Selection rules for charged lepton flavor violating processes from residual flavor groups

Lorenzo Calibbi,[1,*] Claudia Hagedorn,[2,†] Michael A. Schmidt,[3,‡] and James Vandeleur[3,§]

[1]*School of Physics, Nankai University, Tianjin 300071, China*

[2]*Instituto de Física Corpuscular, Universitat de València—CSIC, Parc Científic, Catedrático José Beltrán 2, 46980 Paterna, Spain*

[3]*Sydney Consortium for Particle Physics and Cosmology, School of Physics, The University of New South Wales, Sydney, New South Wales 2052, Australia*

(Received 10 June 2025; accepted 8 August 2025; published 17 October 2025)

We systematically investigate the possible phenomenological impact of residual flavor groups in the charged lepton sector. We consider all possible flavor charge assignments for Abelian residual symmetries up to $\mathbb{Z}_8$. The allowed flavor structures of operators in Standard Model effective field theory (up to dimension six) lead to distinctive and observable patterns of charged lepton flavor violating processes. We illustrate the relevance of such selection rules displaying the current bounds on and the future sensitivities to the new physics scale. These results demonstrate, in particular, the importance and discriminating power of searches for lepton flavor violating τ lepton decays and muonium to antimuonium conversion.

DOI: 10.1103/b6gz-c1c9

I. INTRODUCTION

The Standard Model (SM) has been very successful in describing and precisely accounting for a wide range of phenomena in particle physics and cosmology. Nevertheless, it has several shortcomings that call for its extension: fermion masses and mixing can only be adjusted, but not explained, a mechanism for generating neutrino masses is missing, the baryon asymmetry of the Universe cannot be correctly accommodated, and a viable dark matter candidate is absent.

Discrete flavor symmetries (also in combination with a CP symmetry) have been successfully applied to the lepton sector in particular in order to explain the observed fermion mixing; see, e.g., [1–5] for reviews. A particularly predictive approach assumes that in the charged and neutral lepton sector different residual symmetries, G_ℓ and G_ν, are preserved, whose mismatch corresponds to lepton mixing. Such residual symmetries are Abelian in general, since non-Abelian groups would impose too strong constraints (for example, two exactly degenerate fermion masses). Well-known examples are the flavor symmetries A_4 and S_4 that can predict tri-bi-maximal lepton mixing (see, e.g., [6–8]), if the residual symmetries are $G_\ell = \mathbb{Z}_3$ and $G_\nu = \mathbb{Z}_2 \times \mathbb{Z}_2$.[1] The observation of the reactor mixing angle being of the size of the Cabibbo angle, $\sin^2\theta_{13} \approx 0.022$ [9], requires modifications to this result, and numerous dedicated studies can be found in the literature; see for instance [10–13].

Assuming the existence of a residual symmetry G_ℓ among charged leptons, it is interesting to consider its further phenomenological consequences. As has already been pointed out [14–24], the residual $\mathbb{Z}_3$ symmetry, usually called lepton triality, constrains possible signals of charged lepton flavor violation (cLFV). In A_4 and S_4 models, the charged lepton flavors, e, μ, and τ, typically transform differently under $G_\ell = \mathbb{Z}_3$, e.g., $e \sim 1$, $\mu \sim \omega$, and $\tau \sim \omega^2$ with ω being the third root of unity which corresponds to e having flavor charge 0, μ charge 1, and τ charge 2 modulo 3. Then, radiative cLFV processes such as $\mu \to e\gamma$ are forbidden in the limit of an exact residual symmetry, while the two trilepton decays $\tau \to ee\bar{\mu}$ and $\tau \to \mu\mu\bar{e}$ are compatible with $G_\ell = \mathbb{Z}_3$.

This example shows how residual symmetries can lead to the existence of certain selection rules that leave an observable imprint on the pattern of cLFV processes. This observation is of particular interest given that cLFV is a general prediction of extensions of the SM that address neutrino masses and other unexplained phenomena, while

*Contact author: calibbi@nankai.edu.cn
†Contact author: claudia.hagedorn@ific.uv.es
‡Contact author: m.schmidt@unsw.edu.au
§Contact author: james.vandeleur@unsw.edu.au

[1]In the case of A_4 the second $\mathbb{Z}_2$ symmetry is accidental and not contained in the original flavor group.

at the same time it is subject to stringent experimental constraints; see, e.g., [25–27]. The existence of selection rules could entail that the discovery of cLFV signals does not occur through the processes most searched for. On the other hand, this highlights the power of cLFV searches to discriminate among different residual flavor symmetries and different new physics models more generally. There is the exciting possibility that present and future experiments are capable of unveiling the underlying organizing principle in the charged lepton sector. For earlier discussions along these lines—not necessarily related to flavor symmetries—see, e.g., [28–34].

In this work, we analyze cLFV processes compatible with different residual symmetries G_ℓ among the charged leptons and different possible flavor charge assignments of the flavors e, μ, and τ. Concretely, we study all cyclic groups $\mathbb{Z}_N$ with $N = 2, \ldots, 8$ and all possible flavor charge combinations, including those for which two of the lepton flavors transform in the same way. First, we systematically construct and classify all possible flavor structures (with up to 16 charged lepton fields) that are compatible with the imposed residual symmetry. Then, we apply these flavor structures to SM effective field theory (SMEFT) operators [35–39] and derive phenomenological implications and constraints. In a first step, we work out which cLFV processes, considering up to dimension-six SMEFT operators, are allowed and compare the results among different residual symmetries $G_\ell = \mathbb{Z}_N$, highlighting which patterns of observables could provide evidence for a certain G_ℓ and, potentially, also flavor charge assignment. We only include operators that conserve the (total) lepton number, $\Delta L = 0$, since, in general, both the mechanism and the scale of lepton number violation can differ from those of (charged) lepton flavor violation. In a second step, we set the coefficients of all permitted operators to values motivated by different types of ultraviolet (UV) theories, derive lower limits on the new physics scale Λ using existing experimental bounds, and show the future reach in Λ based on prospective sensitivities. This enables us to assess the role of different cLFV observables in testing the discussed selection rules within the context of the present and future experimental landscape.

The remainder of this paper is organized as follows. In Sec. II we give details about the motivation of this study. Section III is dedicated to the systematic listing of the flavor structures allowed for the different groups G_ℓ and the different flavor charge assignments. Section IV comprises the results for allowed and forbidden cLFV processes for the different residual symmetries $G_\ell = \mathbb{Z}_N$. In Sec. V we use these results and make various assumptions regarding the size of the coefficients of the allowed SMEFT operators, in order to estimate present and future limits on the new physics scale Λ. We summarize and give a brief outlook in Sec. VI. The appendixes contain technical details: a demonstration of the completeness of the study (Appendix A), the list of the flavor structures for G_ℓ larger than $\mathbb{Z}_5$ (Appendix B), all possible flavor charge assignments for $G_\ell = \mathbb{Z}_3$ (Appendix C), as well as the present and future experimental limits employed in the analysis and numerical bounds on Λ (Appendixes D and E).

II. MOTIVATION

In this section, we motivate different choices of the residual symmetry G_ℓ and comment on possible flavor charge assignments.

Typically, the discrete groups employed as flavor symmetries (and thus also their subgroups) are subgroups of $SU(3)$; see [5]. This constrains the possible combinations of flavor charges, as their sum must equal zero modulo N for $G_\ell = \mathbb{Z}_N$. In this study, we do not impose this constraint but do highlight the instances in which the flavor charge assignment can correspond to the action of $\mathbb{Z}_N$ being represented by a special unitary matrix; see Sec. III B.

When explaining fermion mixing as the mismatch of two distinct residual symmetries of a (discrete) flavor group, the predictive power is maximized if the three fermion generations are distinguished by their charge under $G_\ell = \mathbb{Z}_N$, since otherwise at least one free parameter enters the mixing matrix. Indeed, the choice $G_\ell = \mathbb{Z}_3$ is the minimal one that can ensure the absence of such a free parameter. We do not impose this as a constraint in the following, and consequently, we include $G_\ell = \mathbb{Z}_2$ in the list of residual symmetries and also permit combinations of flavor charges for which two lepton flavors have the same charge.

The group $G_\ell = \mathbb{Z}_3$ is not only the minimal possible choice for maximizing the predictive power regarding fermion mixing; it is also frequently the residual symmetry in the charged lepton sector when considering different flavor symmetries and deriving lepton mixing. Indeed, many viable lepton mixing patterns can be obtained from members of the series $\Delta(3n^2)$ [40] and $\Delta(6n^2)$ [41], potentially combined with CP, where $G_\ell = \mathbb{Z}_3$, while the residual symmetry G_ν is a Klein group (or the direct product of $\mathbb{Z}_2$ and CP); for studies on lepton mixing, see, e.g., [42–45].

An example with $G_\ell = \mathbb{Z}_4$ can be obtained from the flavor group S_4, where, together with $G_\nu = \mathbb{Z}_2 \times \mathbb{Z}_2$, it gives rise to bimaxmial mixing [46]. Similarly, for the flavor symmetry A_5 (and CP) the residual symmetries $G_\ell = \mathbb{Z}_5$ and $G_\nu = \mathbb{Z}_2 \times \mathbb{Z}_2$ (or $G_\nu = \mathbb{Z}_2 \times CP$) can lead to the so-called golden ratio mixing [47] (or golden ratio type, if a CP symmetry is also involved [48–50]). The residual symmetry $G_\ell = \mathbb{Z}_7$ instead can play a role in scenarios with the flavor group $PSL(2,7)$ [also called $\Sigma(168)$]; see, e.g., [46]. Further examples of residual symmetries G_ℓ are encountered, e.g., in the analysis of the flavor groups $\Sigma(n\varphi)$ [51]. Hence, it is well motivated to study the phenomenology of residual symmetries in the charged lepton sector other than lepton triality. Still, residual symmetries $G_\ell = \mathbb{Z}_N$ with N small usually occur, and it is, thus, reasonable to only consider $N \leq 8$.

One may also employ direct products of cyclic symmetries as G_ℓ. However, they do not lead to new results as the impact of $G_\ell = \mathbb{Z}_2 \times \mathbb{Z}_2$ can be deduced from the combination of the results for $G_\ell = \mathbb{Z}_2$, and similarly for $G_\ell = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ as well as $G_\ell = \mathbb{Z}_2 \times \mathbb{Z}_4$; see comments in Sec. III E. The group $\mathbb{Z}_2 \times \mathbb{Z}_3$ is isomorphic to $\mathbb{Z}_6$, and therefore the results corresponding to this choice are also covered in the present study.

As happens in models implementing the idea that the mismatch of the residual symmetries G_ℓ and G_ν encodes lepton mixing, none of the residual symmetries are exact. In general, G_ℓ is broken, e.g., by (small) shifts in the alignment of the vacuum of the flavor symmetry breaking fields as well as by contributions of higher-dimensional operators to the charged lepton sector that involve fields that break the original flavor symmetry to G_ν; see, e.g., [6]. Consequently, we expect that a process forbidden by G_ℓ is not strictly absent in a concrete model, but rather has a signal strength that is (highly) suppressed compared to those of processes that are compatible with G_ℓ. In the case of lepton triality, for example, a signal of $\tau \to \mu\mu\bar{\mu}$ should be suppressed compared to $\tau \to ee\bar{\mu}$—see, e.g., [20].

Eventually, we point out that in concrete realizations the branching ratios of two processes compatible with G_ℓ may have different sizes, e.g., the branching ratio of $\tau \to \mu\mu\bar{e}$ is much larger than that of $\tau \to ee\bar{\mu}$ in the supersymmetric version of the well-known A_4 model [21]. Nevertheless, this conclusion depends on whether the flavor symmetry breaking fields mix or not, as shown in [20]. Such a mixing can be achieved via the inclusion of certain soft supersymmetry breaking terms.

In the current study, we do not consider explicit models, and thus we do not take into account corrections to the breaking of the residual symmetry G_ℓ in the charged lepton sector, nor effects that can induce a hierarchy among the signal strengths of different processes allowed by G_ℓ. Rather, we focus on the possible constraints on cLFV processes arising from G_ℓ different from $\mathbb{Z}_3$ and for all possible flavor charge assignments.

III. FLAVOR CHARGE ASSIGNMENTS AND FLAVOR STRUCTURES

We begin by systematically listing the flavor structures of operators with charged lepton fields which are compatible with the (nontrivial) flavor charge assignments for the different choices of G_ℓ.[2] With this at hand, we construct in Sec. IV all allowed SMEFT operators up to dimension six and categorize the cLFV processes that are permitted by a given G_ℓ and flavor charge assignment. Thus, in this section, the Lorentz and other structures of the operators are suppressed so that only their lepton flavor structure is specified. For example, we have the following correspondences:

$$(\bar{\ell}_\mu \gamma_\nu \ell_e)(\bar{\ell}_\mu \gamma^\nu \ell_e) \to ee\mu^\dagger\mu^\dagger \quad \text{and} \quad (\varphi^\dagger\varphi)(\bar{\ell}_\tau e_e \varphi) \to e\tau^\dagger, \tag{1}$$

where ℓ_i is a left-handed lepton doublet of flavor i; e_j a right-handed charged lepton of flavor j; φ denotes the Higgs field; and e, μ, and τ are the three lepton flavors.

A. Constraints on number of charged leptons and flavor charges

Consider an operator that conserves the number of charged leptons and can contain any number of the three flavors of the charged lepton fields. The *individual charged lepton numbers* must obey the condition

$$\begin{aligned}(n_e^- - n_e^+) + (n_\mu^- - n_\mu^+) + (n_\tau^- - n_\tau^+) \\ \equiv \Delta n_e + \Delta n_\mu + \Delta n_\tau = 0,\end{aligned} \tag{2}$$

where $n^-_{e,\mu,\tau}$, $n^+_{e,\mu,\tau}$ are respectively the number of particle and antiparticle fields of charged lepton flavor e, μ, and τ and $\Delta n_{e,\mu,\tau}$ is the change in the corresponding individual charged lepton number.

Similarly, for e, μ, and τ carrying flavor charge α, β, and γ under $G_\ell = \mathbb{Z}_N$, the conservation of *lepton flavor charge* imposes the condition

$$\alpha\Delta n_e + \beta\Delta n_\mu + \gamma\Delta n_\tau = 0 \mod N. \tag{3}$$

These flavor charges are defined modulo N. Any operator with individual charged lepton numbers satisfying Eqs. (2) and (3) is permitted by the residual symmetry G_ℓ and the flavor charge assignment.

B. Notation of flavor charge assignments

We represent an assignment of lepton flavor charges under $G_\ell = \mathbb{Z}_N$ as

$$\mathbb{Z}_N(\alpha, \beta, \gamma), \tag{4}$$

with α, β, and γ being the flavor charges for e, μ, and τ, respectively. Using the two constraints in Eqs. (2) and (3), we can reduce the labeling of the flavor charge assignment from three to two independent parameters. An allowed flavor structure must satisfy Eq. (3),

$$\begin{aligned}0 \mod N &= \alpha\Delta n_e + \beta\Delta n_\mu + \gamma\Delta n_\tau \\ &= \alpha(\Delta n_e + \Delta n_\mu + \Delta n_\tau) + \delta_1\Delta n_\mu + (\delta_1 + \delta_2)\Delta n_\tau,\end{aligned} \tag{5}$$

with $\delta_1 \equiv \beta - \alpha$ and $\delta_2 \equiv \gamma - \beta$ defined as the differences between the assigned flavor charges. Taking into account

[2]We assume that only charged leptons and no other SM particles are charged under G_ℓ.

the conservation of the number of charged leptons [see Eq. (2)], we arrive at

$$0 \mod N = \delta_1 \Delta n_\mu + (\delta_1 + \delta_2)\Delta n_\tau. \tag{6}$$

This leaves only a constraint on the parameters δ_1 and δ_2, while also being independent of Δn_e. Note that the particular permutation of the charged lepton flavors is arbitrary. However, one of the differences of individual charged lepton numbers is determined in terms of the other two by means of conservation of the charged lepton number; see Eq. (2). Therefore, we introduce the following reduced notation:

$$N(\delta_1, \delta_2). \tag{7}$$

It can be derived from the notation used in Eq. (4), e.g., for $\mathbb{Z}_3(0,1,2)$, that we have

$$\delta_1 = \beta - \alpha = 1 - 0 = 1, \quad \delta_2 = \gamma - \beta = 2 - 1 = 1, \tag{8}$$

leading to

$$\mathbb{Z}_3(0,1,2) \to 3(1,1). \tag{9}$$

Importantly, the reduced notation, in general, does not uniquely specify a flavor charge assignment. For example, several of the latter correspond to 3(0, 1), i.e.,

$$\left.\begin{matrix} \mathbb{Z}_3(0,0,1) \\ \mathbb{Z}_3(1,1,2) \\ \mathbb{Z}_3(2,2,0) \end{matrix}\right\} \to 3(0,1). \tag{10}$$

Although the reduced notation in this example represents three different flavor charge assignments, they all permit the same flavor structures of operators.

If the three flavor charges sum to zero modulo N, the action of the $\mathbb{Z}_N$ symmetry on the charged leptons can be represented by a special unitary matrix; see Sec. II. From this point onward, we highlight these instances by displaying the flavor charge assignment in boldface. For example, $\mathbf{4}(\mathbf{0},\mathbf{1})$ corresponds to $\mathbb{Z}_\mathbf{4}(\mathbf{1},\mathbf{1},\mathbf{2})$. Note, however, that $\mathbf{4}(\mathbf{0},\mathbf{1})$ also comprises the flavor charge assignment $\mathbb{Z}_4(0,0,1)$, whose flavor charges do not sum to zero modulo four. The reduced notation cannot separate these cases. Indeed, the constraint on the flavor charges for summing to zero is

$$3\alpha + 2\delta_1 + \delta_2 = 0 \mod N, \tag{11}$$

which not only depends on the parameters δ_1 and δ_2, but also explicitly on the flavor charge of the electron, α. Unless three divides N, 3α can take any value modulo N, and thus there is always an assignment of the three lepton flavor charges that sum to zero modulo N.

C. Equivalences and redundancies in notation

In the case two or more flavor charge assignments permit the same flavor structures, we consider these assignments equivalent. We have already shown that the reduced notation $N(\delta_1, \delta_2)$ represents a set of equivalent flavor charge assignments. However, there are further possibilities for such an equivalence. In the following, our choice of conventions is emphasized in *italics*.

1. Common factors

Inspecting the constraint in Eq. (6), it is clear that applying a common scaling factor to δ_1, δ_2, and N does not change the constraint. Consequently, two flavor charge assignments allow the same flavor structures, if they are related by

$$N'(\delta_1', \delta_2') \leftrightarrow AN(A\delta_1, A\delta_2), \tag{12}$$

with A being a rational number.

The constraint in Eq. (6) is also invariant under the scaling of just δ_1 and δ_2 by a nonzero integer B. The important difference between these two scalings is that A is always invertible, and thus there is a (two-way) equivalence between the flavor charge assignments. On the contrary, B can only be inverted in particular cases, meaning the implication only applies in one direction in general. For example,

$$\mathbf{4}(\mathbf{0},\mathbf{2}) = \mathbf{4}(2 \times \mathbf{0}, 2 \times \mathbf{1}) \tag{13}$$

with $B = 2$ implies that flavor structures allowed by the flavor charge assignment $\mathbf{4}(\mathbf{0},\mathbf{1})$ are also allowed by $\mathbf{4}(\mathbf{0},\mathbf{2})$. However, there is no requirement for all flavor structures permitted by $\mathbf{4}(\mathbf{0},\mathbf{2})$ to be allowed by $\mathbf{4}(\mathbf{0},\mathbf{1})$. In particular, the flavor structure

$$\begin{aligned} ee\tau^\dagger\tau^\dagger \text{ is allowed by } &\mathbb{Z}_4(0,0,2) \to \mathbf{4}(\mathbf{0},\mathbf{2}), \\ \text{but not by } &\mathbb{Z}_4(0,0,1) \to \mathbf{4}(\mathbf{0},\mathbf{1}). \end{aligned} \tag{14}$$

Two flavor charge assignments related by a scaling of δ_1 and δ_2 are only equivalent in cases where this scaling can be inverted, while taking into account the definition of the flavor charges modulo N. Indeed, this inversion is always possible for N prime. For N not prime, the scaling can only be inverted, if B and N are coprime. For example, the flavor charge assignments $\mathbf{5}(\mathbf{1},\mathbf{1})$ and $\mathbf{5}(\mathbf{2},\mathbf{2})$ allow the same set of flavor structures (with $B = 2$ and the inverse scaling factor $B' = 3$), as do $\mathbf{8}(\mathbf{0},\mathbf{1})$ and $\mathbf{8}(\mathbf{0},\mathbf{3})$ (with $B = 3$ and $B' = 3$); however, as argued, $\mathbf{4}(\mathbf{0},\mathbf{1})$ and $\mathbf{4}(\mathbf{0},\mathbf{2})$ do not. *We always use the smallest values of* N, δ_1, *and* δ_2 *that represent a set of equivalent flavor charge assignments.*

2. Permutations

The choice of the labels e, μ, and τ is somewhat arbitrary. Although, for example, $\mathbb{Z}_\mathbf{4}(\mathbf{0},\mathbf{1},\mathbf{3}) \to \mathbf{4}(\mathbf{1},\mathbf{2})$ and $\mathbb{Z}_\mathbf{4}(\mathbf{1},\mathbf{3},\mathbf{0}) \to \mathbf{4}(\mathbf{2},\mathbf{1})$ lead to different sets of allowed flavor structures, they only differ by a permutation of the charged lepton flavors, i.e., $\{e,\mu,\tau\}$ should be traded for $\{\tau,e,\mu\}$. Where they are not important, we suppress permutations of the flavor charges. *By convention, we denote the flavor charge assignments with lowest lexicographic ordering, i.e., with* $\delta_1 \le \delta_2$.

3. Hermitian and complex conjugation

If an operator with a given flavor structure is allowed, its Hermitian conjugate must be as well, implying a sign change of the changes in individual charged lepton numbers $\Delta n_{e,\mu,\tau}$. This amounts to an overall sign change in Eqs. (2) and (3), but does not alter the equalities. So, whenever we list the allowed flavor structures for a certain flavor charge assignment, it should be understood that *operators arising from Hermitian conjugation are implicitly included*.

At the same time, it is clear that using the complex conjugate of the flavor charges α, β, and γ, i.e., $N-\alpha$, $N-\beta$, $N-\gamma$, leads to the same allowed flavor structures.

D. Systematic study of flavor structures

In the following, we systematically study all flavor structures that are allowed for the different flavor charge assignments and for different $G_\ell = \mathbb{Z}_N$.

For each G_ℓ, all possible flavor charge assignments are given by $\mathbb{Z}_N(\alpha,\beta,\gamma)$, where $0 \le \alpha,\beta,\gamma \le N-1$. However, it suffices to only search for the allowed flavor structures of a single permutation of the flavor charges α, β, γ and then permute the labels of the charged leptons in order to obtain the results for the remaining permutations. Thus, we can restrict the search to[3]

$$0 \le \alpha \le \beta, \qquad 0 \le \beta \le \gamma, \qquad 0 \le \gamma \le N-1. \tag{15}$$

In the next step, we consider the flavor structures. Each flavor structure corresponds to a list of the changes in individual charged lepton numbers, $\{\Delta n_e, \Delta n_\mu, \Delta n_\tau\}$. Ignoring trivial flavor structures, such as $ee^\dagger$, since these do not have an effect on the change in individual lepton number, a (negative) positive value of $\Delta n_{e,\mu,\tau}$ corresponds to a net number of (anti)particles. For example, $\{\Delta n_e, \Delta n_\mu, \Delta n_\tau\} = \{2,-1,-1\}$ encodes the flavor structure $ee\mu^\dagger\tau^\dagger$. For each flavor charge assignment, we check the parameter space of flavor structures that is spanned by

$$0 \le \Delta n_e \le N, \quad -N \le \Delta n_\mu \le N, \quad \Delta n_\tau = -\Delta n_e - \Delta n_\mu \tag{16}$$

and filter those flavor structures that satisfy the conservation of flavor charge; see Eq. (3). The conservation of the charged lepton number is ensured by the choice of Δn_τ. A few comments are in order: despite the flavor charge being defined modulo N, we include both 0 and N in the ranges in Eq. (16), as the individual charged lepton numbers are counted without modulo N. For example, with $\mathbb{Z}_2(0,0,1)$, the flavor structure $ee\tau^\dagger\tau^\dagger$, corresponding to $\Delta n_e = 2$, $\Delta n_\mu = 0$, and $\Delta n_\tau = -2$, is allowed and cannot be reduced to/deduced from $e\tau^\dagger$, since the latter is not permitted. Allowed flavor structures with values of $\Delta n_{e,\mu,\tau}$ being outside of the ranges in Eq. (16) can be constructed via Hermitian conjugation (for negative Δn_e) or arise from a combination of flavor structures with smaller values of the changes in individual charged lepton numbers (for $|\Delta n_{e,\mu,\tau}| > N$). The latter is shown in Appendix A.

E. Results

We summarize the results of the flavor structures allowed by the flavor charge assignments for $G_\ell = \mathbb{Z}_N$ with $N \le 8$. Taking into account the equivalences and redundancies in notation outlined in Sec. III C greatly compacts the list of flavor charge assignments to only the inequivalent ones. For $N \le 5$, these can be found in Table I together with their allowed flavor structures. The results for larger N are given in Appendix B. Note that for $N \ge 5$ the different flavor charge assignments allow for at most a single flavor structure with no more than four charged leptons. Given that these correspond to dimension-six SMEFT operators, the study of residual symmetries $G_\ell = \mathbb{Z}_N$ with larger N is phenomenologically less interesting. For this reason, we focus on flavor charge assignments that can be obtained for $N \le 4$ in Secs. IV and V.

For the sake of brevity, in Table I we do not include Hermitian conjugates of the flavor structures, trivial flavor combinations (e.g., $ee^\dagger$) or flavor structures containing such combinations nor flavor structures that arise from combining flavor structures already shown in this table for a given flavor charge assignment. Flavor charge assignments and their corresponding flavor structures that result from a permutation of the labels of the charged leptons of a flavor charge assignment shown in Table I are also excluded. As mentioned in Sec. III B, flavor charge assignments that comprise combinations of flavor charges which sum up to zero modulo N are indicated in boldface. It is interesting to note that all but one of the flavor charge assignments shown in Table I, namely 3(0, 1), have this property. In Appendix C we provide the list of all possible flavor charge assignments for $G_\ell = \mathbb{Z}_3$ as an explicit example of the notation.

[3]Using the reduced notation [see Eq. (7)], it is sufficient to take into account flavor charge assignments with $0 \le \delta_1 \le \delta_2$ and $0 \le \delta_2 \le N-1$. For $G_\ell = \mathbb{Z}_N$ with $N \le 8$, there is little difference in computation time to scan over either set of assignments.

TABLE I. Allowed flavor structures for each inequivalent flavor charge assignment for $G_\ell = \mathbb{Z}_N$ with $N \leq 5$. For further description and details, see text.

Flavor charges	d_ℓ	Flavor structures
$\mathbf{2(0,1)}$	3	$e\mu^\dagger$
	6	$\mu\mu\tau^\dagger\tau^\dagger$
		$e\mu\tau^\dagger\tau^\dagger$
		$ee\tau^\dagger\tau^\dagger$
$3(0,1)$	3	$e\mu^\dagger$
	9	$\mu\mu\mu\tau^\dagger\tau^\dagger\tau^\dagger$
		$e\mu\mu\tau^\dagger\tau^\dagger\tau^\dagger$
		$ee\mu\tau^\dagger\tau^\dagger\tau^\dagger$
		$eee\tau^\dagger\tau^\dagger\tau^\dagger$
$\mathbf{3(1,1)}$	6	$e\mu\tau^\dagger\tau^\dagger$
		$e\mu^\dagger\mu^\dagger\tau$
		$ee\mu^\dagger\tau^\dagger$
	9	$\mu\mu\mu\tau^\dagger\tau^\dagger\tau^\dagger$
		$eee\tau^\dagger\tau^\dagger\tau^\dagger$
		$eee\mu^\dagger\mu^\dagger\mu^\dagger$
$\mathbf{4(0,1)}$	3	$e\mu^\dagger$
	12	$\mu\mu\mu\mu\tau^\dagger\tau^\dagger\tau^\dagger\tau^\dagger$
		$e\mu\mu\mu\tau^\dagger\tau^\dagger\tau^\dagger\tau^\dagger$
		$ee\mu\mu\tau^\dagger\tau^\dagger\tau^\dagger\tau^\dagger$
		$eee\mu\tau^\dagger\tau^\dagger\tau^\dagger\tau^\dagger$
		$eeee\tau^\dagger\tau^\dagger\tau^\dagger\tau^\dagger$
$\mathbf{4(1,1)}$	6	$e\mu^\dagger\mu^\dagger\tau$
		$ee\tau^\dagger\tau^\dagger$
	9	$e\mu\mu\tau^\dagger\tau^\dagger\tau^\dagger$
		$eee\mu^\dagger\mu^\dagger\tau^\dagger$
	12	$\mu\mu\mu\mu\tau^\dagger\tau^\dagger\tau^\dagger\tau^\dagger$
		$eeee\mu^\dagger\mu^\dagger\mu^\dagger\mu^\dagger$
$\mathbf{5(0,1)}$	3	$e\mu^\dagger$
	15	$\mu\mu\mu\mu\mu\tau^\dagger\tau^\dagger\tau^\dagger\tau^\dagger\tau^\dagger$
		$e\mu\mu\mu\mu\tau^\dagger\tau^\dagger\tau^\dagger\tau^\dagger\tau^\dagger$
		$ee\mu\mu\mu\tau^\dagger\tau^\dagger\tau^\dagger\tau^\dagger\tau^\dagger$
		$eee\mu\mu\tau^\dagger\tau^\dagger\tau^\dagger\tau^\dagger\tau^\dagger$
		$eeee\mu\tau^\dagger\tau^\dagger\tau^\dagger\tau^\dagger\tau^\dagger$
		$eeeee\tau^\dagger\tau^\dagger\tau^\dagger\tau^\dagger\tau^\dagger$
$\mathbf{5(1,1)}$	6	$e\mu^\dagger\mu^\dagger\tau$
	9	$ee\mu\tau^\dagger\tau^\dagger\tau^\dagger$
		$eee\mu^\dagger\tau^\dagger\tau^\dagger$
	12	$e\mu\mu\mu\tau^\dagger\tau^\dagger\tau^\dagger\tau^\dagger$
		$eeee\mu^\dagger\mu^\dagger\mu^\dagger\tau^\dagger$
	15	$\mu\mu\mu\mu\mu\tau^\dagger\tau^\dagger\tau^\dagger\tau^\dagger\tau^\dagger$
		$eeeee\tau^\dagger\tau^\dagger\tau^\dagger\tau^\dagger\tau^\dagger$
		$eeeee\mu^\dagger\mu^\dagger\mu^\dagger\mu^\dagger\mu^\dagger$

As indicated in Sec. II, results for residual symmetries G_ℓ that are direct products of cyclic groups are not explicitly discussed. Nevertheless, they can be deduced from the information found in Table I. In general, for a flavor charge assignment of such a direct product, the set of allowed flavor structures can be constructed as the intersection of the sets of allowed flavor structures for the flavor charge assignments of each group in the product, potentially, taking into account appropriate permutations of the labels of the charged leptons. Furthermore, one has to include larger flavor structures allowed by the flavor charge assignments of each group. For instance, reading from Table I, a flavor charge assignment $\mathbf{2(0,1)} \times 3(0,1)$ for $G_\ell = \mathbb{Z}_2 \times \mathbb{Z}_3$ (corresponding to $\mathbb{Z}_6$) allows the flavor structure $e\mu^\dagger$ (and those that can be constructed following the above prescription). Additionally, it permits flavor structures with twelve charged leptons made from combinations of the allowed flavor structures with six and four fields, respectively, for each group in the product. Such flavor structures are, indeed, found in Apepndix B for the flavor charge assignment 6(0, 1) for $G_\ell = \mathbb{Z}_6$.

In Table I, we have introduced the quantity d_ℓ as the lepton mass dimension which is defined as $d_\ell = (\text{no. charged lepton fields}) \times 3/2$. In this way, the results for the flavor structures can be more easily applied to SMEFT operators. One can clearly see that the largest value of d_ℓ for which a given flavor charge assignment results in nondecomposable flavor structures is $3N$. Moreover, such flavor structures always exist. They contain N copies of flavor charged leptons that therefore trivially conserve flavor charge. For flavor charge assignments with two equal flavor charges, such as $\mathbf{2(0,1)}$ or $\mathbf{5(0,1)}$, the flavor structures with the largest value of d_ℓ are the only ones that differentiate among them.

IV. CLASSIFICATION OF ALLOWED CLFV PROCESSES

We are now in the position to determine which combination of cLFV processes one can expect from a certain residual symmetry G_ℓ and flavor charge assignment of the charged leptons. To do so, we translate the flavor structures studied in Sec. III into SMEFT operators [35–39]. We choose SMEFT, since it is the most general framework for studying beyond SM effects, mediated by new physics heavier than the electroweak scale. As commented in Sec. II, the group G_ℓ in general does not correspond to an exact residual symmetry of the theory. As such, references to the "observation"/"nonobservation" of a process in the following discussion should be understood as amplification/suppression of this process in comparison to others. Analogously, the "implication"/"exclusion" of a certain residual symmetry G_ℓ and flavor charge assignment has to be interpreted as these being favored/disfavored by experimental results.

A. Flavor structures of dimension-six SMEFT operators

For any realistic observation of cLFV, we are restricted to SMEFT operators at dimension six or lower. The dimension-five Weinberg operator violates the lepton number [52], and thus it is beyond the scope of the present study. Moreover, it does not contribute to cLFV processes other than through the modification of the lepton mixing matrix where it suffers from suppression by the smallness of neutrino masses. Consequently, we work within a theory defined by the effective Lagrangian

$$\mathcal{L}_{\rm SMEFT} = \mathcal{L}_{\rm SM} + \frac{1}{\Lambda^2}\sum_x C_x \mathcal{O}_x^{d=6}, \tag{17}$$

where Λ is the new physics scale and C_x denotes the Wilson coefficient (WC) of a given dimension-six operator $\mathcal{O}_x^{d=6}$. For simplicity, we assume a common new physics scale Λ for all dimension-six operators.

Dimension-six SMEFT operators comprising leptons either contain two or four of them [53]. Typical SMEFT operators with two charged leptons are the dipole operators $(\overline{\ell}_i\sigma^{\mu\nu}e_j)\varphi B_{\mu\nu}$ and $(\overline{\ell}_i\sigma^{\mu\nu}e_j)\varphi W_{\mu\nu}$, with the $U(1)_Y$ and $SU(2)_L$ field strength tensors $B_{\mu\nu}$ and $W_{\mu\nu}$, respectively. Upon electroweak symmetry breaking via the vacuum expectation value, $\langle\varphi\rangle = v/\sqrt{2}$ and $v \approx 246$ GeV, these operators combine into the photon dipole operators $\frac{v}{\sqrt{2}}(\overline{e}_{Li}\sigma^{\mu\nu}e_j)F_{\mu\nu}$ with e_{Li} being a left-handed charged lepton of flavor i and $F_{\mu\nu}$ the electromagnetic field strength tensor. At low energy, these operators induce radiative decays of heavier charged leptons into lighter ones, i.e., the processes $\mu \to e\gamma$, $\tau \to \mu\gamma$ or $\tau \to e\gamma$.

The operators containing four leptons, such as $(\overline{\ell}_i\gamma_\nu\ell_j)(\overline{\ell}_k\gamma^\nu\ell_l)$, can be divided into three categories depending on their flavor structure. First, we have nontrivial four-lepton operators. These correspond to the flavor structures labeled with lepton mass dimension $d_\ell = 6$ in Table I and include operators that lead to processes such as $\tau \to ee\bar{\mu}$, muonium to antimuonium conversion $\mathrm{M} \to \bar{\mathrm{M}}$ (that is, a bound state $\mathrm{M} = \mu^+\mathrm{e}^-$ converting into $\bar{\mathrm{M}} = \mu^-\mathrm{e}^+$), and scattering processes such as $ee \to \tau\tau$ (standing for $e^\pm e^\pm \to \tau^\pm\tau^\pm$). Second, there are semitrivial four-lepton operators with one pair of charged leptons forming a trivial flavor combination (e.g., $ee^\dagger$). Although they can lead to more complicated processes such as $\tau \to \mu e\bar{e}$, these operators are equivalent to two-lepton operators in terms of the information they provide about the residual symmetry G_ℓ and the flavor charge assignment. Note that operators with a flavor structure such as $ee\mu^\dagger\mu^\dagger$ can be constructed as the duplicate of a two-lepton flavor structure, namely $e\mu^\dagger$; however, they do not necessarily admit the same set of G_ℓ and flavor charge assignments; cf. Table I. Such operators are still considered nontrivial four-lepton operators and studied independently from their two-lepton counterparts. Finally, there exist trivial four-lepton operators that are constructed from two trivial two-lepton flavor structures such as $ee^\dagger\mu\mu^\dagger$. They are of little interest to the analysis at hand as they carry no information about the residual symmetry G_ℓ and the flavor charge assignment of the charged leptons.

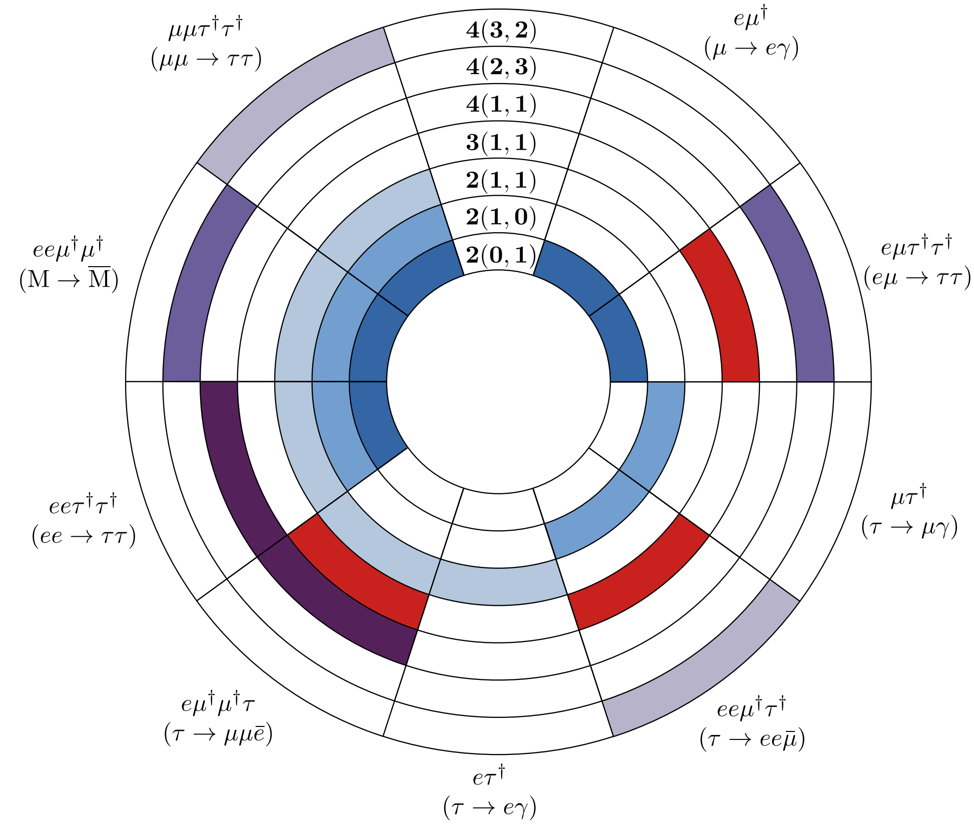
B. Implications for G_ℓ and flavor charge assignments

Focusing on flavor charge assignments that allow for at least two (independent and nontrivial) flavor structures with up to four leptons, $d_\ell \leq 6$ (see Table I), Fig. 1 illustrates the residual symmetries $G_\ell = \mathbb{Z}_N$ and flavor charge assignments compatible with these flavor structures. In doing so, we take into account the possibility of permutations among the charged lepton flavors. This figure highlights the restrictions that combinations of different observed cLFV processes could place. For example, it shows that if both trilepton decays $\tau \to \mu\mu\bar{e}$ and $\tau \to ee\bar{\mu}$ are observed, the only viable nontrivial flavor charge assignment is $\mathbf{3}(\mathbf{1},\mathbf{1})$.[4] The table at the bottom of Fig. 1 lists the restrictions on the flavor charge assignments that result from a single flavor structure (corresponding to the observation of one type of cLFV process). For instance, we clearly see that if at least one $\mu \to e$ transition is observed (the flavor structure $e\mu^\dagger$ is allowed), the muon and the electron have to have the same flavor charge [see also $\mathbf{2}(\mathbf{0},\mathbf{1})$ in the figure], or no (approximate) residual symmetry G_ℓ is present among charged leptons.

From Fig. 1, one can see that in most instances, the observation of cLFV processes corresponding to only two different flavor structures is sufficient to constrain the size of the minimal equivalent residual symmetry to $\mathbb{Z}_N$ with $N = 2$, 3 or 4, considering only lepton mass dimension $d_\ell \leq 6$. All processes allowed by each of the flavor charge assignments for $G_\ell = \mathbb{Z}_4$ are also allowed by a corresponding flavor charge assignment for $G_\ell = \mathbb{Z}_2$. In order to distinguish these, the relevant radiative cLFV decays would have to be scrutinized.

Several combinations of observed cLFV processes, corresponding to different types of flavor structures, could be used to (strongly) disfavor the existence of a residual symmetry G_ℓ—one that at least partially distinguishes between the three charged leptons. Arguably, the simplest such combination consists of two radiative cLFV decays. Figure 1 shows that cLFV processes corresponding to a maximum of five different flavor structures with $d_\ell \leq 6$ could be observed (which would be in the case of $G_\ell = \mathbb{Z}_2$) before excluding the existence of any (approximate) residual symmetries considered.

[4]Also flavor charge assignments equivalent to $\mathbf{3}(\mathbf{1},\mathbf{1})$ as per Sec. III C are viable (except that the permutation of the lepton flavors e, μ, and τ is fixed). Concretely, for $G_\ell = \mathbb{Z}_3$ the flavor charge of each charged lepton cannot be uniquely determined, since the flavor charge assignment $\mathbf{3}(\mathbf{1},\mathbf{1})$ corresponds to three different flavor charge combinations and, furthermore, $\mathbf{3}(\mathbf{2},\mathbf{2})$ leads to the same allowed flavor structures; see Appendix C and Sec. III. Thus, we can only conclude that the three charged leptons possess distinct flavor charges.



Restrictions from single flavor structure

$e\mu^\dagger$ - $N(0,a)$	$ee\mu^\dagger\mu^\dagger$ - $2N(N,a)$	$ee\mu^\dagger\tau^\dagger$ - $N(N-a,2a)$
$\mu\tau^\dagger$ - $N(a,0)$	$\mu\mu\tau^\dagger\tau^\dagger$ - $2N(a,N)$	$e\mu^\dagger\mu^\dagger\tau$ - $N(a,a)$
$e\tau^\dagger$ - $N(a,N-a)$	$ee\tau^\dagger\tau^\dagger$ - $2N(a,N-a)$	$e\mu\tau^\dagger\tau^\dagger$ - $N(2a,N-a)$

FIG. 1. Illustration of compatibility between flavor charge assignments and flavor structures with $d_\ell \leq 6$, taking into account permutations of the charged lepton flavors. Colored segments indicate compatibility. Only flavor charge assignments that allow for at least two (independent and nontrivial) flavor structures with up to four leptons, $d_\ell \leq 6$ (see Table I) are shown. For each flavor structure an example of a cLFV process is given in parentheses. The table lists restrictions on the flavor charge assignments from observing (at least) one cLFV process corresponding to a certain flavor structure, again accounting for permutations among the charged lepton flavors. The parameter a is an integer.

The flavor charge assignments shown in Tables I and II that do not explicitly appear in Fig. 1, can be discussed according to the categories found in the table at the bottom of this figure. Concretely, the flavor charge assignments 3(0, 1), $\mathbf{4(0,1)}$, $\mathbf{5(0,1)}$, 6(0, 1), $\mathbf{7(0,1)}$, and $\mathbf{8(0,1)}$ all only generate the flavor structure $e\mu^\dagger$ (take $a=1$ in the mentioned table), when restricting to dimension-six SMEFT operators. The analogous flavor structures, $\mu\tau^\dagger$ and $e\tau^\dagger$, are achieved by permuting the lepton flavors e, μ, and τ. Consequently, the flavor structure $\mu\tau^\dagger$ is allowed by the flavor charge assignments 3(1, 0), $\mathbf{4(1,0)}$, $\mathbf{5(1,0)}$, 6(1, 0), $\mathbf{7(1,0)}$, and $\mathbf{8(1,0)}$, being of the form $N(a,0)$ with $a=1$. Likewise, for the flavor structure $e\tau^\dagger$ viable flavor charge assignments are 3(1, 2), $\mathbf{4(1,3)}$, $\mathbf{5(1,4)}$, 6(1, 5), $\mathbf{7(1,6)}$, and $\mathbf{8(1,7)}$, meaning $N(a,N-a)$ for $a=1$ in agreement with the table at the bottom of Fig. 1. The flavor structure $e\mu^\dagger\mu^\dagger\tau$ is allowed by the flavor charge assignments $\mathbf{5(1,1)}$, $\mathbf{6(1,1)}$, $\mathbf{7(1,1)}$, and $\mathbf{8(1,1)}$, all being of the form $N(a,a)$ with $a=1$. Applying permutations among the lepton flavors e, μ, and τ, the flavor structures $ee\mu^\dagger\tau^\dagger$ (up to Hermitian conjugation) and $e\mu\tau^\dagger\tau^\dagger$ arise. Examples of flavor charge assignments allowing $ee\mu^\dagger\tau^\dagger$ as only (nontrivial) flavor structure with $d_\ell \leq 6$ are $\mathbf{5(4,2)}$, $\mathbf{6(5,2)}$, $\mathbf{7(6,2)}$, and $\mathbf{8(7,2)}$, whereas $e\mu\tau^\dagger\tau^\dagger$ is permitted by, for example, the flavor charge assignments $\mathbf{5(2,4)}$, $\mathbf{6(2,5)}$, $\mathbf{7(2,6)}$, and $\mathbf{8(2,7)}$. The two flavor charge assignments 6(1, 2) and $\mathbf{8(1,3)}$ both lead to the flavor structure $ee\tau^\dagger\tau^\dagger$, cf. flavor charge assignment $2N(a,N-a)$ with $a=1$ in the table at the bottom of Fig. 1. Analogously, the two flavor structures $ee\mu^\dagger\mu^\dagger$ and $\mu\mu\tau^\dagger\tau^\dagger$, arising from permuting the lepton flavors e, μ, and τ, are allowed by $6(3,1)$ and $\mathbf{8}(\mathbf{4},\mathbf{1})$ as well as $6(1,3)$ and $\mathbf{8}(\mathbf{1},\mathbf{4})$, respectively, which follow the general forms $2N(N,a)$ and $2N(a,N)$, always with $a=1$; see the table at the bottom of Fig. 1. Lastly, we observe that the flavor charge assignments $\mathbf{7}(\mathbf{1},\mathbf{2})$ and $\mathbf{8}(\mathbf{1},\mathbf{2})$ do not allow for (nontrivial) flavor structures with $d_\ell \leq 6$. It is, thus, not possible to test these with the cLFV processes discussed in this study.

We reiterate that while observations have the potential to constrain G_ℓ and the flavor charge assignments, these alone are not sufficient in order to arrive at any conclusion, since, at the same time, the cLFV processes corresponding to forbidden flavor structures have to be (strongly) suppressed.

V. SENSITIVITY OF CLFV SEARCHES TO NEW PHYSICS SCALE

In the preceding section, we have presented the inequivalent flavor charge assignments for $G_\ell = \mathbb{Z}_N$ that are compatible with flavor structures up to lepton mass dimension $d_\ell \leq 6$. We now examine the bounds on the new physics scale Λ that can be derived from various current and prospective cLFV experiments; see, e.g., [25–27] for reviews. In Sec. V A we focus on low-energy cLFV experiments, while we comment on possible cLFV searches at high-energy colliders in Sec. V B. The results for flavor charge assignments not explicitly mentioned in Fig. 1 can be found, for completeness, in Appendix E.

A. Low-energy cLFV experiments

The tightest experimental limits on cLFV processes exist in the $\mu - e$ sector, with the strongest bound recently released by the MEG-II experiment on the decay $\mu^+ \to e^+\gamma$, i.e., $\mathrm{BR}(\mu^+ \to e^+\gamma) < 1.5 \times 10^{-13}(90\%\mathrm{CL})$ [54]; see Appendix D. This result is to be further improved by MEG-II itself [55]. The decay $\mu^+ \to e^+e^+e^-$ is best constrained by measurements from SINDRUM, $\mathrm{BR}(\mu^+ \to e^+e^+e^-) < 1.0 \times 10^{-12}(90\%\mathrm{CL})$ [56], and probes the same flavor transition as the decay $\mu^+ \to e^+\gamma$, although $\mu^+ \to e^+e^+e^-$ is currently less sensitive to the cLFV dipole operators. Upcoming searches by the Mu3e Collaboration [57] are expected to improve this limit by three to four orders of magnitude, matching the prospective MEG-II sensitivity to the cLFV dipole operators, and largely exceeding MEG-II if other SMEFT operators are the dominant sources of cLFV. Strong constraints in the $\mu - e$ sector are also provided by searches for $\mu - e$ conversion in nuclei N, $\mu^-\mathrm{N} \to e^-\mathrm{N}$, with the strongest limit currently given by the SINDRUM-II experiment for $\mu - e$ conversion in gold, $\mathrm{CR}(\mu^-\mathrm{Au} \to e^-\mathrm{Au}) < 7 \times 10^{-13}(90\%\mathrm{CL})$ [58]. This already stringent bound is foreseen to be improved by about four orders of magnitude by the Mu2e [59] and COMET [60] Collaborations using aluminium targets. Such an improvement corresponds to more than one order of magnitude increase in the sensitivity to the new physics scale Λ; see Appendix D. In the long run, the Mu2e-II experiment can possibly achieve an even stronger limit by employing nuclei heavier than aluminium [61]. A several orders of magnitude weaker constraint on the new physics scale Λ in the $\mu - e$ sector is obtained from the bound on the muonium to antimuonium conversion probability, $\mathrm{P}(\mathrm{M} \to \bar{\mathrm{M}}) < 8.2 \times 10^{-11}(90\%\mathrm{CL})$ [62]. In the case in which $\mu \to e$ transitions are not allowed by the residual flavor symmetry G_ℓ and the flavor charge assignment of the charged leptons, the limit on $\mathrm{P}(\mathrm{M} \to \bar{\mathrm{M}})$ can be the dominant constraint in the $\mu - e$ sector. Moreover, this bound is expected to be improved by about three orders of magnitude by the proposed MACE experiment [63].

The various cLFV τ lepton decays give rise to the next strongest constraints on the new physics scale Λ. These are, however, a few orders of magnitude weaker. The mass of the τ lepton allows for many possible decay channels, and thus experiments searching for them are vital in discriminating among residual flavor symmetries G_ℓ and flavor charge assignments that forbid $\mu \to e$ transitions. The current best limits on cLFV τ lepton decays, reaching the level $\mathcal{O}(10^{-8})$, come from Belle, *BABAR*, and Belle II [64–71]. All of these are expected to be improved by one to two orders of magnitude by future searches performed at Belle II [72–74].

Other tests of cLFV can, in principle, be performed at high-energy collider experiments searching for cLFV scattering processes and cLFV decays of heavy SM bosons. Current constraints from these sources, where they exist, are significantly weaker than those from muon and τ lepton decays; see Sec. V B.

For each of the mentioned processes, we can calculate the corresponding observable in terms of the WCs of the contributing SMEFT operators and the new physics scale Λ [cf. Eq. (17)] by means of the formulas collected in [75]. We then translate the experimental upper bound on the observable into a lower bound on Λ, making different well-motivated assumptions about the WCs. The results are shown in Fig. 2.

We primarily consider two scenarios in which new physics contributes at the tree and the one-loop level, respectively. In the case of tree-level new physics, we set all WCs of the operators allowed by a certain residual symmetry G_ℓ and flavor charge assignment to $C_x = 1$, except for those of the dipole operators that are generated at the (one-)loop level within any realistic UV-complete model. Hence, we set $C_d = e/(16\pi^2) \approx 0.002$, where $e = \sqrt{4\pi\alpha} \approx 0.30$ is the electromagnetic coupling at low energy. In the case that all new physics effects arise at the one-loop level, the WCs of the dipole operators remain the same, while all other (nonzero) WCs become suppressed by a loop factor, $C_x = 1/(16\pi^2) \approx 0.006$.

It is worth also considering a third scenario, in which the WCs of the dipole operators are further suppressed by the Yukawa coupling of the decaying lepton, representing a

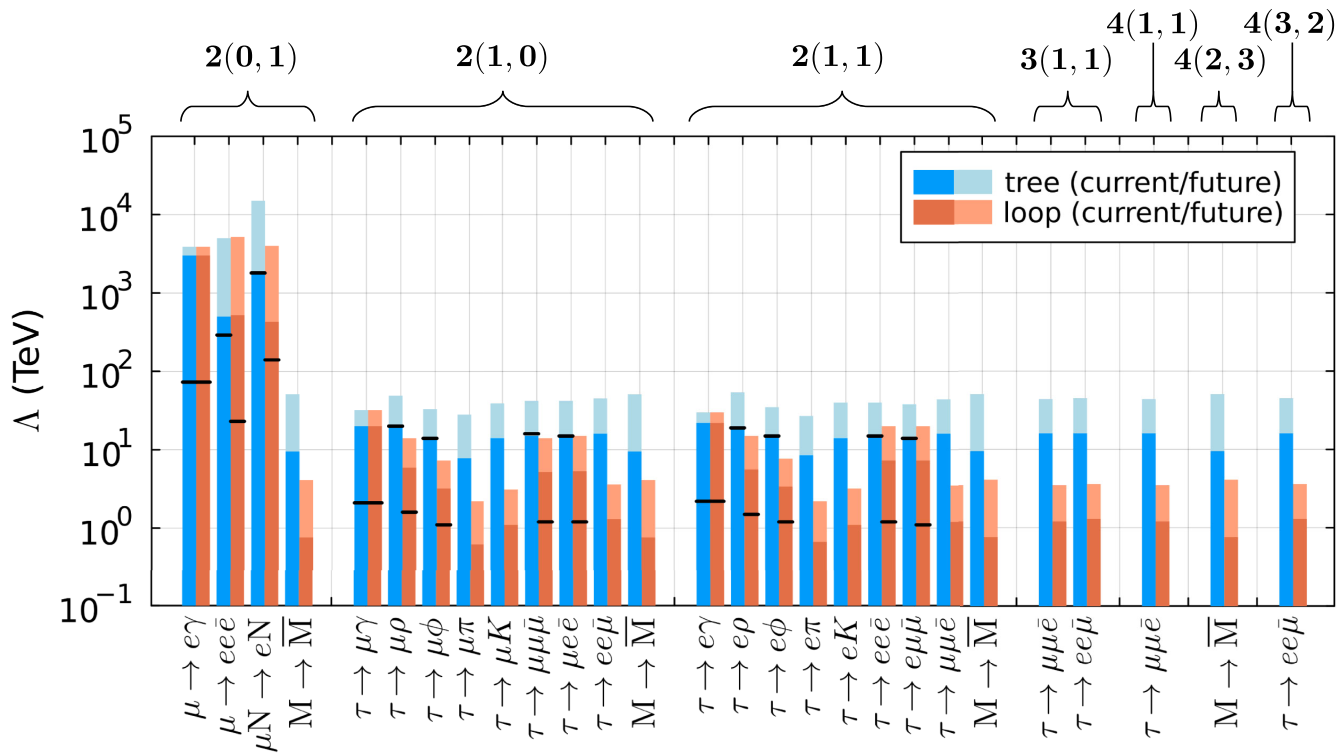


FIG. 2. Current lower bounds on and future sensitivities to the new physics scale Λ (in TeV), based on experimental limits for various processes at 90% CL. The results are grouped according to the residual flavor symmetry G_ℓ and flavor charge assignment of the charged leptons; cf. Table I and Fig. 1. Present bounds (dark colors) and future sensitivities (light colors) are shown for the two different scenarios with tree-level (blue) and loop-level contributions (orange). The results for the scenario with chirally suppressed contributions are indicated by a horizontal black line, where relevant. See text for details and Appendix D for the numerical values of the new physics scale Λ in the different considered scenarios.

mass insertion as required to perform a chirality flip. This leads to $C_d = \sqrt{2} m_\ell e/(16\pi^2 v)$. The effects of this further suppression are shown by black lines in Fig. 2 for the processes with contributions from the dipole operators. This type of suppression is negligible, if the WCs of the nondipole operators are already dominant, as is the case for several τ lepton decays.

Finally, note that we consider neither renormalization group (RG) running of the WCs from the new physics scale Λ to the relevant energy scale of a given process nor the matching of SMEFT operators to low-energy effective field theory operators at the electroweak scale. Since in the current approach all operators allowed by a certain residual symmetry G_ℓ and flavor charge assignment are present in the UV with a comparable strength and, furthermore, G_ℓ prohibits forbidden operators to arise through RG running, this simplifying assumption only makes us overlook corrections to the WCs, e.g., of the order $\alpha_{(s)}/(4\pi)$ [76]. These are numerically negligible, even when stemming from perturbative QCD.

In Fig. 2, we display the lower bound on and future sensitivity to the new physics scale Λ, obtained from various relevant cLFV searches. Comparing the values of Λ across the three different scenarios (with tree-level, loop-level, or chirally suppressed contributions) provides an idea of the model dependence of the results. The (current/prospective) experimental bounds and numerical values for Λ are tabulated in Appendix D.

Following the discussion in Sec. IV, the processes in Fig. 2 are grouped according to the residual flavor symmetry $G_\ell = \mathbb{Z}_N$ and the flavor charge assignments they are compatible with. In this way, we can contrast the resulting selection rules with the reach of present and future cLFV experiments.

Since in all considered cases the present experimental limits entail Λ larger than the electroweak scale, the adoption of SMEFT is justified. As expected, the limits on Λ arising from $\mu \to e$ transitions are far stronger than those of other processes, predominately τ lepton decays, which is consistent with similar analyses available in the literature; see, e.g., [53,75,77–85].[5] However, we see that

[5]Nevertheless, numerical differences are expected with respect to these studies because of the different treatment of the WCs of the dipole operators and, more importantly, the choice of including all allowed SMEFT operators, whereas previous studies typically display results obtained by considering a single operator at a time.

$\mu \to e$ transitions are only allowed by flavor charge assignments such as $\mathbf{2}(\mathbf{0},\mathbf{1})$ that forbid cLFV τ lepton decays. On the contrary, flavor charge assignments such as $\mathbf{2}(\mathbf{1},\mathbf{0})$ can be best probed with τ lepton decays, showing that cLFV searches for processes involving τ leptons can be the key in order to underpin the existence of a certain residual symmetry G_ℓ and flavor charge assignment. In particular, Fig. 2 shows that if one cLFV τ lepton decay is observed at Belle II, while not observing several more, this would disfavor $G_\ell = \mathbb{Z}_2$ as residual symmetry among charged leptons. Moreover, cLFV searches for the trilepton decays $\tau \to ee\bar{\mu}$ and $\tau \to \mu\mu\bar{e}$ are suitable to test flavor charge assignments associated with the residual symmetries $G_\ell = \mathbb{Z}_3$ and $G_\ell = \mathbb{Z}_4$. In general, along with stronger constraints on muonium to antimuonium conversion, stronger constraints on τ lepton decays could provide information on the plausibility of all the discussed flavor charge assignments. Furthermore, we note that, although several residual symmetries G_ℓ and flavor charge assignments allow muonium to antimuonium conversion, this quantitative study shows that stringent bounds from $\mu \to e$ transitions render this process unobservable for the flavor charge assignment $\mathbf{2}(\mathbf{0},\mathbf{1})$ and $G_\ell = \mathbb{Z}_2$, while it could be observed, but alongside cLFV τ lepton decays, if either the flavor charge assignment $\mathbf{2}(\mathbf{1},\mathbf{0})$ or $\mathbf{2}(\mathbf{1},\mathbf{1})$ is (approximately) realized in nature. Additionally, experimental evidence of muonium to antimuonium conversion alone is compatible with (and could provide support for) the flavor charge assignment $\mathbf{4}(\mathbf{2},\mathbf{3})$ and $G_\ell = \mathbb{Z}_4$.

B. High-energy colliders

The limits in Fig. 2 do not take into account SMEFT operators corresponding to flavor structures in Table I that contain two *same sign* τ leptons. These would be directly tested, when searching for the scattering processes

$$e^\pm e^\pm \to \tau^\pm\tau^\pm, \qquad e^\pm\mu^\pm \to \tau^\pm\tau^\pm, \qquad \mu^\pm\mu^\pm \to \tau^\pm\tau^\pm. \tag{18}$$

There are no current colliders at which any of these could be probed. One may argue that tests of the SMEFT operators involving two τ leptons are not strictly necessary. As mentioned, the flavor charge assignments belonging to $G_\ell = \mathbb{Z}_4$ may be distinguished from the corresponding ones for $G_\ell = \mathbb{Z}_2$ by scrutinizing the relevant radiative cLFV decays. However, the preceding discussion has also shown that the observation of cLFV processes corresponding to distinct flavor structures is necessary to build up potential evidence of a certain residual symmetry G_ℓ and flavor charge assignment. If observed in combination with, respectively, $\tau \to \mu\mu\bar{e}$, muonium to antimuonium conversion, or $\tau \to ee\bar{\mu}$, the three processes in Eq. (18) would be a solid indication for one of the flavor charge assignments for $G_\ell = \mathbb{Z}_4$, as illustrated by Fig. 1.

An environment in which part of the processes shown in Eq. (18) could possibly be searched for is the proposed μTRISTAN project [86]. The proposal involves two operation modes: (i) employing two 1 TeV μ^+ beams to deliver $\mu^+\mu^+$ collisions with center-of-mass (c.m.) energy $\sqrt{s} = 2$ TeV; (ii) colliding a 1 TeV μ^+ beam with a 30 GeV e^- beam achieving a Higgs factory with $\sqrt{s} \simeq 346$ GeV. The first operation mode could directly be used to search for $\mu^+\mu^+ \to \tau^+\tau^+$ and $\mu^+\mu^+ \to e^+e^+$. The flavor structure corresponding to the latter process is, however, already probed by muonium to antimuonium conversion. A modification of the second operation mode employing an e^+ beam instead of an e^- beam could, in principle, be used to constrain the process $e^+\mu^+ \to \tau^+\tau^+$. When induced, as in the case at hand, by four-lepton SMEFT operators, e.g., $(\bar{\ell}_\tau\gamma_\nu\ell_\mu)(\bar{\ell}_\tau\gamma^\nu\ell_\mu)$, these scattering cross sections increase with the c.m. energy squared:[6]

$$\sigma(\mu^+\mu^+ \to \tau^+\tau^+) = \frac{s}{2\pi}\frac{|C_x|^2}{\Lambda^4} \simeq 25\ \mathrm{fb}\left(\frac{\sqrt{s}}{2\ \mathrm{TeV}}\right)^2\left(\frac{10\ \mathrm{TeV}}{\Lambda/\sqrt{|C_x|}}\right)^4, \tag{19}$$

while for $\sigma(e^+\mu^+ \to \tau^+\tau^+)$ this formula should be divided by a factor of four, because the initial-state particles are not identical. From here, we can see that μTRISTAN would have the capability to test the scattering $\mu^+\mu^+ \to \tau^+\tau^+$ up to $\Lambda \approx 30$ TeV for $C_x = 1$.[7] Hence, it probes scales comparable to the expected sensitivities of searches for cLFV τ lepton decays and muon to antimuonium conversion.

Alternatively, SMEFT operators with flavor structures such as $ee\tau^\dagger\tau^\dagger$ and $\mu\mu\tau^\dagger\tau^\dagger$ could be tested in heavy boson decays. The example of the four-body Z decay $Z \to \tau\tau\bar{e}\,\bar{e}$, and the analogous decay $Z \to \tau\tau\bar{\mu}\,\bar{\mu}$, is discussed in [31]. However, currently there are no experimental limits on these processes, and requiring them not to be in conflict with the well-measured (and well-predicted) Z decay width only leads to the extremely weak constraint $\Lambda \gtrsim 1$ GeV. If future Tera-Z factories, such as CEPC [89] and FCC-ee [90], can set a limit $\mathrm{BR}(Z \to \tau\tau\bar{e}\,\bar{e}) < 10^{-12}$, the constraint would be strongly improved, $\Lambda \gtrsim 250$ GeV [31]. Still, such a sensitivity seems to be insufficient to exceed the current

[6]Notice that this is consistent with unitarity as long as $\sqrt{s} \ll \Lambda$, that is, in the regime of validity of SMEFT. For a discussion of processes induced by cLFV SMEFT operators in the context of high-energy e^+e^- colliders, we refer to [87]. See also [23,88] for studies on possible searches for cLFV at μTRISTAN.

[7]Since the signature is free of (irreducible) physics backgrounds, we conservatively assume a selection/detection efficiency of $\varepsilon_{\mathrm{eff}} = 10\%$, and that evidence for the process can be achieved for a number of signal events $n_s = \sigma(\mu^+\mu^+ \to \tau^+\tau^+) \times \varepsilon_{\mathrm{eff}} \times \mathcal{L} = 3$, where the expected integrated luminosity is $\mathcal{L} = 100\ \mathrm{fb}^{-1}$ [86].

limits from τ lepton decays, shown in Fig. 2, and is just barely within the regime of validity of SMEFT.

VI. SUMMARY AND OUTLOOK

Predictive models of lepton flavor with a discrete flavor (and CP) symmetry often feature a residual symmetry G_ℓ for charged leptons, which restricts the allowed cLFV processes. We have analyzed the resulting selection rules and phenomenological consequences for all possible flavor charge assignments and $G_\ell = \mathbb{Z}_N$ with $N \leq 8$. These reduce to a small set of unique flavor charge assignments. For each, we have provided a complete categorization of the allowed flavor structures; cf. Tables I and II. In the phenomenological analysis, we have matched each flavor structure with no more than four charged leptons to SMEFT operators, up to dimension six; see Fig. 1. We have found that the observation of cLFV processes corresponding to more than one of these flavor structures can constrain any combination of flavor charge assignment and (approximate) residual symmetry $G_\ell = \mathbb{Z}_N$ to be equivalent to one with $N \leq 4$. The results illustrate the patterns of cLFV processes that, if observed, could point to the existence of a certain residual symmetry G_ℓ and flavor charge assignment for the charged leptons.

The quantitative analysis in Sec. V has demonstrated that evidence for the obtained selection rules can indeed be achieved with current and near-future experiments. The lower limits on and prospective sensitivities to the new physics scale Λ shown in Fig. 2 can be interpreted as the scale of a possible UV completion of the lepton flavor theory. At the same time, they indicate the order of magnitude of the breaking scale of the employed flavor (and CP) symmetry. An important observation is that flavor charge assignments which distinguish among the electron and the muon do not allow $\mu \to e$ transitions. As a consequence, searches for cLFV τ lepton processes and muonium to antimuonium conversion can be crucial to provide complementary information.

Let us comment on directions in which this study is worth extending. On the one hand, one could consider concrete models with G_ℓ being one of the analyzed symmetries, and possibly focus on specific UV completions. On the other hand, one could study effective operators that also violate lepton number (in general, assuming a different scale Λ_{LNV}) since these may have interesting phenomenological consequences. For example, neutrinoless double beta decay could be forbidden and μ^+ to e^- conversion in nuclei allowed, such that evidence for lepton number violation could be first observed at conversion experiments such as Mu2e [59] and COMET [60]. Furthermore, the consideration of lepton number violating processes may distinguish flavor charge assignments of the charged leptons that lead to the same set of allowed lepton number conserving processes. Additionally, one could apply a similar logic to the quark sector and discuss processes of quark flavor violation. In doing so, one possibility is to apply the residual symmetry of the charged leptons, and potentially also their flavor charge assignment, to quarks.[8] Another option is to have different residual symmetries for up and down quarks, as for example in [92,93], which generally also differ from the symmetry G_ℓ.

ACKNOWLEDGMENTS

L. C. and C. H. thank the School of Physics of the University of New South Wales (UNSW) for its hospitality and financial support during the initial stage of this project. L. C. acknowledges financial support from the National Natural Science Foundation of China (NSFC) under Grant No. 12035008. C. H. is supported by the Spanish MINECO through the Ramón y Cajal programme RYC2018-024529-I, by the national Grant No. PID2023-148162NB-C21, by the Generalitat Valenciana through PROMETEO/2021/083, by the MCIU/AEI Severo-Ochoa Project No. CEX2023-001292-S as well as by the European Union's Horizon 2020 research and innovation program under the Marie Skłodowska-Curie Grant Agreement No. 860881 (HIDDeν network) and under the Marie Skłodowska-Curie Staff Exchange Grant Agreement No. 101086085 (ASYMMETRY). M. S. acknowledges support by the Australian Research Council Discovery Project No. DP200101470. J. V. is supported by an Australian Government Research Training Program (RTP) scholarship.

DATA AVAILABILITY

The data supporting this study's findings are available within the article.

APPENDIX A: COMPLETENESS OF STUDY

Here, we show that permitted flavor structures with at least one $|\Delta n_{e,\mu,\tau}| > N$ can be decomposed into a combination of allowed flavor structures with $|\Delta n_{e,\mu,\tau}| \leq N$. Consider an arbitrary flavor charge assignment $\mathbb{Z}_N(\alpha, \beta, \gamma)$ and an allowed flavor structure encoded in $\{\Delta n'_e, \Delta n'_\mu, \Delta n'_\tau\}$. Since this flavor structure is allowed, the constraints in Eqs. (2) and (3) are fulfilled, i.e.,

$$\begin{aligned} &\Delta n'_e + \Delta n'_\mu + \Delta n'_\tau = 0 \quad \text{and} \\ &\alpha \Delta n'_e + \beta \Delta n'_\mu + \gamma \Delta n'_\tau = 0 \mod N. \end{aligned} \tag{A1}$$

For integers a, b, c, we can write the changes in the individual charged lepton numbers as

[8] Consequently, quark mixing would vanish at the level of exact residual symmetries (see, e.g., [91]), and a small breaking could explain the smallness of the quark mixing angles.

$$\begin{aligned}\Delta n'_e &= aN + \Delta n_e,\\ \Delta n'_\mu &= bN + \Delta n_\mu,\\ \Delta n'_\tau &= cN + \Delta n_\tau,\end{aligned}\tag{A2}$$

where $0 \leq \Delta n_e \leq N$ and $-N \leq \Delta n_\mu, \Delta n_\tau \leq N$. We have to show that integers a, b, c, and Δn_e, Δn_μ, Δn_τ exist such that the latter also correspond to an allowed flavor structure, i.e.,

$$\Delta n_e + \Delta n_\mu + \Delta n_\tau = 0,\tag{A3}$$

$$\alpha\Delta n_e + \beta\Delta n_\mu + \gamma\Delta n_\tau = 0 \quad \mathrm{mod}\ N.\tag{A4}$$

Plugging Eq. (A2) into the second equation in Eq. (A1), we immediately find

$$\alpha\Delta n_e + \beta\Delta n_\mu + \gamma\Delta n_\tau = 0 \quad \mathrm{mod}\ N,\tag{A5}$$

meaning that $\Delta n_{e,\mu,\tau}$ conserve the flavor charge (independent of the values of a, b, and c). Plugging Eq. (A2) into the first equation in Eq. (A1), we instead get

$$(a + b + c) + \frac{\Delta n_e + \Delta n_\mu + \Delta n_\tau}{N} = 0.\tag{A6}$$

We discuss the implications of Eq. (A6) on a case by case basis. First, note that if

$$\Delta n_e + \Delta n_\mu + \Delta n_\tau \neq zN\tag{A7}$$

with z being an integer, it must hold that $a + b + c = 0$ and $\Delta n_e + \Delta n_\mu + \Delta n_\tau = 0$ separately, as the former is integer valued, and the latter, when divided by N, is not. If so, also $\Delta n_{e,\mu,\tau}$ fulfill the constraint on the charged lepton number; see Eq. (A3). On the other hand, if

$$\Delta n_e + \Delta n_\mu + \Delta n_\tau = zN,\tag{A8}$$

then $a + b + c = -z$ must be true, and by definition of $\Delta n_{e,\mu,\tau}$, the integer z can only take values in the interval $-2 \leq z \leq 3$. For $z = 0$, $\Delta n_{e,\mu,\tau}$ satisfy the condition on the charged lepton number immediately. If $z = 1(-1)$, we can assume that $\Delta n_\mu \geq 0$ ($\Delta n_\mu \leq 0$). We can then reparametrize Δn_μ such that $\Delta n_\mu \to \Delta n_\mu - N$ and $b \to b + 1$ ($\Delta n_\mu \to \Delta n_\mu + N$ and $b \to b - 1$). This reparametrization leads back to the case of $z = 0$. For $z = 2(-2)$, both Δn_μ and Δn_τ must be positive (negative). In a similar fashion to the previous case, we can modify $\Delta n_{\mu,\tau}$, i.e., $\Delta n_\mu \to \Delta n_\mu - N$ and $\Delta n_\tau \to \Delta n_\tau - N$ together with $b \to b + 1$ and $c \to c + 1$ ($\Delta n_\mu \to \Delta n_\mu + N$ and $\Delta n_\tau \to \Delta n_\tau + N$ together with $b \to b - 1$ and $c \to c - 1$), and apply again the results for $z = 0$. In the case where $\Delta n_e = N(0)$, we could also remove (add) N from (to) Δn_e and one of either Δn_μ or Δn_τ. Finally, for $z = 3$, necessarily $\Delta n_{e,\mu,\tau} = N$. Then, we may choose to reparametrize $\Delta n_{e,\mu,\tau}$ by removing either N from each $\Delta n_{e,\mu,\tau}$ such that $\Delta n_{e,\mu,\tau} = 0$, or we may remove N from Δn_e and $2N$ from either Δn_μ or Δn_τ, or remove N from either Δn_μ or Δn_τ and $2N$ from the other one, while not changing Δn_e. All of these reparametrizations restore the situation of $z = 0$.

APPENDIX B: ALLOWED FLAVOR STRUCTURES FOR $G_\ell = \mathbb{Z}_{6,7,8}$ (EXTENSION OF TABLE I)

In Table II we display the allowed flavor structures for $G_\ell = \mathbb{Z}_{6,7,8}$, complementing the information available in Table I.

TABLE II. Allowed flavor structures for $G_\ell = \mathbb{Z}_N$, $6 \leq N \leq 8$. We use a shorthand notation for repeated lepton flavors, e.g., $e^2 = ee$. For further details, see Table I and main text.

Flavor charges	d_ℓ	Flavor structures
6(0, 1)	3	$e\mu^\dagger$
	18	$\mu^6(\tau^\dagger)^6$
		$e\mu^5(\tau^\dagger)^6$
		$e^2\mu^4(\tau^\dagger)^6$
		$e^3\mu^3(\tau^\dagger)^6$
		$e^4\mu^2(\tau^\dagger)^6$
		$e^5\mu(\tau^\dagger)^6$
		$e^6(\tau^\dagger)^6$
6(1, 1)	6	$e(\mu^\dagger)^2\tau$
	9	$e^3(\tau^\dagger)^3$
	12	$e^2\mu^2(\tau^\dagger)^4$
		$e^4(\mu^\dagger)^2(\tau^\dagger)^2$
	15	$e\mu^4(\tau^\dagger)^5$
		$e^5(\mu^\dagger)^4\tau^\dagger$
	18	$\mu^6(\tau^\dagger)^6$
		$e^6(\mu^\dagger)^6$
6(1, 2)	6	$e^2(\tau^\dagger)^2$
	9	$\mu^3(\tau^\dagger)^3$
		$e^2(\mu^\dagger)^3\tau$
	12	$e^4(\mu^\dagger)^3\tau^\dagger$
	18	$e^6(\mu^\dagger)^6$
7(0, 1)	3	$e\mu^\dagger$
	21	$\mu^7(\tau^\dagger)^7$
		$e\mu^6(\tau^\dagger)^7$
		$e^2\mu^5(\tau^\dagger)^7$
		$e^3\mu^4(\tau^\dagger)^7$
		$e^4\mu^3(\tau^\dagger)^7$
		$e^5\mu^2(\tau^\dagger)^7$
		$e^6\mu(\tau^\dagger)^7$
		$e^7(\tau^\dagger)^7$
7(1, 1)	6	$e(\mu^\dagger)^2\tau$
	12	$e^3\mu(\tau^\dagger)^4$
		$e^4\mu^\dagger(\tau^\dagger)^3$
	15	$e^2\mu^3(\tau^\dagger)^5$
		$e^5(\mu^\dagger)^3(\tau^\dagger)^2$
	18	$e\mu^5(\tau^\dagger)^6$
		$e^6(\mu^\dagger)^5\tau^\dagger$
	21	$\mu^7(\tau^\dagger)^7$
		$e^7(\tau^\dagger)^7$
		$e^7(\mu^\dagger)^7$
7(1, 2)	9	$e\mu^2(\tau^\dagger)^3$
		$e^2(\mu^\dagger)^3\tau$
		$e^3\mu^\dagger(\tau^\dagger)^2$
	15	$e(\mu^\dagger)^5\tau^4$
		$e^4\mu(\tau^\dagger)^5$
		$e^5(\mu^\dagger)^4\tau^\dagger$
	21	$\mu^7(\tau^\dagger)^7$
		$e^7(\tau^\dagger)^7$
		$e^7(\mu^\dagger)^7$
8(0, 1)	3	$e\mu^\dagger$
	24	$\mu^8(\tau^\dagger)^8$
		$e\mu^7(\tau^\dagger)^8$
		$e^2\mu^6(\tau^\dagger)^8$
		$e^3\mu^5(\tau^\dagger)^8$
		$e^4\mu^4(\tau^\dagger)^8$
		$e^5\mu^3(\tau^\dagger)^8$
		$e^6\mu^2(\tau^\dagger)^8$
		$e^7\mu(\tau^\dagger)^8$
		$e^8(\tau^\dagger)^8$
8(1, 1)	6	$e(\mu^\dagger)^2\tau$
	12	$e^4(\tau^\dagger)^4$
	15	$e^3\mu^2(\tau^\dagger)^5$
		$e^5(\mu^\dagger)^2(\tau^\dagger)^3$
	18	$e^2\mu^4(\tau^\dagger)^6$
		$e^6(\mu^\dagger)^4(\tau^\dagger)^2$
	21	$e\mu^6(\tau^\dagger)^7$
		$e^7(\mu^\dagger)^6\tau^\dagger$
	24	$\mu^8(\tau^\dagger)^8$
		$e^8(\mu^\dagger)^8$
8(1, 2)	9	$e^2\mu(\tau^\dagger)^3$
		$e^2(\mu^\dagger)^3\tau$
	12	$\mu^4(\tau^\dagger)^4$
		$e^4(\mu^\dagger)^2(\tau^\dagger)^2$
	18	$e^6\mu^\dagger(\tau^\dagger)^5$
		$e^6(\mu^\dagger)^5\tau^\dagger$
	24	$e^8(\tau^\dagger)^8$
		$e^8(\mu^\dagger)^8$
8(1, 3)	6	$e^2(\tau^\dagger)^2$
	12	$e(\mu^\dagger)^4\tau^3$
		$e^3(\mu^\dagger)^4\tau$
	15	$e\mu^4(\tau^\dagger)^5$
		$e^5(\mu^\dagger)^4\tau^\dagger$
	24	$\mu^8(\tau^\dagger)^8$
		$e^8(\mu^\dagger)^8$

APPENDIX C: FLAVOR CHARGE ASSIGNMENTS FOR $G_\ell = \mathbb{Z}_3$

In order to clarify the number of possible flavor charge assignments, we list all of them for $G_\ell = \mathbb{Z}_3$ in Table III.

TABLE III. All flavor charge assignments for $G_\ell = \mathbb{Z}_3$. P_1, P_2, and P_3 label the permutations of the charged lepton flavors.

P_1		P_2		P_3	
$N(\delta_1,\delta_2)$	Explicit form	$N(\delta_1,\delta_2)$	Explicit form	$N(\delta_1,\delta_2)$	Explicit form
3(0,1)	$\mathbb{Z}_3(0,0,1)$	3(1, 2)	$\mathbb{Z}_3(0,1,0)$	3(2,0)	$\mathbb{Z}_3(1,0,0)$
	$\mathbb{Z}_3(1,1,2)$		$\mathbb{Z}_3(1,2,1)$		$\mathbb{Z}_3(2,1,1)$
	$\mathbb{Z}_3(2,2,0)$		$\mathbb{Z}_3(2,0,2)$		$\mathbb{Z}_3(0,2,2)$
3(0,2)	$\mathbb{Z}_3(0,0,2)$	3(2, 1)	$\mathbb{Z}_3(0,2,0)$	3(1,0)	$\mathbb{Z}_3(0,1,1)$
	$\mathbb{Z}_3(1,1,0)$		$\mathbb{Z}_3(1,0,1)$		$\mathbb{Z}_3(1,2,2)$
	$\mathbb{Z}_3(2,2,1)$		$\mathbb{Z}_3(2,1,2)$		$\mathbb{Z}_3(2,0,0)$

$P_{1,2,3}$	
$N(\delta_1,\delta_2)$	Explicit form
$\mathbf{3(1,1)}$	$\mathbb{Z}_\mathbf{3}(\mathbf{0},\mathbf{1},\mathbf{2})$
	$\mathbb{Z}_\mathbf{3}(\mathbf{1},\mathbf{2},\mathbf{0})$
	$\mathbb{Z}_\mathbf{3}(\mathbf{2},\mathbf{0},\mathbf{1})$
$\mathbf{3(2,2)}$	$\mathbb{Z}_\mathbf{3}(\mathbf{0},\mathbf{2},\mathbf{1})$
	$\mathbb{Z}_\mathbf{3}(\mathbf{1},\mathbf{0},\mathbf{2})$
	$\mathbb{Z}_\mathbf{3}(\mathbf{2},\mathbf{1},\mathbf{0})$

APPENDIX D: NUMERICAL VALUES FOR Λ FROM CURRENT AND FUTURE CLFV SEARCHES

In Sec. V, the calculation of lower limits on and future sensitivities to the new physics scale Λ is discussed, and the results are displayed in Fig. 2. In this appendix, the experimental limits employed and the numerical values of Λ are tabulated in Tables IV and V.

TABLE IV. Numerical values of the current lower bounds on and future sensitivities to the new physics scale Λ. All experimental limits, current and expected, are given at 90% CL. Three different scenarios are considered: one with tree-level contributions, $C_x = 1$, except the WCs of the dipole operators, $C_d = e/(16\pi^2)$, leading to the limits denoted as Λ_{T}; one with loop-level contributions, $C_x = 1/(16\pi^2)$, resulting in Λ_{L}; a third one with chirally suppressed WCs of the dipole operators, $C_d = \sqrt{2} m_\ell e/(16\pi^2 v)$, in each of the former two scenarios, corresponding to $\Lambda_{\mathrm{T}\chi}$ and $\Lambda_{\mathrm{L}\chi}$, respectively.

			Current (Λ in TeV)			Future (Λ in TeV)		
$N(\delta_1, \delta_2)$		Observable	Constraint	$\Lambda_{\mathrm{T}}(\Lambda_{\mathrm{T}\chi})$	$\Lambda_{\mathrm{L}}(\Lambda_{\mathrm{L}\chi})$	Constraint	Λ_{T}	Λ_{L}
$\mathbf{2(0,1)}$	$e\mu^\dagger$	$\mathrm{BR}(\mu \to e\gamma)$	1.5×10^{-13} [54]	3000 (73)	3000 (73)	6×10^{-14} [55]	3900	3900
		$\mathrm{BR}(\mu \to ee\bar{e})$	1.0×10^{-12} [56]	500 (290)	520 (23)	10^{-16} [57]	5000	5200
		$\mathrm{CR}(\mu\mathrm{Au} \to e\mathrm{Au})$	7×10^{-13} [58]	1800 (1800)	430 (140)			
		$\mathrm{CR}(\mu\mathrm{Al} \to e\mathrm{Al})$				6×10^{-17} [59,60]	15 000	4000
	$ee\mu^\dagger\mu^\dagger$	$\mathrm{P}(\mathrm{M} \to \bar{\mathrm{M}})$	8.2×10^{-11} [62]	9.5	0.76	10^{-13} [63]	51	4.1
$\mathbf{2(1,0)}$	$\mu\tau^\dagger$	$\mathrm{BR}(\tau \to \mu\gamma)$	4.2×10^{-8} [66]	20 (2.1)	20 (2.1)	6.9×10^{-9} [73]	32	32
		$\mathrm{BR}(\tau \to \mu\rho)$	1.7×10^{-8} [67]	21 (20)	5.9 (1.6)	5.5×10^{-10} [73]	49	14
		$\mathrm{BR}(\tau \to \mu\phi)$	2.3×10^{-8} [67]	14 (14)	3.2 (1.1)	8.4×10^{-10} [73]	33	7.3
		$\mathrm{BR}(\tau \to \mu\pi)$	1.1×10^{-7} [64]	7.8	0.62	7.1×10^{-10} [73]	28	2.2
		$\mathrm{BR}(\tau \to \mu K)$	2.3×10^{-8} [69]	14	1.1	4.0×10^{-10} [73]	39	3.1
		$\mathrm{BR}(\tau \to \mu\mu\bar{\mu})$	1.9×10^{-8} [71]	16 (16)	5.3 (1.3)	3.6×10^{-10} [73]	42	14
		$\mathrm{BR}(\tau \to \mu e\bar{e})$	1.8×10^{-8} [70]	15 (15)	5.3 (1.2)	2.9×10^{-10} [73]	42	15
	$ee\mu^\dagger\mu^\dagger$	$\mathrm{P}(\mathrm{M} \to \bar{\mathrm{M}})$	8.2×10^{-11} [62]	9.5	0.76	10^{-13} [63]	51	4.1
	$ee\mu^\dagger\tau^\dagger$	$\mathrm{BR}(\tau \to ee\bar{\mu})$	1.5×10^{-8} [70]	16	1.3	2.3×10^{-10} [73]	45	3.6
$\mathbf{2(1,1)}$	$e\tau^\dagger$	$\mathrm{BR}(\tau \to e\gamma)$	3.3×10^{-8} [68]	22 (2.2)	22 (2.2)	9.0×10^{-9} [73]	30	30
		$\mathrm{BR}(\tau \to e\rho)$	2.2×10^{-8} [67]	20 (19)	5.6 (1.5)	3.8×10^{-10} [73]	54	15
		$\mathrm{BR}(\tau \to e\phi)$	2.0×10^{-8} [67]	15 (15)	3.4 (1.2)	7.4×10^{-10} [73]	35	7.7
		$\mathrm{BR}(\tau \to e\pi)$	8.0×10^{-8} [65]	8.5	0.67	7.3×10^{-10} [73]	27	2.2
		$\mathrm{BR}(\tau \to eK)$	2.6×10^{-8} [69]	14	1.1	4.0×10^{-10} [73]	40	3.2
		$\mathrm{BR}(\tau \to ee\bar{e})$	2.7×10^{-8} [70]	15 (15)	7.3 (1.2)	4.7×10^{-10} [73]	40	20
		$\mathrm{BR}(\tau \to e\mu\bar{\mu})$	2.7×10^{-8} [70]	14 (14)	7.3 (1.1)	4.5×10^{-10} [73]	38	20
	$ee\mu^\dagger\mu^\dagger$	$\mathrm{P}(\mathrm{M} \to \bar{\mathrm{M}})$	8.2×10^{-11} [62]	9.5	0.76	10^{-13} [63]	51	4.1
	$e\mu^\dagger\mu^\dagger\tau$	$\mathrm{BR}(\tau \to \mu\mu\bar{e})$	1.7×10^{-8} [70]	16	1.2	2.6×10^{-10} [73]	44	3.5
$\mathbf{3(1,1)}$	$e\mu^\dagger\mu^\dagger\tau$	$\mathrm{BR}(\tau \to \mu\mu\bar{e})$	1.7×10^{-8} [70]	16	1.2	2.6×10^{-10} [73]	44	3.5
	$ee\mu^\dagger\tau^\dagger$	$\mathrm{BR}(\tau \to ee\bar{\mu})$	1.5×10^{-8} [70]	16	1.3	2.3×10^{-10} [73]	45	3.6
$\mathbf{4(1,1)}$	$e\mu^\dagger\mu^\dagger\tau$	$\mathrm{BR}(\tau \to \mu\mu\bar{e})$	1.7×10^{-8} [70]	16	1.2	2.6×10^{-10} [73]	44	3.5
$\mathbf{4(2,3)}$	$ee\mu^\dagger\mu^\dagger$	$\mathrm{P}(\mathrm{M} \to \bar{\mathrm{M}})$	8.2×10^{-11} [62]	9.5	0.76	10^{-13} [63]	51	4.1
$\mathbf{4(3,2)}$	$ee\mu^\dagger\tau^\dagger$	$\mathrm{BR}(\tau \to ee\bar{\mu})$	1.5×10^{-8} [70]	16	1.3	2.3×10^{-10} [73]	45	3.6

TABLE V. Compilation of current limits on and future sensitivities to the new physics scale Λ arising from high-energy probes of cLFV physics. All calculations assume the scenario with tree-level contributions; see Sec. V B for details. The notation follows that of Table IV.

			Current (Λ in TeV)		Future (Λ in TeV)	
$N(\delta_1, \delta_2)$		Observable	Constraint	Λ_{T}	Constraint	Λ_{T}
$\mathbf{2(a,b)}^{\times}$, $\mathbf{4(3,2)}$	$\mu\mu\tau^\dagger\tau^\dagger$	$\sigma(\mu^+\mu^+ \to \tau^+\tau^+)$			0.3 fb [86]	30
		$\mathrm{BR}(Z \to \tau\tau\bar{\mu}\,\bar{\mu})$	2×10^{-3} [31]	0.001	10^{-12} [31]	0.25
$\mathbf{2(a,b)}^{\times}$, $\mathbf{4(1,1)}$	$ee\tau^\dagger\tau^\dagger$	$\mathrm{BR}(Z \to \tau\tau\bar{e}\,\bar{e})$	2×10^{-3} [31]	0.001	10^{-12} [31]	0.25
$\mathbf{2(0,1)}, \mathbf{3(1,1)}, \mathbf{4(2,3)}$	$e\mu\tau^\dagger\tau^\dagger$	$\mathrm{BR}(Z \to \tau\tau\bar{e}\,\bar{\mu})$	2×10^{-3} [31]	0.001	10^{-12} [31]	0.21

$^{\times}\mathbf{2(a,b)}$ stands for $\mathbf{2(0,1)}$, $\mathbf{2(1,0)}$, and $\mathbf{2(1,1)}$.

APPENDIX E: CONSTRAINTS ON Λ FOR REMAINING FLAVOR CHARGE ASSIGNMENTS

We present for the flavor charge assignments not appearing in Fig. 1 the tables analogous to Tables IV and V (see Tables VI and VII) as well as the figure corresponding to Fig. 2 (see Fig. 3).

TABLE VI. Numerical values of the current lower bounds on and future sensitivities to the new physics scale Λ for flavor charge assignments listed in the table at the bottom of Fig. 1. Conventions are the same as in Table IV.

			Current (Λ in TeV)			Future (Λ in TeV)		
$N(\delta_1,\delta_2)$		Observable	Constraint	$\Lambda_T(\Lambda_{T\chi})$	$\Lambda_L(\Lambda_{L\chi})$	Constraint	Λ_T	Λ_L
$N(0,a)$	$e\mu^\dagger$	$\mathrm{BR}(\mu\to e\gamma)$	1.5×10^{-13} [54]	3000 (73)	3000 (73)	6×10^{-14} [55]	3900	3900
		$\mathrm{BR}(\mu\to ee\bar{e})$	1.0×10^{-12} [56]	500 (290)	520 (23)	10^{-16} [57]	5000	5200
		$\mathrm{CR}(\mu\mathrm{Au}\to e\mathrm{Au})$	7×10^{-13} [58]	1800 (1800)	430 (140)			
		$\mathrm{CR}(\mu\mathrm{Al}\to e\mathrm{Al})$				6×10^{-17} [59,60]	15 000	4000
$N(a,0)$	$\mu\tau^\dagger$	$\mathrm{BR}(\tau\to\mu\gamma)$	4.2×10^{-8} [66]	20 (2.1)	20 (2.1)	6.9×10^{-9} [73]	32	32
		$\mathrm{BR}(\tau\to\mu\rho)$	1.7×10^{-8} [67]	21 (20)	5.9 (1.6)	5.5×10^{-10} [73]	49	14
		$\mathrm{BR}(\tau\to\mu\phi)$	2.3×10^{-8} [67]	14 (14)	3.2 (1.1)	8.4×10^{-10} [73]	33	7.3
		$\mathrm{BR}(\tau\to\mu\pi)$	1.1×10^{-7} [64]	7.8	0.62	7.1×10^{-10} [73]	28	2.2
		$\mathrm{BR}(\tau\to\mu K)$	2.3×10^{-8} [69]	14	1.1	4.0×10^{-10} [73]	39	3.1
		$\mathrm{BR}(\tau\to\mu\mu\bar{\mu})$	1.9×10^{-8} [71]	16 (16)	5.3 (1.3)	3.6×10^{-10} [73]	42	14
		$\mathrm{BR}(\tau\to\mu e\bar{e})$	1.8×10^{-8} [70]	15 (15)	5.3 (1.2)	2.9×10^{-10} [73]	42	15
$N(a,N-a)$	$e\tau^\dagger$	$\mathrm{BR}(\tau\to e\gamma)$	3.3×10^{-8} [68]	22 (2.2)	22 (2.2)	9.0×10^{-9} [73]	30	30
		$\mathrm{BR}(\tau\to e\rho)$	2.2×10^{-8} [67]	20 (19)	5.6 (1.5)	3.8×10^{-10} [73]	54	15
		$\mathrm{BR}(\tau\to e\phi)$	2.0×10^{-8} [67]	15 (15)	3.4 (1.2)	7.4×10^{-10} [73]	35	7.7
		$\mathrm{BR}(\tau\to e\pi)$	8.0×10^{-8} [65]	8.5	0.67	7.3×10^{-10} [73]	27	2.2
		$\mathrm{BR}(\tau\to eK)$	2.6×10^{-8} [69]	14	1.1	4.0×10^{-10} [73]	40	3.2
		$\mathrm{BR}(\tau\to ee\bar{e})$	2.7×10^{-8} [70]	15 (15)	7.3 (1.2)	4.7×10^{-10} [73]	40	20
		$\mathrm{BR}(\tau\to e\mu\bar{\mu})$	2.7×10^{-8} [70]	14 (14)	7.3 (1.1)	4.5×10^{-10} [73]	38	20
$2N(N,a)$	$ee\mu^\dagger\mu^\dagger$	$\mathrm{P}(\mathrm{M}\to\bar{\mathrm{M}})$	8.2×10^{-11} [62]	9.5	0.76	10^{-13} [63]	51	4.1
$N(N-a,2a)$	$ee\mu^\dagger\tau^\dagger$	$\mathrm{BR}(\tau\to ee\bar{\mu})$	1.5×10^{-8} [70]	16	1.3	2.3×10^{-10} [73]	45	3.6
$N(a,a)$	$e\mu^\dagger\mu^\dagger\tau$	$\mathrm{BR}(\tau\to\mu\mu\bar{e})$	1.7×10^{-8} [70]	16	1.2	2.6×10^{-10} [73]	44	3.5

TABLE VII. Compilation of current limits on and future sensitivities to the new physics scale Λ arising from high-energy probes of cLFV physics for flavor charge assignments listed in the table at the bottom of Fig. 1. Conventions are the same as in Table V.

			Current (Λ in TeV)		Future (Λ in TeV)	
$N(\delta_1,\delta_2)$		Observable	Constraint	Λ_T	Constraint	Λ_T
$2N(a,N)$	$\mu\mu\tau^\dagger\tau^\dagger$	$\sigma(\mu^+\mu^+\to\tau^+\tau^+)$			0.3 fb [86]	30
		$\mathrm{BR}(Z\to\tau\tau\bar{\mu}\,\bar{\mu})$	2×10^{-3} [31]	0.001	10^{-12} [31]	0.25
$2N(a,N-a)$	$ee\tau^\dagger\tau^\dagger$	$\mathrm{BR}(Z\to\tau\tau\bar{e}\,\bar{e})$	2×10^{-3} [31]	0.001	10^{-12} [31]	0.25
$N(2a,N-a)$	$e\mu\tau^\dagger\tau^\dagger$	$\mathrm{BR}(Z\to\tau\tau\bar{e}\,\bar{\mu})$	2×10^{-3} [31]	0.001	10^{-12} [31]	0.21

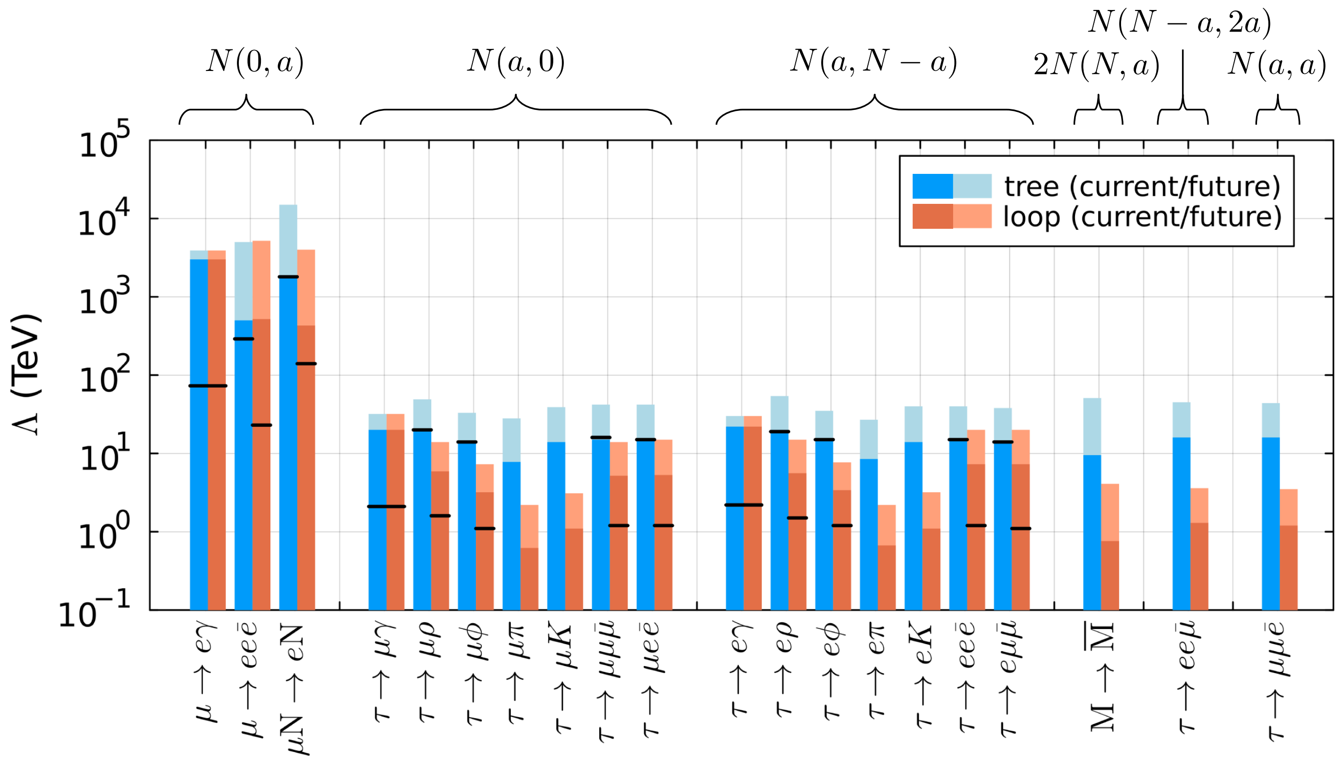


FIG. 3. Current lower bounds on and future sensitivities to the new physics scale Λ (in TeV), based on experimental limits for various processes at 90% CL, for flavor charge assignments listed in the table at the bottom of Fig. 1. Conventions are the same as in Fig. 2.

[1] S. F. King, Models of neutrino mass, mixing and *CP* violation, J. Phys. G **42**, 123001 (2015).
[2] F. Feruglio and A. Romanino, Lepton flavor symmetries, Rev. Mod. Phys. **93**, 015007 (2021).
[3] Z.-z. Xing, Flavor structures of charged fermions and massive neutrinos, Phys. Rep. **854**, 1 (2020).
[4] T. Kobayashi, H. Ohki, H. Okada, Y. Shimizu, and M. Tanimoto, *An Introduction to Non-Abelian Discrete Symmetries for Particle Physicists* Lecture Notes in Physics Vol. 1 (Springer Berlin, Heidelberg, Germany, 2022).
[5] W. Grimus and P. O. Ludl, Finite flavour groups of fermions, J. Phys. A **45**, 233001 (2012).
[6] G. Altarelli and F. Feruglio, Tri-bimaximal neutrino mixing, A_4 and the modular symmetry, Nucl. Phys. **B741**, 215 (2006).
[7] X.-G. He, Y.-Y. Keum, and R. R. Volkas, A_4 flavor symmetry breaking scheme for understanding quark and neutrino mixing angles, J. High Energy Phys. 04 (2006) 039.
[8] C. S. Lam, Determining horizontal symmetry from neutrino mixing, Phys. Rev. Lett. **101**, 121602 (2008).
[9] I. Esteban, M. C. Gonzalez-Garcia, M. Maltoni, I. Martinez-Soler, J. a. P. Pinheiro, and T. Schwetz, NuFit-6.0: Updated global analysis of three-flavor neutrino oscillations, J. High Energy Phys. 12 (2024) 216.
[10] Y. Lin, A predictive A_4 model, Charged lepton hierarchy and tri-bimaximal sum rule, Nucl. Phys. **B813**, 91 (2009).
[11] Y. Shimizu, M. Tanimoto, and A. Watanabe, Breaking tri-bimaximal mixing and large θ_{13}, Prog. Theor. Phys. **126**, 81 (2011).
[12] S. F. King and C. Luhn, A_4 models of tri-bimaximal-reactor mixing, J. High Energy Phys. 03 (2012) 036.
[13] I. de Medeiros Varzielas and D. Pidt, UV completions of flavour models and large θ_{13}, J. High Energy Phys. 03 (2013) 065.
[14] F. Feruglio, C. Hagedorn, Y. Lin, and L. Merlo, Lepton flavour violation in models with A_4 flavour symmetry, Nucl. Phys. **B809**, 218 (2009).
[15] C. Csaki, C. Delaunay, C. Grojean, and Y. Grossman, A model of lepton masses from a warped extra dimension, J. High Energy Phys. 10 (2008) 055.
[16] E. Ma, Quark and lepton flavor triality, Phys. Rev. D **82**, 037301 (2010).
[17] Q.-H. Cao, A. Damanik, E. Ma, and D. Wegman, Probing lepton flavor triality with Higgs boson decay, Phys. Rev. D **83**, 093012 (2011).

[18] C. Hagedorn and M. Serone, Leptons in holographic composite Higgs models with non-Abelian discrete symmetries, J. High Energy Phys. 10 (2011) 083.

[19] M. Holthausen, M. Lindner, and M. A. Schmidt, Lepton flavor at the electroweak scale: A complete A_4 model, Phys. Rev. D **87**, 033006 (2013).

[20] S. Pascoli and Y.-L. Zhou, Flavon-induced connections between lepton flavour mixing and charged lepton flavour violation processes, J. High Energy Phys. 10 (2016) 145.

[21] Y. Muramatsu, T. Nomura, and Y. Shimizu, Mass limit for light flavon with residual Z_3 symmetry, J. High Energy Phys. 03 (2016) 192.

[22] I. Bigaran, X.-G. He, M. A. Schmidt, G. Valencia, and R. Volkas, Lepton-flavor-violating tau decays from triality, Phys. Rev. D **107**, 055001 (2023).

[23] G. Lichtenstein, M. A. Schmidt, G. Valencia, and R. R. Volkas, Complementarity of μTRISTAN and Belle II in searches for charged-lepton flavour violation, Phys. Lett. B **845**, 138144 (2023).

[24] G. Lichtenstein, M. A. Schmidt, G. Valencia, and R. R. Volkas, Z and Higgs boson decays with doubly-charged scalars at one-loop: Current constraints, future sensitivities, and application to lepton-triality models, Nucl. Phys. **B1013**, 116850 (2025).

[25] L. Calibbi and G. Signorelli, Charged lepton flavour violation: An experimental and theoretical introduction, Riv. Nuovo Cimento **41**, 71 (2018).

[26] M. Ardu and G. Pezzullo, Introduction to charged lepton flavor violation, Universe **8**, 299 (2022).

[27] S. Davidson, B. Echenard, R. H. Bernstein, J. Heeck, and D. G. Hitlin, Charged lepton flavor violation, arXiv:2209.00142.

[28] G. Blankenburg, G. Isidori, and J. Jones-Perez, Neutrino masses and LFV from minimal breaking of $U(3)^5$ and $U(2)^5$ flavor symmetries, Eur. Phys. J. C **72**, 2126 (2012).

[29] J. Jones-Pérez Split-family SUSY, $U(2)^5$ flavour symmetry and neutrino physics, Eur. Phys. J. C **74**, 2772 (2014); Eur. Phys. J. C **74**, 2886(E) (2014).

[30] J. Heeck, Interpretation of lepton flavor violation, Phys. Rev. D **95**, 015022 (2017).

[31] J. Heeck and M. Sokhashvili, Lepton flavor violation by two units, Phys. Lett. B **852**, 138621 (2024).

[32] A. Greljo, A. Palavrić, and A. E. Thomsen, Adding flavor to the SMEFT, J. High Energy Phys. 10 (2022) 010.

[33] A. Palavrić, Discrete leptonic flavor symmetries: UV mediators and phenomenology, Phys. Rev. D **110**, 115025 (2024).

[34] J. Heeck, M. Sokhashvili, and A. Thapa, Lepton flavor violation by three units, arXiv:2505.17178.

[35] W. Buchmuller and D. Wyler, Effective Lagrangian analysis of new interactions and flavor conservation, Nucl. Phys. **B268**, 621 (1986).

[36] B. Grzadkowski, M. Iskrzynski, M. Misiak, and J. Rosiek, Dimension-six terms in the standard model Lagrangian, J. High Energy Phys. 10 (2010) 085.

[37] B. Henning, X. Lu, and H. Murayama, How to use the standard model effective field theory, J. High Energy Phys. 01 (2016) 023.

[38] I. Brivio and M. Trott, The standard model as an effective field theory, Phys. Rep. **793**, 1 (2019).

[39] G. Isidori, F. Wilsch, and D. Wyler, The standard model effective field theory at work, Rev. Mod. Phys. **96**, 015006 (2024).

[40] C. Luhn, S. Nasri, and P. Ramond, The flavor group $\Delta(3n^2)$, J. Math. Phys. (N.Y.) **48**, 073501 (2007).

[41] J. A. Escobar and C. Luhn, The flavor group $\Delta(6n^2)$, J. Math. Phys. (N.Y.) **50**, 013524 (2009).

[42] S. F. King, T. Neder, and A. J. Stuart, Lepton mixing predictions from $\Delta(6n^2)$ family symmetry, Phys. Lett. B **726**, 312 (2013).

[43] G.-J. Ding and S. F. King, Generalized CP and $\Delta(3n^2)$ family symmetry for semi-direct predictions of the PMNS matrix, Phys. Rev. D **93**, 025013 (2016).

[44] C. Hagedorn, A. Meroni, and E. Molinaro, Lepton mixing from $\Delta(3n^2)$ and $\Delta(6n^2)$ and CP, Nucl. Phys. **B891**, 499 (2015).

[45] G.-J. Ding, S. F. King, and T. Neder, Generalised CP and $\Delta(6n^2)$ family symmetry in semi-direct models of leptons, J. High Energy Phys. 12 (2014) 007.

[46] R. de Adelhart Toorop, F. Feruglio, and C. Hagedorn, Finite modular groups and lepton mixing, Nucl. Phys. **B858**, 437 (2012).

[47] F. Feruglio and A. Paris, The Golden Ratio prediction for the solar angle from a natural model with A_5 flavour symmetry, J. High Energy Phys. 03 (2011) 101.

[48] A. Di Iura, C. Hagedorn, and D. Meloni, Lepton mixing from the interplay of the alternating group A_5 and CP, J. High Energy Phys. 08 (2015) 037.

[49] P. Ballett, S. Pascoli, and J. Turner, Mixing angle and phase correlations from A_5 with generalized CP and their prospects for discovery, Phys. Rev. D **92**, 093008 (2015).

[50] C.-C. Li and G.-J. Ding, Lepton mixing in A_5 family symmetry and generalized CP, J. High Energy Phys. 05 (2015) 100.

[51] C. Hagedorn, A. Meroni, and L. Vitale, Mixing patterns from the groups $\Sigma(n\varphi)$, J. Phys. A **47**, 055201 (2014).

[52] S. Weinberg, Baryon and lepton nonconserving processes, Phys. Rev. Lett. **43**, 1566 (1979).

[53] A. Crivellin, S. Najjari, and J. Rosiek, Lepton flavor violation in the standard model with general dimension-six operators, J. High Energy Phys. 04 (2014) 167.

[54] K. Afanaciev *et al.* (MEG II Collaboration), New limit on the $\mu^+ \to e^+\gamma$ decay with the MEG II experiment, arXiv:2504.15711.

[55] A. M. Baldini *et al.* (MEG II Collaboration), The search for $\mu^+ \to e^+\gamma$ with 10^{-14} sensitivity: The upgrade of the MEG experiment, Symmetry **13**, 1591 (2021).

[56] U. Bellgardt *et al.* (SINDRUM Collaboration), Search for the decay $\mu^+ \to e^+e^+e^-$, Nucl. Phys. **B299**, 1 (1988).

[57] G. Hesketh, S. Hughes, A.-K. Perrevoort, and N. Rompotis (Mu3e Collaboration), The Mu3e experiment, in Snowmass 2021 (2022), arXiv:2204.00001.

[58] W. H. Bertl *et al.* (SINDRUM II Collaboration), A search for muon to electron conversion in muonic gold, Eur. Phys. J. C **47**, 337 (2006).

[59] L. Bartoszek *et al.* (Mu2e Collaboration), Mu2e technical design report, arXiv:1501.05241.

[60] R. Abramishvili *et al.* (COMET Collaboration), COMET Phase-I technical design report, Prog. Theor. Exp. Phys. **2020**, 033C01 (2020).

[61] K. Byrum *et al.* (Mu2e-II Collaboration), Mu2e-II: Muon to electron conversion with PIP-II, in Snowmass 2021 (2022), arXiv:2203.07569.

[62] L. Willmann *et al.*, New bounds from searching for muonium to anti-muonium conversion, Phys. Rev. Lett. **82**, 49 (1999).

[63] A.-Y. Bai *et al.*, Conceptual design of the muonium-to-antimuonium conversion experiment (MACE), arXiv:2410.18817.

[64] B. Aubert *et al.* (*BABAR* Collaboration), Search for lepton flavor violating decays $\tau^\pm \to \ell^\pm\pi^0$, $\ell^\pm\eta$, $\ell^\pm\eta'$, Phys. Rev. Lett. **98**, 061803 (2007).

[65] Y. Miyazaki *et al.* (Belle Collaboration), Search for lepton flavor violating τ^- decays into $\ell^-\eta$, $\ell^-\eta'$ and $\ell^-\pi^0$, Phys. Lett. B **648**, 341 (2007).

[66] A. Abdesselam *et al.* (Belle Collaboration), Search for lepton-flavor-violating tau-lepton decays to $\ell\gamma$ at Belle, J. High Energy Phys. 10 (2021) 19.

[67] N. Tsuzuki *et al.* (Belle Collaboration), Search for lepton-flavor-violating τ decays into a lepton and a vector meson using the full Belle data sample, J. High Energy Phys. 06 (2023) 118.

[68] B. Aubert *et al.* (*BABAR* Collaboration), Searches for lepton flavor violation in the decays $\tau^\pm \to e^\pm\gamma$ and $\tau^\pm \to \mu^\pm\gamma$, Phys. Rev. Lett. **104**, 021802 (2010).

[69] Y. Miyazaki *et al.* (Belle Collaboration), Search for lepton flavor violating τ^- decays into $\ell^- K_s^0$ and $\ell^- K_s^0 K_s^0$, Phys. Lett. B **692**, 4 (2010).

[70] K. Hayasaka *et al.*, Search for lepton flavor violating tau decays into three leptons with 719 million produced $\tau^+\tau^-$ pairs, Phys. Lett. B **687**, 139 (2010).

[71] I. Adachi *et al.* (Belle-II Collaboration), Search for lepton-flavor-violating $\tau^- \to \mu^-\mu^+\mu^-$ decays at Belle II, J. High Energy Phys. 09 (2024) 062.

[72] S. Banerjee *et al.*, Snowmass 2021 white paper: Charged lepton flavor violation in the tau sector, arXiv:2203.14919.

[73] L. Aggarwal *et al.* (Belle-II Collaboration), Snowmass white paper: Belle II physics reach and plans for the next decade and beyond, arXiv:2207.06307.

[74] S. Banerjee, Searches for lepton flavor violation in tau decays at Belle II, Universe **8**, 480 (2022).

[75] L. Calibbi, X. Marcano, and J. Roy, Z lepton flavour violation as a probe for new physics at future e^+e^- colliders, Eur. Phys. J. C **81**, 1054 (2021).

[76] R. Alonso, E. E. Jenkins, A. V. Manohar, and M. Trott, Renormalization group evolution of the standard model dimension six operators III: Gauge coupling dependence and phenomenology, J. High Energy Phys. 04 (2014) 159.

[77] M. Carpentier and S. Davidson, Constraints on two-lepton, two quark operators, Eur. Phys. J. C **70**, 1071 (2010).

[78] G. M. Pruna and A. Signer, The $\mu \to e\gamma$ decay in a systematic effective field theory approach with dimension 6 operators, J. High Energy Phys. 10 (2014) 014.

[79] F. Feruglio, P. Paradisi, and A. Pattori, Lepton flavour violation in composite Higgs models, Eur. Phys. J. C **75**, 579 (2015).

[80] A. Crivellin, S. Davidson, G. M. Pruna, and A. Signer, Renormalisation-group improved analysis of $\mu \to e$ processes in a systematic effective-field-theory approach, J. High Energy Phys. 05 (2017) 117.

[81] S. Davidson and A. Saporta, Constraints on $2\ell 2q$ operators from $\mu - e$ flavour-changing meson decays, Phys. Rev. D **99**, 015032 (2019).

[82] M. Ardu and S. Davidson, What is leading order for LFV in SMEFT?, J. High Energy Phys. 08 (2021) 002.

[83] M. Ardu, S. Davidson, and M. Gorbahn, Sensitivity of $\mu \to \mathrm{e}$ processes to τ flavor change, Phys. Rev. D **105**, 096040 (2022).

[84] S. Jahedi and A. Sarkar, Exploring optimal sensitivity of lepton flavor violating effective couplings at the e^+e^- colliders, Phys. Rev. D **110**, 095021 (2024).

[85] L. Calibbi, T. Li, X. Marcano, and M. A. Schmidt, Indirect constraints on lepton-flavor-violating quarkonium decays, Phys. Rev. D **106**, 115039 (2022).

[86] Y. Hamada, R. Kitano, R. Matsudo, H. Takaura, and M. Yoshida, μTRISTAN, Prog. Theor. Exp. Phys. **2022**, 053B02 (2022).

[87] W. Altmannshofer, P. Munbodh, and T. Oh, Probing lepton flavor violation at circular electron-positron colliders, J. High Energy Phys. 08 (2023) 026.

[88] K. Fridell, R. Kitano, and R. Takai, Lepton flavor physics at $\mu^+\mu^+$ colliders, J. High Energy Phys. 06 (2023) 086.

[89] W. Abdallah *et al.* (CEPC Study Group), CEPC Technical design report: Accelerator, Radiat. Detect. Technol. Methods **8**, 1 (2024); Radiat. Detect. Technol. Methods **9**, 184(E) (2025).

[90] A. Abada *et al.* (FCC Collaboration), FCC physics opportunities: Future circular collider conceptual design report volume 1, Eur. Phys. J. C **79**, 474 (2019).

[91] F. Feruglio, C. Hagedorn, Y. Lin, and L. Merlo, Tri-bimaximal neutrino mixing and quark masses from a discrete flavour symmetry, Nucl. Phys. **B775**, 120 (2007); Nucl. Phys. **B836**, 127(E) (2010).

[92] A. Blum, C. Hagedorn, and M. Lindner, Fermion masses and mixings from dihedral flavor symmetries with preserved subgroups, Phys. Rev. D **77**, 076004 (2008).

[93] C. Hagedorn and D. Meloni, D_{14}—A common origin of the Cabibbo angle and the lepton mixing angle θ^l_{13}, Nucl. Phys. **B862**, 691 (2012).