x),$$

for every $f \in \mathcal{H}$ and $\|x\| < 1$. Applying this equality to the constant function 1 gives $\psi(x) \neq 0$ for every $\|x\| < 1$. Then

$$(M_\psi)^{-1}(f)(x) = \frac{1}{\psi(x)} f(x),$$

for every $f \in \mathcal{H}$ and $x \in B$. This shows that $M_{\frac{1}{\psi}} = (M_\psi)^{-1}$ which, by [7, Proposition 2.3], implies that $\frac{1}{\psi} \in \mathcal{H}$.

The converse implication also follows from [7, Proposition 2.3]. ☐

The *spectrum* of an operator $T : E \to E$, where E is an lcHs space, is denoted by

$$\sigma(T; E) := \{\lambda \in \mathbb{C} : \lambda \cdot \mathrm{id} - T \text{ is not invertible}\}.$$

Proposition 3.2. *Let $\mathcal{H}$ be either $H^\infty(B)$, $H_b(B)$, or $H(B)$. If $\psi \in \mathcal{H}$ then*

$$\psi(B) \subseteq \sigma(M_\psi; \mathcal{H}) \subseteq \overline{\psi(B)}. \tag{8}$$

Moreover, the following hold:

(a) $\sigma(M_\psi; H^\infty(B)) = \overline{\psi(B)}$,
(b) $\sigma(M_\psi; H(B)) = \psi(B)$,
(c) $\sigma(M_\psi; H_b(B)) = \{\lambda \in \mathbb{C} : \inf_{\|x\|<r} |\lambda - \psi(x)| = 0$, *for some* $0 < r < 1\}$.

Proof. In order to prove the lower inclusion in Equation (8) let us take some $x_0 \in B$, and see that the operator $\psi(x_0)\,\mathrm{id}_{\mathcal{H}} - M_\psi$ (on $\mathcal{H}$) is not surjective. Note that $\psi(x_0)\cdot f - M_\psi(f) = (\psi(x_0) - \psi)\cdot f$ for every $f \in \mathcal{H}$ and, then $\psi(x_0)\cdot f(x_0) - M_\psi(f)(x_0) = 0$ for every f. But any non-zero constant function (which belongs to $\mathcal{H}$) cannot have a preimage, giving our claim.

To show the upper inclusion, take $\lambda \notin \overline{\psi(B)}$ and let us show that $\lambda \cdot \mathrm{id}_{\mathcal{H}} - M_\psi$ is invertible. Note first that

$$\lambda \cdot \mathrm{id}_{\mathcal{H}} - M_\psi = M_{\lambda-\psi}. \tag{9}$$

Take $\varepsilon > 0$ such that $|\lambda - \psi(x)| > \varepsilon$ for every $x \in B$, then the function $\frac{1}{\lambda-\psi}$ is bounded and clearly holomorphic. Hence, it belongs to $\mathcal{H}$. Lemma 3.1 gives that $M_{\lambda-\psi}$ is invertible and completes the proof of Equation (8).

Now, we see (a). Since $H^\infty(B)$ is a Banach space the spectrum of any operator acting on $H^\infty(B)$ is closed and the conclusion follows from Equation (8).

For $M_\psi : H(B) \to H(B)$, we consider two cases. If ψ is constant, then $\psi(B) = \overline{\psi(B)}$ and by Equation (8), we have that (b) trivially holds. If ψ is not constant, in view of Lemma 3.1 and Equation (9), it is enough to show that for $\lambda \notin \psi(B)$ the function $\frac{1}{\lambda-\psi}$ is in $H(B)$. To see this, let K be a compact subset of B. Then $\lambda \in \mathbb{C} \setminus \psi(K)$, which is an open set, and there exists $\varepsilon_K > 0$ such that $|\lambda - \psi(x)| > \varepsilon_K$ for every $x \in K$. Hence,

$$\sup_{x\in K} \left| \frac{1}{\lambda - \psi(x)} \right| < \infty,$$

and $\frac{1}{\lambda-\psi}$ is a continuous function which is bounded on every compact subset of B. We conclude that $\frac{1}{\lambda-\psi}$ is holomorphic and this completes the proof of (b).

The proof of (c) follows from Lemma 3.1. □

If the space X is finite dimensional, we have $H_b(B) = H(B)$ and then the spectrum $\sigma(M_\psi; H_b(B)) = \psi(B)$ is fully characterized. However, we do not know what happens in the infinite-dimensional case.

Lemma 3.3. *Let $\mathcal{H}$ be either $H^\infty(B)$, $H_b(B)$, or $H(B)$. If $M_\psi : \mathcal{H} \to \mathcal{H}$ is compact, then $\psi \in \mathcal{H}$ is constant.*

Proof. Since M_ψ is compact, by [13, Chapter 5, Part 2, Theorem 4], the spectrum of M_ψ is either finite or the closure of a null sequence.

Now, assume that ψ is not constant in B. We can find $y_0 \in B$ such that $\psi(0) \neq \psi(y_0)$. Taking $\|y_0\| < r < 1$, by [20, Proposition 5.8] the set $\psi(B(0,r))$ is open in $\mathbb{C}$ and, by Equation (8), it is contained in a discrete set, which is a contradiction. □

As a consequence of the previous results, we can see that there are no non-trivial compact multiplication operators on $H_b(B)$ and $H^\infty(B)$.

Proposition 3.4. *Let $\mathcal{H}$ be either $H_b(B)$ or $H^\infty(B)$. Then, $M_\psi : \mathcal{H} \to \mathcal{H}$ is compact if and only if ψ is 0.*

Proof. Assume $M_\psi : H^\infty(B) \to H^\infty(B)$ is compact. Lemma 3.3 gives $\psi(x) = c \in \mathbb{C}$ for every $\|x\| < 1$. Since we can write the multiplication operator as C_{c,id_B}, by [7, Theorem 3.5], we have

$$0 = \lim_{r\to 1^-} \sup_{\|x\|>r} |\psi(x)| = \lim_{r\to 1^-} \sup_{\|x\|>r} |c| = |c|,$$

which gives the claim.

If $M_\psi : H_b(B) \to H_b(B)$ is compact then, by Lemma 3.3, we have that $\psi(x) = c \in \mathbb{C}$ for every $\|x\| < 1$. Assume $c \neq 0$. Then, the operator C_{c,id_B} is not compact in $H_b(B)$ since $\mathrm{id}_B(B) = B$ is not contained in rB for some $0 < r < 1$ (see [7, Theorem 3.17]). This is a contradiction. □

Let E and F be locally convex Hausdorff spaces. We say that an operator $T : E \to F$ is *Montel* if it maps bounded sets in E into relatively compact sets in F. In particular, every compact operator T is Montel but the converse does not hold in general. We characterize when the operator $M_\psi : H_b(B) \to H_b(B)$ is Montel with a straightforward application of a result from [7].

Proposition 3.5. *Let $\psi \in H_b(B)$. Then $M_\psi : H_b(B) \to H_b(B)$ is Montel if and only if X is finite dimensional.*

Proof. Assume that $M_\psi : H_b(B) \to H_b(B)$ is Montel. Since id_B is holomorphic of bounded type and open, [7, Proposition 3.19] gives that X is finite dimensional.

Conversely, since ψ and id_B are of bounded type we have that the set $(\psi \cdot \mathrm{id}_B)(rB)$ is bounded in X for every $0 < r < 1$. Now if X is finite dimensional the set $(\psi \cdot \mathrm{id}_B)(rB)$ is relatively compact in X for every $0 < r < 1$. An application of [7, Theorem 3.16] finishes the proof. □

3.2 | Ergodic properties

We move now to the study of ergodic properties, following [9]. We observe that the iterates of M_ψ are given by

$$M_\psi^n(f) = \psi^n \cdot f = M_{\psi^n}(f),$$

where ψ^n now denotes the nth power of the function instead of the nth composition.

Remark 3.6. Let $\psi \in \mathcal{H}$, where $\mathcal{H}$ is either $H(B)$, $H_b(B)$ or $H^\infty(B)$. The map $\mathrm{id}_B : B \to B$ clearly has stable orbits and B-stable orbits. Then, by Propositions 2.4 and 2.8 the operator $M_\psi : \mathcal{H} \to \mathcal{H}$ is always topologizable.

In order to characterize power boundedness of M_ψ, it is enough to characterize when the set $(\psi^n)_n$ is bounded in $H(B)$, $H_b(B)$, and $H^\infty(B)$ (see Proposition 2.5, Theorem 2.9, and Proposition 2.11). Then, as in [9, Proposition 2.3], the operator M_ψ is power bounded if and only if $\psi(B) \subseteq \overline{\mathbb{D}}$.

For the study of the mean ergodicity of M_ψ we first make an observation. If $M_\psi : \mathcal{H} \to \mathcal{H}$ is mean ergodic then, $\big((M_\psi)_{[n]}(f)(x)\big)_n$ is convergent for every $f \in \mathcal{H}$, $x \in B$. A simple computation gives

$$(M_\psi)_{[n]}(f)(x) = \frac{f(x)}{n} \sum_{m=0}^{n-1} \psi(x)^m, \tag{10}$$

for every $f \in \mathcal{H}$, $x \in B$. Now, if there is $x_0 \in B$ such that $|\psi(x_0)| > 1$ it is enough to take the constant function $f = 1$ to obtain that $\big((M_\psi)_{[n]}(f)(x_0)\big)_n$ is not convergent. Hence, M_ψ is not mean ergodic. This altogether gives the following result.

Proposition 3.7. *Let $\psi \in \mathcal{H}$. If $M_\psi : \mathcal{H} \to \mathcal{H}$ is mean ergodic, then $\psi(B) \subseteq \overline{\mathbb{D}}$. In particular, M_ψ is power bounded.*

As a straightforward consequence of these results, together with [5, Proposition 3.3] (see also [15, Proposition 3.1]) we have the following characterization:

Corollary 3.8. *Let $\psi \in H(B)$. Consider $M_\psi : H(B) \to H(B)$. Then, the following are equivalent:*

(a) M_ψ is power bounded.
(b) M_ψ is uniformly mean ergodic.
(c) M_ψ is mean ergodic.
(d) $\psi(B) \subseteq \overline{\mathbb{D}}$.

Now, observe that if $\psi(B) \subseteq \overline{\mathbb{D}}$ holds and there is $x_0 \in B$ such that $\psi(x_0) = 1$, then by the maximum principle (see [20, Proposition 5.9]) we obtain $\psi = 1$. In particular, $M_\psi = M_1 = \mathrm{id}_{\mathcal{H}}$ which is obviously uniformly mean ergodic and mean ergodic. That is, if M_ψ is mean ergodic and nontrivial $\psi(B) \subseteq \overline{\mathbb{D}} \setminus \{1\}$ holds. Avoiding this case we obtain a similar result to [9, Proposition 2.8].

Theorem 3.9. *Let $\mathcal{H}$ be either $H_b(B)$ or $H^\infty(B)$. Let $\psi \in \mathcal{H} \setminus \{1\}$. Then, the following are equivalent:*

(a) M_ψ is uniformly mean ergodic in $\mathcal{H}$.
(b) M_ψ is mean ergodic in $\mathcal{H}$.
(c) $\psi(B) \subseteq \overline{\mathbb{D}} \setminus \{1\}$ and $\frac{1}{1-\psi} \in \mathcal{H}$.

In particular, for $\mathcal{H} = H^\infty(B)$ condition (c) is equivalent to $\psi(B) \subseteq \overline{\mathbb{D}} \setminus \{1\}$ and $1 \notin \overline{\psi(B)}$.

Proof. That (a) implies (b) is trivial.

Assume now that (b) holds. Since $\mathcal{H}$ is barreled we can apply [1, Theorem 2.4] to obtain

$$\mathcal{H} = \ker(\mathrm{id}_{\mathcal{H}} - M_\psi) \oplus \overline{\mathrm{im}(\mathrm{id}_{\mathcal{H}} - M_\psi)} = \ker(M_{1-\psi}) \oplus \overline{\mathrm{im}(M_{1-\psi})}.$$

As we have just mentioned $\psi(B) \subseteq \overline{\mathbb{D}} \setminus \{1\}$ holds, which implies $\ker(M_{1-\psi}) = \{0\}$. Then, $\mathcal{H} = \overline{\mathrm{im}(M_{1-\psi})}$ and, by [7, Proposition 2.3] the operator $M_{1-\psi}$ is continuous. The closed graph theorem implies

$$\mathcal{H} = \mathrm{im}(M_{1-\psi}).$$

In other words, $M_{1-\psi} = \mathrm{id}_{\mathcal{H}} - M_\psi$ is an isomorphism of $\mathcal{H}$ onto itself. In particular, it is invertible and, by Lemma 3.1, we have $\frac{1}{1-\psi} \in \mathcal{H}$.

Assume (c) is satisfied for $\mathcal{H} = H_b(B)$. Let $V \subset H_b(B)$ be a bounded set and fix $0 < r < 1$. Then, by Equation (10), we have

$$\begin{aligned}\sup_{f\in V}\sup_{x\in rB}\left|(M_\psi)_{[n]}(f)(x)\right| &= \sup_{f\in V}\sup_{x\in rB}\left|\frac{f(x)}{n}\frac{1-\psi(x)^n}{1-\psi(x)}\right| \\ &\leq \left(\sup_{f\in V}\sup_{x\in rB}|f(x)|\sup_{x\in rB}\left|\frac{1}{1-\psi(x)}\right|\right)\frac{2}{n},\end{aligned}$$

which clearly tends to 0 as n tends to ∞. Hence, M_ψ is uniformly mean ergodic on $H_b(B)$. For $\mathcal{H} = H^\infty(B)$, the proof of (c) implies (a) follows the same steps, replacing the sets rB with the whole open unit ball B. Whenever $\mathcal{H} = H^\infty(B)$ see Lemma 3.1 and Proposition 3.2 for the equivalent statement of (c). □

Observe that condition (c) for $\mathcal{H} = H^\infty(B)$ gives that power boundedness does not imply mean ergodicity (see Example 4.4). If we consider (c) for $\mathcal{H} = H(B)$ we can see that it is equivalent to $\psi(B) \subseteq \overline{\mathbb{D}} \setminus \{1\}$ (see Proposition 3.2) or, simply, $\psi(B) \subseteq \overline{\mathbb{D}}$ if we take ψ different from the constant function 1. This explains why the notions of (uniform) mean ergodicity and power boundedness coincide in $H(B)$ (see Corollary 3.8). However, the situation in $H_b(B)$ is not totally clear. Note that the spectrum of M_ψ in $H_b(B)$ is not completely described in terms of $\psi(B)$, thus the question of whether or not every power-bounded multiplication operator $M_\psi : H_b(B) \to H_b(B)$ is mean ergodic remains open.

4 | EXAMPLES

Examples on ergodic properties of $C_{\psi,\varphi}$ can be found in [16] for the special case of $\psi \equiv 1$. Here, we present a composition operator that is power bounded in $H_b(B_{c_0})$, $H^\infty(B_{c_0})$ and on $H(B_{c_0})$, but when taking a particular weight, the weighted composition operator is not power bounded in any of these spaces.

Example 4.1. Let $w \in \mathcal{H}$ defined by $w(x) = 1 + x_1$ (where $\mathcal{H}$ is either $H^\infty(B_{c_0})$, $H_b(B_{c_0})$, or $H(B_{c_0})$). Consider $\Sigma : B_{c_0} \to B_{c_0}$ the usual backward shift given by

$$\Sigma(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots). \tag{11}$$

Then, $C_\Sigma : \mathcal{H} \to \mathcal{H}$ is power bounded but the operator $C_{w,\Sigma} : \mathcal{H} \to \mathcal{H}$ is not power bounded.

Since Σ has stable orbits and B_{c_0}-stable orbits, $C_\Sigma : \mathcal{H} \to \mathcal{H}$ is power bounded (see [16, Theorems 4.3 and 5.2]. To see that $C_{w,\Sigma} : \mathcal{H} \to \mathcal{H}$ is not power bounded it is enough to show that the set $(w^{[n]})_n$ is not bounded in $\mathcal{H}$ (see Proposition 2.5, Theorem 2.9, and Proposition 2.11). Consider $y = \left(\frac{1}{i+1}\right)_i \in B_{c_0}$, then for any $n \in \mathbb{N}$ we have

$$\begin{aligned} w_{[n]}(y) &= \prod_{m=0}^{n-1} w(\Sigma^m(y)) = \prod_{m=0}^{n-1} w\left(\left(\frac{1}{i+1+m}\right)_i\right) = \prod_{m=0}^{n-1}\left(1 + \frac{1}{2+m}\right) \\ &= \prod_{m=0}^{n-1}\left(\frac{3+m}{2+m}\right) = \frac{n+2}{2}. \end{aligned}$$

And the set $(w^{[n]})_n$ is unbounded in $\mathcal{H}$.

Examples 4.2. The following composition operators are not power bounded but when we take a particular ψ, the operators $C_{\psi,\varphi}$ become power bounded.

(a) Consider $\phi : B_{c_0} \to B_{c_0}$ defined by

$$\phi(x) = \left(\frac{x_1}{2} + \frac{1}{2}, 0, 0, \dots\right)$$

and the weight $\psi(x) = x_2$. Then, $C_{\psi,\phi} : H_b(B_{c_0}) \to H_b(B_{c_0})$ is power bounded, but $C_\phi : H_b(B_{c_0}) \to H_b(B_{c_0})$ is not. In fact, $\psi \circ \phi \equiv 0$ and Remark 2.2 gives that the iterates of $C_{\psi,\phi}$ are eventually 0. On the other hand, observe that for any $x \in B_{c_0}$ the sequence $(\phi^n(x))_n$ converges to $(1, 0, 0, \dots)$. Then, ϕ does not have B_{c_0}-stable orbits (see [16, Section 3]). By [16, Theorem 5.2], we obtain that C_ϕ is not power bounded.

(b) Consider $F : B_{c_0} \to B_{c_0}$ the usual forward shift given by

$$F(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots) \tag{12}$$

and the weight $\psi(x) = x_2$. Then, $C_{\psi,F} : H(B_{c_0}) \to H(B_{c_0})$ is power bounded, but $C_F : H(B_{c_0}) \to H(B_{c_0})$ is not. In fact, $\psi \circ F^2 \equiv 0$ but F does not have stable orbits on B_{c_0} (see Remark 2.2, [16, Example 3.4], and [16, Theorem 5.2]).

Note that in both examples the symbol is not an open map (recall Remark 2.2).

It is not difficult to construct multiplication operators that are neither mean ergodic nor power bounded, but when taking a particular symbol the associated weighted composition operator satisfies both properties.

Example 4.3. Let $\varphi : \mathbb{D} \to \mathbb{D}$ and $\psi \in H^\infty(\mathbb{D})$ defined by

$$\varphi(z) = \frac{z}{2} \quad \text{and} \quad \psi(z) = 2z.$$

Then, M_ψ is not power bounded nor mean ergodic in $H^\infty(\mathbb{D})$ or $H(\mathbb{D})$ but $C_{\psi,\varphi}$ is power bounded and mean ergodic in $H^\infty(\mathbb{D})$ and $H(\mathbb{D})$. We check these facts first.

Observe that $\|\psi\|_\infty > 1$. Then, M_ψ cannot be power bounded nor mean ergodic (see Proposition 3.7).

Clearly, φ has $\mathbb{D}$-stable orbits. Now, we check that the set $(\psi_{[n]})_n$ is bounded in $H^\infty(\mathbb{D})$ and therefore bounded in $H(\mathbb{D})$. Indeed, we have

$$\begin{aligned} |\psi_{[n]}(z)| &= \left| \prod_{m=0}^{n-1} (\psi \circ \varphi^m)(z) \right| = \left| \prod_{m=0}^{n-1} 2(z \cdot 2^{-m}) \right| \\ &= \left| z^n 2^{\sum_{m=0}^{n-1} 1-m} \right| \leq |z^n| 2^{3-n} \leq 2^{3-n}, \end{aligned} \tag{13}$$

for all $z \in \mathbb{D}$. By Theorem 2.9 and Proposition 2.11 we obtain the power boundedness of $C_{\psi,\varphi}$. Additionally, Equation (13) implies that $\lim_{n\to\infty} \|\psi_{[n]}\|_\infty = 0$. For any $0 < r < 1$ since $\varphi^n(r\mathbb{D}) \subseteq r\mathbb{D}$ for every $n \in \mathbb{N}$, for an arbitrary $f \in H(\mathbb{D})$ we have

$$\lim_{n\to\infty} \sup_{|z|<r} |\psi_{[n]}(z) \cdot f(\varphi^n(z))| \leq \sup_{|z|<r} |f(z)| \lim_{n\to\infty} \|\psi_{[n]}\|_\infty = 0.$$

Then, $C^n_{\psi,\varphi}$ converges pointwise to 0 in $H(\mathbb{D})$ and $C_{\psi,\varphi}$ is mean ergodic in $H(\mathbb{D})$. The mean ergodicity of $C_{\psi,\varphi}$ in $H^\infty(\mathbb{D})$ can be checked with an analogous procedure.

There are weighted composition operators that are power bounded and not mean ergodic in $H^\infty(B)$. In fact, we give an example of multiplication operator on $H^\infty(\mathbb{D})$ with this property.

Example 4.4. Let $M_\psi : H^\infty(\mathbb{D}) \to H^\infty(\mathbb{D})$ be given by $\psi(z) = \frac{1+z}{2}$.

Then, M_ψ is power bounded but not (uniformly) mean ergodic because $1 \in \overline{\psi(\mathbb{D})}$ (see Theorem 3.9).

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