Unit 1

General overview of finance

Dept. Financial and Actuarial Economics

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1. Finance: concept and relevance

Finance: object of study

- Finance (or *financial economics)* **is** dedicated to

  1. the **analysis of financial decisions** of individuals and companies

  2. (including the **influence of the institutional environment),**

  3. as well as the **impact of such decisions on financial markets.**

1 and 3. On financial decisions and their repercussions on the markets

Examples:

- How should individuals select assets for their investment portfolios?
- How should households and businesses choose the form of financing that best suits them?
- How do they actually make those decisions and why?
- How can a company use the various contracts and financial instruments available to manage the various risks linked to its activity?
- How do companies decide between alternative investment projects?
- Why do American Treasury bills (*T- bills*) pay a different interest rate than Spanish ones (letras del Tesoro)?
- What determines the price of shares?

Key financial decisions: goals

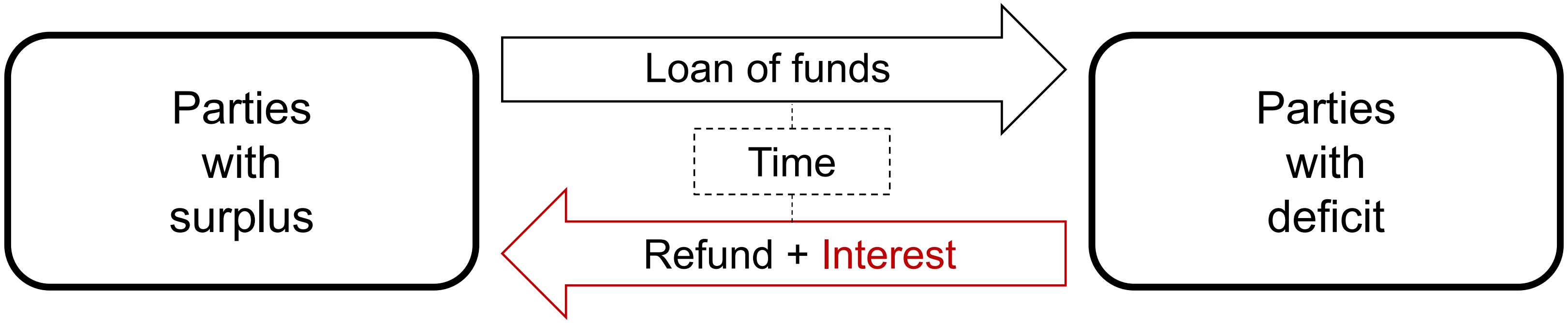
- The **main financial decisions** deal with:

1. The **transfer of financial resources** (cash flows) between economic agents **over time**
   - Example: investment and debt (financing) decisions

2. **Risk** management
   - Example: insurance and risk management decisions for an investment portfolio

1. Fund transfer

In economics, the first objective traditionally stands out:



- **Two** basic ideas:
  1. Lending money from parties (*economic agents*) with a surplus of funds to those with a deficit (financing needs).
  2. The return of borrowed funds occurs **after a period of time** and involves the payment of a reward called **interest**.

2. Risk management

In **financial economics**, the second objective is also fundamental:

- Many **financial decisions are made with uncertainty about the future**, so future events beyond our control could turn against us.
- In this uncertain environment, **economic and financial decisions carry risks** for our future well-being.
  - Example: In an investment, funds that are certain in the present are sacrificed in exchange for other (often uncertain) future funds, so that we could end up losing all or part of the money invested.
- Therefore, it may be **advisable or even necessary to manage these risks**.
  - Example: In the management of investments, attention is paid not only to the expected profitability but also to the associated risk.
  - In fact, there are **contracts and financial instruments specially designed** to assist individuals and companies in their management of a variety of financial risks.
    - Examples: Insurance contracts and what are called *derivative assets*, which we will see later.

Financial decisions: essential aspects

- **Essential** elements or attributes involved in financial decisions are:
  1. **Time**
     - What gives finance its **distinctive character within economics** is the consideration of the time factor. Indeed, finance is fundamentally interested in non-simultaneous exchanges of capital.
  2. **Risk**
     - Hand in hand with the question of time comes the recognition that the (future) world is generally **uncertain** and, therefore, involves the consideration of **risk.**
- In some decisions, the time factor predominates (such as when you borrow money at a fixed interest rate or when you invest in treasury bills until they mature); in others, both are important (stock investment); while finally, in yet others, risk is the fundamental aspect (in contracting insurance).
  - Formally, **financial economics** can be defined as the *discipline that studies the allocation of scarce resources over time, usually in an environment of uncertainty.*

2. On the influence of the institutional environment

- By ***institutional environment*** we assume:
    a. the financial system; and
    b. the action of the state (government).

a. People rely decisively on the financial system and its institutions to implement their financial decisions.

> *<u>Financial system</u>: Set of institutions (banks, insurers, etc.) dedicated to offering products and services to satisfy the financial needs of economic agents, together with their regulatory and supervisory bodies.*

b. Furthermore, the **action of the state** (government) significantly influences financial decisions,

> because in addition to **regulating** the financial system, it can provide certain financial services, such as those included in **government safety-net programmes, and can prohibit or favour certain products, services or actions, through the general regulatory framework (**within which the **tax system** stands out).

Relevance of finance

- Ultimately, financial decisions seek to *contribute to* ***economic well-being,*** relying on a well-functioning financial system.
- **Knowing finances is necessary to**:

  a. that **individuals/families and companies** efficiently use financial resources and more easily achieve their economic objectives; and

  b. make **public policy decisions.**
     - For example, about the financial system and its regulation

Recognition of research in finance

- The relevance of research in **financial economics** has been recognized on several occasions with the prize popularly known as the **Nobel Prize in Economics**:

  1990: **H. Markowitz, M. Miller and W. Sharpe**
  - Portfolio theory, capital structure in companies, relationship between risk and return of assets in the market

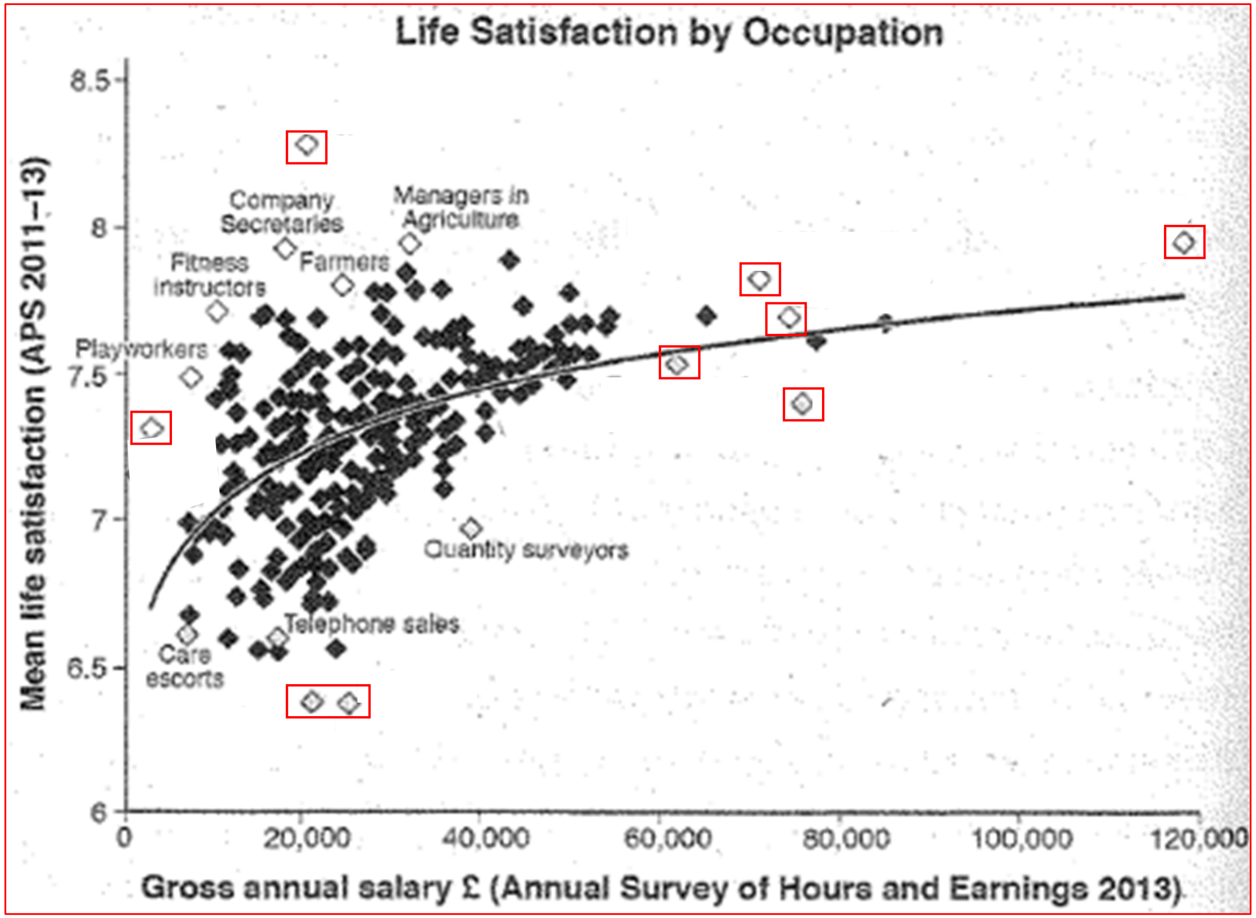
  1997: **R. Merton and M. Scholes**
  - Valuation of options (derivative instruments)

  2013: **E. Fama, L.P. Hansen, R. Shiller**
  - Theories and empirical analisis to explain the behaviour of financial markets

  2017: **R. Thaler**
  - Behavioural Economics/Finance (Behavioural)

  2022: **B. Bernanke, P. Dybvig and D. Diamond**
  - Banking crises

Market value of finance jobs and personal satisfaction



Source: Halpern (2015)

Figure 37. Life satisfaction versus earnings by occupation among UK workers, 2013 data. (Analysis by Cabinet Office analyst Ewen McKinnon.)

2. Main fields of study

Fields of study

Personal finances (or household finances)	• Financial decisions of **individuals** (households)
Corporate finance (or business)	• Financial decisions of non-financial companies (+ Management of financial companies and their regulation)
Market finance	• Effects of the behaviour of economic agents in **financial markets** (especially on prices) (+ Organization and regulation of markets)

3. Financial system: functions and relevance

Financial system: definition

Set of institutions dedicated to offering products and services to satisfy the financial needs of economic agents, together with their regulatory and supervisory bodies.

- *Institutions*:
  - Financial markets
  - Financial companies (intermediaries)
- *Products and services:*
  - Contracts and financial instruments
  - Financial services
- *Regulatory and supervisory bodies:*
  - Government
  - Entities with supervision and regulation functions

Main functions of the financial system

(Bodie, Merton and Cleeton, 2009, section 2.3)

- The financial system fulfills a series of essential functions in advanced economies.
- In addition to the **basic functions** of **facilitating**

  (1) the **transfer of funds** between economic agents and over time and

  (2) risk transfer,

  it also provides other **additional features,** such as:

  (3) information on prices and yields and

  (4) payment clearing and settlement mechanisms,

  among others.

Economic relevance of the Spanish financial sector

- Financial intermediation and contracts have existed for centuries around the world, under different political and economic systems, although they have experienced a very notable development in the prosperous market economies of the modern era.

  “financial capitalism” is sometimes used to highlight the current relevance of finance (Shiller, 2012).

- So *the current weight of the financial sector in advanced countries is substantial.*

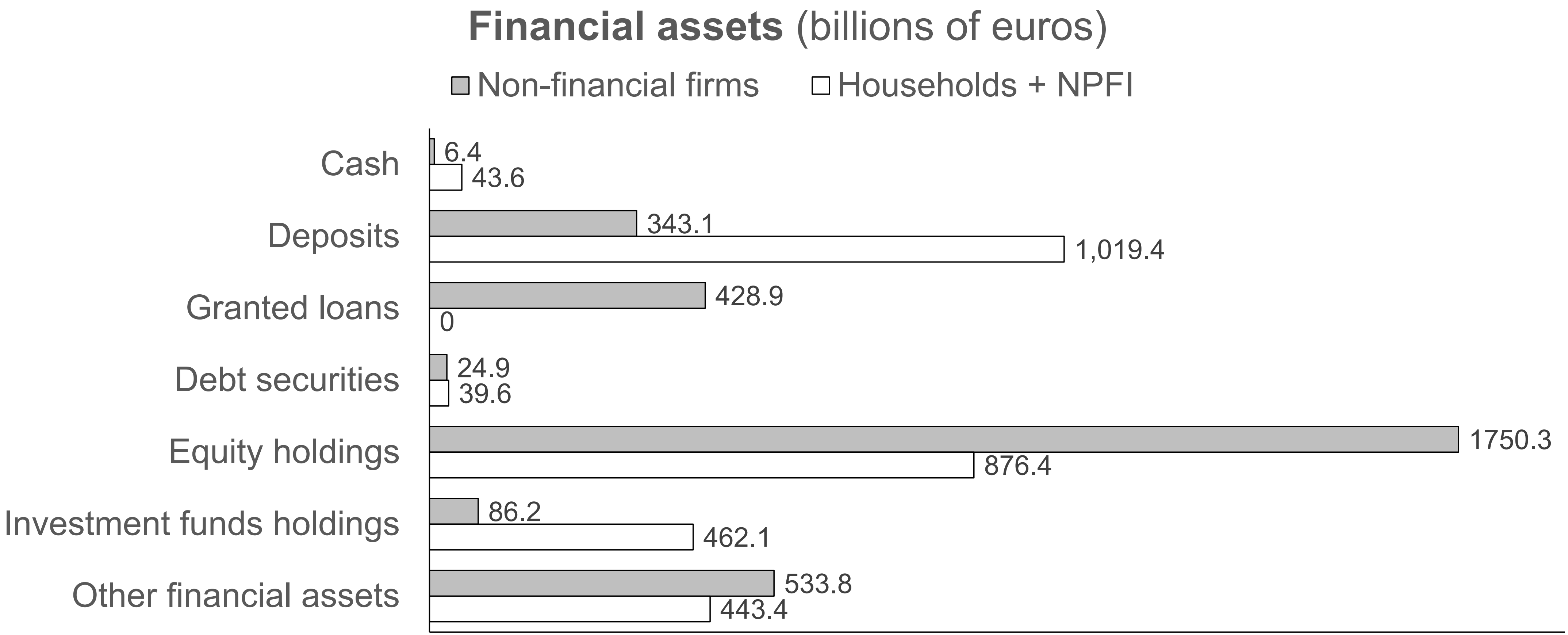
  The following slides show some related figures for the **Spanish case.**

1. Financial assets and liabilities, Spanish private sector (2024-1Q)

- **Financial sector** (excluding Bank of Spain):
  - Financial assets = 4,409,490 million of euros (4,297.72 billion euros or 4.297 trillion euros) = 297.8% annual GDP
  - Financial liabilities = 4.429 trillion euros = 299.2% annual GDP
- **Non-financial companies:**
  - Financial assets = 3.2 trillion euros
  - Financial liabilities = 4.9 trillion euros
- **Households + NPI:**
  - Financial assets = 2.9 trillion euros = 46% of gross housing wealth
  - Financial liabilities = 0.7 trillion euros

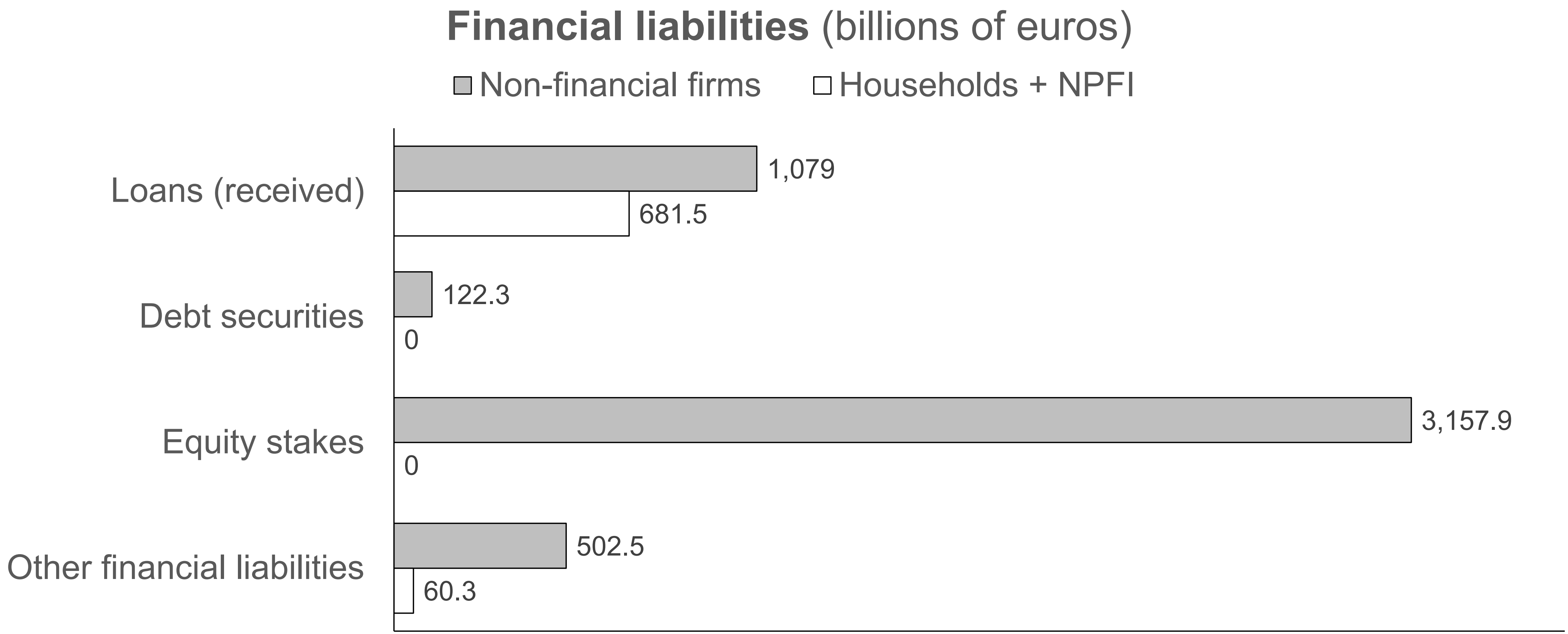
Data source: Financial accounts, Bank of Spain

2. Spanish non-financial sector financial balances: assets, 2024-1Q



Data source: Financial accounts, Bank of Spain

3. Spanish non-financial sector financial balances: liabilities, 2024-1Q



Data source: Financial accounts, Bank of Spain

Relevance of financial crises

- The relevance of finance is also unfortunately revealed when financial crises occur.
- The so-called **Global Financial Crisis** that began in 2007 is the latest landmark example of the high economic costs that can accompany financial crises:
  - Financial systems (including banks) struggled to carry out their essential functions, contributing to economic decline and unemployment around the world (*The Squam Lake Report,* 2010).
  - In **Spain,** the banking sector weakened as the economy contracted and in 2012 it required a rescue package of tens of billions of euros from the Eurozone countries.

- Studies by Reinhart, Rogoff and other researchers have documented that financial crises that have occurred throughout the history of capitalism share certain common patterns.

  In particular, recessions are much more severe when they are accompanied by banking crises and these are preceded by an escalation in private debt (of households and non-financial companies). (Mian and Sufi, 2015)

- In the same way that economic crises focus on the weaknesses (real or apparent) of the market economy and economic discipline, financial crises, such as the one mentioned above, provoke similar reactions regarding the financial system and finances, towards which the angriest criticism is sometimes directed.

Conclusions about the financial system

- Understanding the financial system and its functioning is more important today than ever, due to the great development it has experienced in recent decades and the great influence it has on our economic well-being: for better, when it performs its functions correctly; and for worse, when it does not.

- Both its understanding, including everything positive that it can contribute to our daily economic activity, and the proposals for regulatory reforms, to ensure that it continues to develop so that it increasingly fulfills its functions, should rest on the findings of the body of knowledge of financial theory and its evidence.

  - For more details on the role that the financial system plays in our economies, see Shiller (2002); and for proposals for reform of the financial system after the 2007 crisis, based on the academic research available at that time, see *The Squam Lake Report* (2010).

4. Concept of risk in finance

Concept of risk in finance: sources

- The concept of **risk** in finance refers to some ***unwanted effect of uncertainty.***
- There are multiple **sources of risk** that can affect individuals and companies. <u>Examples</u>:
  - Risks related to people's (lack of) health: illness, accident, disability or unexpected death.
  - Risk of damage to durable consumer goods (house, vehicle, etc.): fire, theft, accident, and so on.
  - Risk of legal liabilities for damage or loss resulting from our actions (or those of others for whom we are responsible).
  - Risks derived from unexpected changes in economic and financial variables such as: prices of goods or assets, yields, interest rates, exchange rates, inflation, etc.
  - Risk derived from the counterparty in a contract failing to fulfill their commitments.

Concept of risk in finance: basic ideas (1)

- Two basic ideas:

1. Often, **the risk** linked to uncertainty (i.e. the unintended consequences of the lack of certain knowledge about what will happen in the future) **depends on both the specific situation (context) and the person/company.**

   <u>Examples</u>:

   - An unexpected increase in input prices can negatively affect the profit margin of a company but not one which can pass this price increase on to its products without affecting its sales.
   - A real estate price slump hurts the owner of a property who plans to sell it in the future but does the very opposite to a potential buyer.

Concept of risk in finance: basic ideas (2)

2. In finance, the unintended consequences of uncertainty are varied, so **risk is measured in various ways,** without any of them being completely universal.

   - Frequently, **deviations from an expected result are considered to be undesirable no matter in which direction they occur.**

     Example: In investment management, individuals focus on the variability of the expected outcome, so risk-averse individuals will be willing to accept a lower expected rate of return on an investment when the investment offers a more predictable outcome.

Concept of risk in finance: basic ideas

- In many other risky situations, the possible results can be classified as undesirable **when their evolution is in a specific direction.**

  Example: Whoever is going to buy a property will normally consider that they are exposed to the risk of rising real estate prices and not their fall. For this reason, for example, property tenants may be interested in signing a rental contract with an option to buy, which protects them against that specific contingency.

5. Types of risk management strategies through the financial system

Risk management strategies

- There are a **variety of** concrete risk management **strategies** that **employ various types of contracts and financial instruments** to achieve their objectives.
- To apply them properly to each specific purpose and correctly evaluate their results, it is important to keep in mind that **each strategy has**:
  - **a. its** specific risk management **objective** and implementation **mechanism**, and
  - **b. its** distinctive **advantages** (benefits) **and drawbacks** (costs).
- Various strategies are studied across the range of finance courses, some of them which are more advanced than this one. To begin with, in this section we will examine some basic general ideas for **three basic groups of strategies.**

Basic types of risk management strategies

- When we talk about risk management strategies through the financial system, it is useful to distinguish amongst **three** related **concepts** (or *dimensions)* of risk transfer:

  A. Diversification

  B. Hedging (coverage)

  C. Insurance

A. Diversification

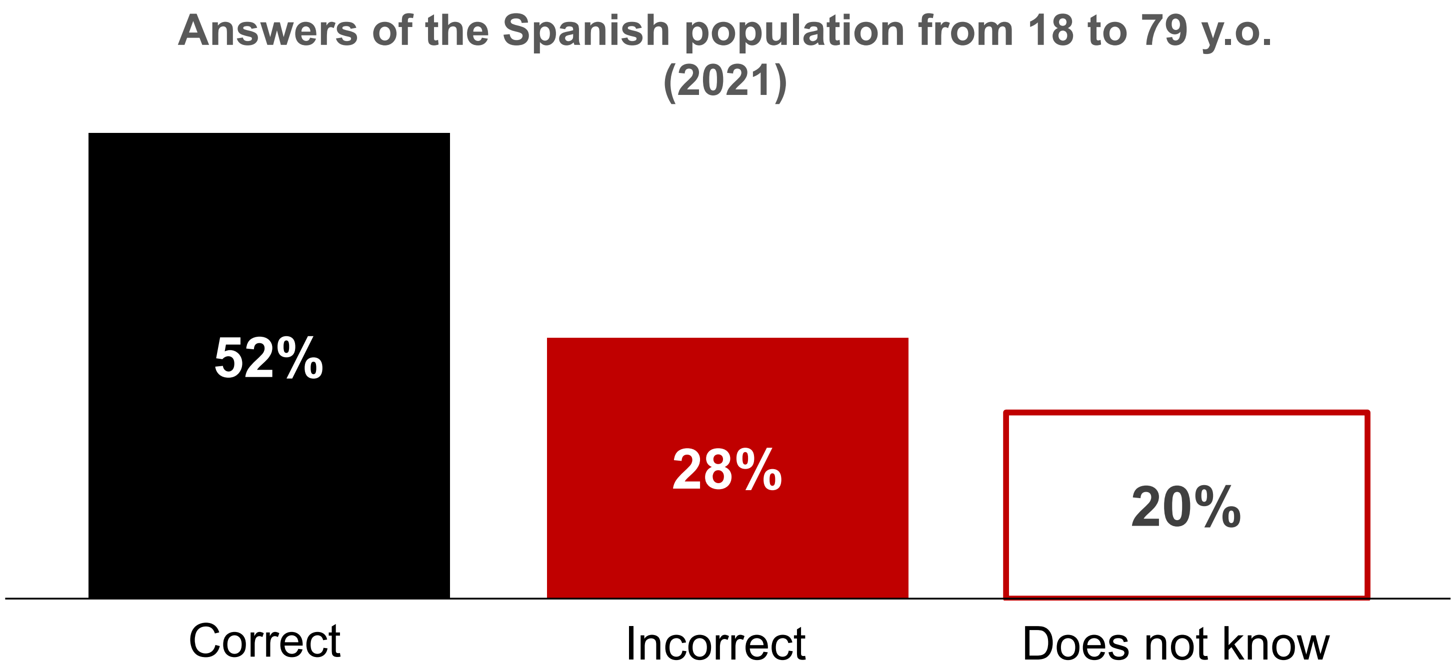
On the lack of knowledge of diversification

- Despite its simplicity, diversification is an often misunderstood concept.

- Answer whether the following statement is true or false:

  "IT IS GENERALLY POSSIBLE TO REDUCE THE RISK OF INVESTING IN THE STOCK MARKET BY PURCHASING A WIDE VARIETY OF STOCKS."

Evidence of lack of knowledge of diversification: the Spanish case



- Knowledge increases among people with higher levels of education and income.

Data source: ECF 2021, Bank of Spain

A. Diversification

Diversifying *consists in reducing the possibility of extreme results by grouping risky alternatives, rather than concentrating on just one.*

- **It is summed up in** the old saying:

“Don't put all your eggs in one basket.”

- It is a concept with applications in risk reduction at both an individual and collective level.
- In this section we will present its basic implications.

Exercise 1.1: diversification of investment in technology companies

Suppose you want to invest a million euros in the technology business, because you think it has good future prospects.

- You have the possibility of investing in two projects (project 1 and project 2) for the creation of companies dedicated to each developing a new and different application.
  - Suppose that, for each project, if it is successful, you would triple the money invested, but if it fails it would mean losing all the money.
- Let's compare two investment possibilities:
  a. Concentrate all your investment in one project.
  b. Diversify, investing half the money in each project.

Exercise 1.1

- Possible results:
  a. If you invest the million in just one of the projects, you could end up
     1. with 3 million euros, in case of success, or
     2. with nothing, in case of failure.

  b. If you invest half a million in each project, you could end up
     1. with 3 million euros, if both projects are successful;
     2. with nothing, if both projects fail; or
     3. with 1.5 million euros, if one of the two projects is successful and the other fails.

Exercise 1.1

Two basic related conclusions:

1. **Diversification allows you to reduce the possibility of extreme results:**
   - Now there is the possibility of achieving 1.5 million, which reduces the possibilities of the two more extreme alternatives.
   - The possibility of ending up with zero euros has been reduced:
     - When you concentrate your investment on a single project, losing everything depends only on *that* project failing.
     - When you diversify, losing everything requires *both* projects to fail.
   - Although the possibility of ending up with 3 million has *also* been reduced:
     - *Both* projects need to be a success.

Exercise 1.1

The second conclusion, equally important, is more subtle and therefore less understood:

2. **Diversification will not reduce the possibility of extreme results if the success or failure of the projects occurs at the same time (perfect and positive correlation):**
    - In that case, the possibility of ending up with 1.5 million euros by diversifying, derived from succeeding in one project and failing in the other, would disappear.
    - The remaining alternatives would be to end up with 0 euros, when both projects fail at once; or with 3 million, when both are successful at the same time (obtaining 1.5 million, triple the amount invested, in each one).

- In later units (numbers 6 and 7) you will find more aspects of diversification: what risk it eliminates and how to make the diversification decision optimally (efficiently) in stock portfolios.

Diversification: practical implications for evaluating investment results

The fundamental conclusion of the previous example:

> *By diversifying, you generally reduce the chances of extreme outcomes.*

It has **implications when evaluating *ex-post* the results of investment** (both private and institutional), which are not always well understood.

Exercise 1.1 (continued)

- Let's go back to the situation in the previous example, in which you have the possibility of investing in two projects (numbers 1 and 2) for the creation of companies dedicated to developing new applications.
  - In each project, if it is successful, the money invested is tripled, but if it fails, all the money is lost.
- Now let's consider three investors (A, B and C), each of whom has 1 million euros to invest:
  - Investor A invests all the money in project 1.
  - Investor B invests all the money in project 2.
  - Investor C invests half of the money in project 1 and the other half in project 2.
- Let's compare the possible outcomes.

Exercise 1.1 (continued)

First, the difference in results between investors occurs when one project succeeds and the other fails:

- In each of those two cases, one of the non-diversified investors, A or B, ends up with 3 million, while the other loses everything.

  (The position of winner or loser changes in each case.)

- Meanwhile, diversified investor C would obtain an intermediate final figure of 1.5 million *in both cases.*

Exercise 1.1 (continued)

Conclusions:

- On the one hand, focusing on the two scenarios in which differences occur between investors, we observe that **non-diversified investors** *end up standing out, either as the investor with the best results or as the one with the worst results* (exchanging roles in each case).

  That is, when one of the projects succeeds and the other fails:

  - One of the non-diversified investors would stand out as the big winner compared to the rest of the investors, obtaining more than the other two (3 million, compared to 1.5 or 0), which *could lead one to think that they have greater investment skills than the others;*
  - while the other non-diversified investor would stand out for being the only loser of the three, *which could make one think they have no investment skills at all.*

Exercise 1.1 (continued)

- On the other hand, **the diversified investor** *is less likely to stand out from the rest as the investor with the best results, although the same can be said when compared in terms of worst results:*
  - In the example, such an investor will get their best and worst results (ending up with 3 million or 0 euros, respectively, when both projects succeed or fail) coinciding with the other two investors;
  - while when one project is successful and the other fails, the diversified investor has intermediate results, just as one of the other two investors gets better results and the other, worse ones.

Exercise 1.1 (continued)

Practical implications:

1. *When judging the outcome of an investment* ***ex-post****, care must be taken to attribute the best results to superior investment skills:*

   It could be due to the ex-ante decision not to diversify, which could provide either large gains or large ex-post losses, depending simply on chance.

2. In the words of Bodie, Merton and Cleeton (2009, p. 278):

   "Of course, one always prefers to be a big winner and called a genius. But if that can only be accomplished by a decision ex-ante that results in either being a big winner or a big loser ex-post, then perhaps it is preferable to choose an alternative that leaves you in the middle."

B. Hedging (coverage)

B. Hedging

- **Hedging** refers to *protecting oneself from fluctuations in a position, using other contracts or financial instruments.*
- To be more specific, let's consider the following typical context in finance:

  There is a future financial result (the sale price of a crop, the purchase price of a commodity, the value of some shares in three months, etc.) that depends on the fluctuations of a variable, or **risk factor** (the price of a commodity or asset), an exchange rate, an interest rate, and so on) that is beyond our control.

  - Therefore, hedging aims to reduce fluctuations in the result of the whole position by incorporating other contracts or financial instruments into our portfolio.
  - This strategy *entails, along with protection against possible unfavourable changes in the risk factor, giving up the benefits of any favourable evolution of the risk factor.*

Exercise 1.2: hedging with a forward contract

- A very old way to hedge or protect yourself against unexpected future fluctuations in the price of a commodity is through forward contracts.
- For example:
  - Farmers are subject to risk arising from changes in the price of their future harvest.
  - A farmer can cover the future sales price of their crop, to be harvested in six months, by agreeing with an interested buyer today on a fixed sales price for delivery in six months.
- Terminology: The fixed price agreed for future delivery is called **the *forward price*,** to distinguish it from the price for immediate delivery, called **the *spot price*.**

Exercise 1.2 (cont.): analysis of the effects of forward contracting

a. This **completely eliminates the risk of selling at a lower price than the agreed one** (a very low price could even lead to losses if it does not cover production costs):

   - If the price of the product on the harvest date is lower than the fixed price agreed, the farmer will get the agreed price.

b. Now, due to the **obligation** acquired to have to sell at the agreed price, **the farmer also waives the possibility of benefiting from increases in the price** of the product above the agreed price:

   - If the price of the crop at the time of harvest within six months is finally higher than the agreed price, the farmer will be obliged to sell it at the agreed price.

Exercise 1.3: forward gold purchase hedge

Suppose a jewellery company wishes to make commemorative medals, for which it will need to use 2,500 ounces of gold within three months. The price of the medals is fixed.

The producer decides to hedge against the risk of an increase in the price of gold during the next three months, contracting the purchase of gold in a forward contract at €1,850 per ounce, for delivery within three months (a price that guarantees the company a profit in making the medals).

Analyze:

1. The amount finally paid for the purchase of gold, in the event that the (spot) price of gold within three months was finally (a) €2,000 per ounce or (b) €1,700. Compare these amounts with what the company would have had to pay if it had not been covered.

2. Suppose that you were in charge in the company of making the decision (ex-ante) to hedge and you are criticized (ex-post) for the decision made when finally the (spot) price of gold ends up being €1,700 in three months' time. What would you answer? What lesson would you draw for future occasions?

Exercise 1.4: hedging a stock portfolio with forward contracts

An investor owns 500 Repsol shares worth 14.79 euros each. Suppose they are not interested in selling them now for tax reasons, but plan to do so in three months to cover a planned expense.

Suppose that it is currently possible to negotiate forward contracts for the purchase / sale of shares at a delivery price of 15.08 euros per share in three months' time.

Determine what strategy the investor would follow using forward contracts to hedge the risk of stock price variations, and what the effects of this strategy would be.

- Note: Standardized contracts for the purchase / sale of Repsol shares for future delivery, called *futures,* are negotiated in Spain at MEFF. Futures will be seen in Unit 7.

C. Insurance

C. Insurance

Insuring *means protecting oneself from the possibility of a loss arising from the uncertain occurrence of a harmful event by* ***paying a price in advance for the protection.***

- In the specific case (usual in finance) that *the risk comes from the future evolution of an uncertain variable, insurance offers protection against its possible unfavourable evolution,* ***while allowing you to benefit from its possible favourable evolution.***
  - These effects can be achieved with what are known as *option contracts*, as illustrated in Exercise 1.7.

Exercise 1.5: insurance policy

Suppose you have a vehicle.

- The law requires you to have third-party liability insurance, so that in the event that you damage other people's property or injure them, the insurance company would be responsible for paying the corresponding expenses and compensation.
- In addition, you can also insure your vehicle against personal damage such as broken windows or fire, so that the insurer would also compensate you for all or part of the repair cost in such circumstances (as agreed in the policy).
- In this way, you are **insured against future contingencies** (risks of damage, theft, etc.), the possible expenses of which could amount to several thousand euros, **in exchange for paying a price (*premium*)** of a few hundred euros **in advance**.

Exercise 1.6: bank guarantee

Suppose you own a rental property.

- You could require the tenant to pay a **bank guarantee**, thereby insuring yourself **against non-payment** of rent.
  - For you, this means insuring against the risk that the other party to a contract fails to meet their commitments (risk of default or insolvency, also called *credit risk*).
- The bank (*guarantor*) would compensate you for any loss suffered in the event that the tenant does not pay the rent.
- The bank charges the tenant the **guarantee price** in advance.

Exercise 1.7: insurance using an option contract

Suppose you have a business that imports goods from the UK and you know that you will have to pay £100,000 in three months' time.

- You know the price of the pound today (say €1.18 = £1), but you don't know what its price will be three months from now. That is, you are exposed to **exchange rate risk.**
- Today, you could pay a price (*premium*) **in advance so as to obtain the right** (but not the obligation) **to buy** £100,000 three months in the future at a fixed price of €1.18 per pound, for example.

(This right to buy an asset in the future is called a *call option.*)

Exercise 1.7 (cont.)

Thus:

- You would be **insured against increases in the euro price** of the pound, *above the agreed purchase price* of €1.18, within three months:

  If the exchange rate in three months' time was higher than €1.18, you would exercise the right contained in the contract to buy the pounds for €1.18 each, paying a total of 118,000 euros.

- At the same time, **you would maintain the possibility of benefiting from drops in the price** of the pound, *below the agreed purchase price,* within three months:

  If the exchange rate in three months' time fell below €1.18, and since you are not obliged to execute the option, you would not buy the pounds at the contract price of €1.18; instead you would prefer to do so at the current, lower exchange rate (*spot rate*) at that time.

- In both cases, to calculate the total cost of the insurance, you must add in the initial price paid for it (*option premium*).

Exercise 1.8: insuring a stock portfolio with put options

Let's return to the initial situation set out in Exercise 1.4:

An investor owns 500 Repsol shares that are currently worth 14.79 euros each. Suppose they are not interested in selling them now for tax reasons, but plan to do so in three months to cover a planned expense.

Let us now consider that it is currently possible to negotiate contracts that grant *the right (but not the obligation) to sell* the shares within three months at an agreed sale price of €15.00 per share. The price (premium) of the right is €0.16 per share, to be paid in advance. This right to sell is called a *put option.*

Determine what strategy the investor would follow using put option contracts, to insure against the risk of a decline in stock price, and what the effects of such a fall would be.

- Note: Standardized option contracts are negotiated at MEFF (Unit 7).

Exercise 1.9: hedging versus insurance of a stock portfolio

Based on the examples in Exercises 1.4 and 1.8, make a general comparison of the risk management of a stock portfolio through:

(a) hedging with forward/futures contracts and;

(b) insurance with options.

- Note: Both forward and option contracts are examples of **derivative contracts**, specially designed to contribute to efficient risk management, which will be seen in more detail in Units 3 and 7.

Practical implications for evaluating the results of risk management strategies

- The previous exercises illustrate an important idea about risk management:
  - Each risk management strategy has its advantages (or benefits in terms of risk reduction) along with certain drawbacks (or associated costs).
    - For example, in the case of hedging with forward contracts, you eliminate the possibility of favourable movements.
  - *The primary mission of risk management is to decide which alternative to choose (including the possibility of doing nothing), by comparing benefits and costs of risk reduction.*

Practical implications for evaluating the results of risk management strategies

- Furthermore and by their nature, risk management decisions are made under conditions of uncertainty, which means that multiple ex-post outcomes are possible.
  - Therefore, there is no point in evaluating a strategy (or the manager's skills, for that matter) *based on the information available when the uncertainty has already been resolved.*
  - What matters is whether the decision made was the best possible using the information available when it was made.

    In the words of Bodie, Merton and Cleeton (2009, p. 269): *The appropriateness of a risk-management decision should be judged in the light of the information available at the time the decision is made.*

6. Main analytical tools of modern financial economics

Main analytical tools of financial theory

- Standard financial theory uses **three main tools of analysis** (also used in other branches of economics):
  A. Absence of arbitrage opportunities
  B. Individual optimization (personal or corporate)
  C. Market equilibrium

- In particular, the postulate of **the *absence of arbitrage* plays a fundamental role in modern finance,** both because of the convincing arguments on which it is based and because of the practical usefulness of its implications.

A. Absence of arbitrage

Exercise 1.10: arbitrage

In finance, ***arbitrage*** refers to the strategy of negotiation with assets that allows us to profit without any risk or initial net investment.

Example: suppose that, at a given time, you observe that the shares of a given company were trading on two exchanges at different prices.

- You could buy shares on the *cheap* stock market and sell them on the *expensive* one.
- This negotiation would bring you a guaranteed immediate profit equal to the difference between the two prices, for each share traded; so you would leap at the opportunity: It would be a *free meal*!
- In fact, everyone would try to use it like a limitless free money machine.
- Obviously, this would not be a sustainable situation and, sooner rather than later, prices would change, making the arbitrage opportunity disappear.

Arbitrage: basic ideas

The previous example illustrates **two basic ideas** regarding arbitrage:

1. **Pure arbitrage is different from speculation:**
   - In speculative trading strategies (and often in investing, for that matter), you *hope* to make a profit, but it is not guaranteed, and *you could eventually lose money.*
     - Example: Investment in shares

       When you speculate or invest in a company's shares over a period of time, you do so in the hope that they will rise in value, but they could fall and lead to a loss.
   - On the other hand, when you invest, you have to pay money at the beginning, even if you invest in a risk-free asset.
     - Example: Investment in treasury bills

       When you invest in treasury bills from a solvent country, you are virtually guaranteed to earn the interest agreed, but you first have to make an initial outlay.

Arbitrage: basic ideas

2. **Arbitrage opportunities are usually not available for very long:**
   - Arbitrages are *free money machines* and, therefore, are the desired trading strategy of *any* investor.
   - However, in the financial markets some professionals are very actively involved in exploiting them: the ***arbitrageurs*.**
   - In practice, it is the actions of these professionals, with their privileged position in the markets (in terms of quick and direct connections to those markets, quick access to information and very low transaction costs), that **cause the arbitrage opportunities to disappear very quickly.**

Absence of arbitrage: basic ideas

- In finance, the importance of arbitrage lies in the **implications** of the following axiom: *consider that there are no arbitrage opportunities available in the market;* that is, *postulate the* **absence of arbitrage.**

- **Two basic ideas**:

    1. **The implications of the absence of arbitrage are relatively robust,** due to the solid assumptions on which it is based.
    2. The absence of arbitrage has **fundamental implications in valuation,** one of the basic fields of finance.

1. Robustness of the absence of arbitrage and its implications

The principle of the absence of arbitrage is robust because of the basic argument on which it is based.

- The basic idea behind the absence of arbitrage is:

  To make arbitrage opportunities disappear, **all you need is *a few rational economic agents who prefer more money to less and are unsatiated,* even in a market populated mostly by irrational ones.**

- Therefore, the absence of arbitrage is a more primitive situation than equilibrium:

  As long as arbitrage opportunities exist, *any* rational economic agent who prefers more to less will continue to trade (to derive money from that situation) and a stable position in the market will not be reached.

2. Absence of arbitrage and relative valuation

The absence of arbitrage has important consequences for the relative valuation of assets.

- In particular, one of the most important implications of the absence of arbitrage is the ***law of one price,*** whereby two assets that are perfect substitutes for each other must be traded at the same price.

- The continuation of the previous exercise showcases the basic idea.
  - This idea is developed and illustrated by a variety of applications for different assets and financial markets in Units 3, 4 and 7.

Exercise 1.10 (cont.): arbitrage

Let's go back to the situation set out in Exercise 1.10:

Remember that to take advantage of the available arbitrage opportunity you must buy shares where they are cheap and sell them where they are expensive.

- Due to the laws of supply and demand, the price in the *cheap* stock market would rise and the price in the *expensive* stock market would fall, causing *the stock prices of the same company to tend to equalize.*

Therefore:

- The actions of arbitrageurs have *consequences for the relative valuation of asset prices.*

B. Individual optimization

B. Individual optimization (personal or corporate)

- In financial economics, *agents* (individuals, companies) are usually characterized as **optimizers**:
  - *They choose optimally, in their own best interest, after correctly evaluating their criteria and the alternatives as well as the information available.*

- Examples of use:
  - Individuals: portfolio selection or risk management by minimizing return variance

    (Optimal portfolio selection is presented in Unit 7.)

  - Companies: choice of investment projects, maximizing the market value of the company for shareholders

    (This and other business decisions based on this optimizing criterion will be covered later on in a more advanced subject called Corporate Finance.)

C. Market equilibrium

C. Market equilibrium

- In financial economics, competitive equilibrium is used as a **mechanism for determining negotiation and prices in financial markets.**

  Remember: *In equilibrium, all individuals, acting independently in their own best interest, are satisfied, and aggregate demand equals aggregate supply.*

- Examples of use:
  - Characterization of the risk premium in the market

    (This topic is covered in Unit 7.)

Final summary

Final summary

1. Finance studies how individuals and companies use financial assets and markets to meet their objectives.
2. A well-functioning financial system contributes to our economic well-being by helping borrowers to obtain funds from lenders and transferring risks, among other functions.
3. Today, the functioning of economies cannot be properly understood without first understanding the operation of the financial system.
   - This is quite evident when economic crises are related to financial ones.

Final summary (cont.)

4. Financial decisions cover (short or long) periods of time and are usually made in an environment of uncertainty, which entails risk.

5. Various types of risk management strategies (diversification, insurance and hedging) can be put into practice through the financial system, each with its own objectives and characteristics.

6. Finance analyzes financial decisions and their repercussions on negotiation and market prices, under the conditions of absence of arbitrage, individual optimization and market equilibrium.

Final summary (cont.)

7. The absence of arbitrage constitutes one of the fundamental pillars of finance, since it rests on a few realistic assumptions about economic agents and has many important practical applications in valuation, one of the most important fields of finance.

Recommended bibliography

- The content of this topic can be found, in various forms, in several excellent textbooks.
- For more details, the following manual is recommended, which corresponds quite well to the content, level and approach of these notes, and from which some specific ideas have been taken:

Finance Economics, 2nd ed. (2009) by Z. Bodie, R. Merton and D. Cleeton, Pearson. Chapters 1, 2, 10 and 11.

Or the same chapters in the first edition, translated into Spanish: *Finanzas,* 1ª ed. (2000), de Z. Bodie y R. Merton, Pearson.

Unit 2

Time value of money and discount of known cash flows

Dept. Financial and Actuarial Economics

VNIVERSITAT DE VALÈNCIA

Facultat d'Economia

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        - YTM (or IRR) as expected rate of return of a bond
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Appendix: demonstrations and summary tables of annuity formulas

Introduction: compound interest

- The main objective of this unit is to understand the concept of a **compound interest rate** and show its varied **applications** in finance.

- Despite its theoretical and practical importance (for example, it is extensively used to calculate interest in many common financial contracts such as loans and also in calculating investment returns), it is an often <u>poorly understood concept</u>.

On the ignorance of compound interest

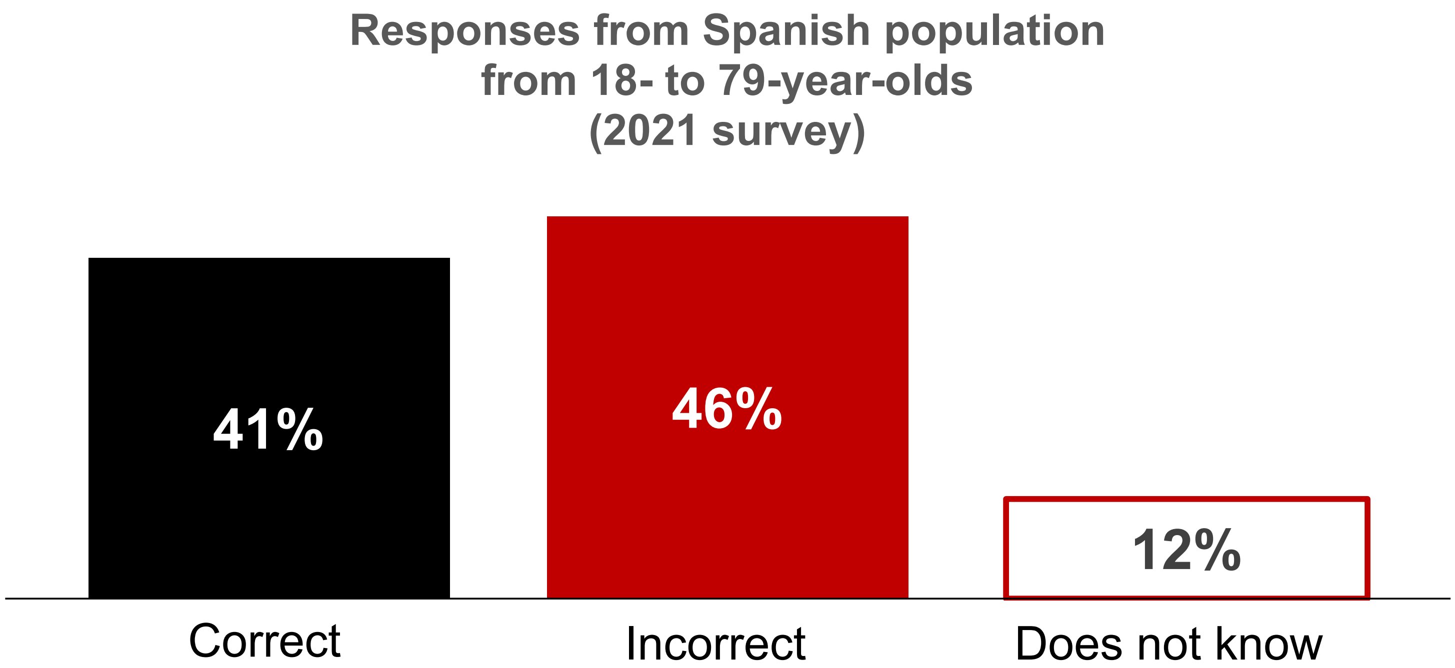
- Answer the following question:

  Suppose you deposit 100 euros in a savings account with a fixed interest of 2% per year. There are no fees or taxes on this account. If you make no further deposits into this account or withdraw any money, after paying interest each year, how much money will be in the account after five years?

  a) More than 110 euros.
  b) Exactly 110 euros.
  c) Less than 110 euros.
  d) It is impossible to say with the information given.

  ¿Sabes qué es el tipo de interés compuesto? - Cliente Bancario, Banco de España (bde.es)

Evidence on the lack of knowledge of the concept of compound interest rates: Spanish case



- Knowledge increases among people with higher levels of education and income.

Data source: ECF 2021, Bank of Spain

Introduction: time value of money

- *Financial decisions require comparing sums of money (or financial capitals) exchanged over time.*

  <u>Example</u>: Investments require evaluating whether the benefits expected in the future justify the money invested at the beginning.

- The time value of money refers to the fact that we do not value one monetary unit (a euro, dollar, yen, etc.) available today the same as one monetary unit available in the future.

  <u>Example:</u> A reward is required for giving up a sum of money in hand today, in exchange for getting it back in the future, as we do when we lend money to others or make financial investments. The difference in value between future money and current money reflects the reward (interest or return) demanded.

Introduction: time value of money

- This unit provides the basic tools for valuing money over time (including calculating the difference in time value, or reward, between sums of money given and received on different dates), essential in making financial decisions.
  - In subsequent units (and subjects), the much more complex issue of the determinants of the difference in time value – that is, interest rates and the return required on investments – will be covered. These are related to time preference and the risks associated with the recovery of money invested.

1. Capitalization (or accumulation) and future value

Interest and interest rates

- The most basic concept relating to cash funds exchanged at different points in time is:

  Interest: payment or reward for the use of funds over a period of time.

- In practice (for example, to compare savings or financing alternatives), it is useful to measure interest in relative terms:

  Interest rate: amount of interest paid per unit of time as a fraction of the balance.

- Historically, the calculation of interest has been complicated by the definition and use of different rates.

- This topic focuses on compound interest rates, since they are the most widely used. However, to understand them properly, it is a helpful learning strategy to start with simple interest rates.

Simple interest

Exercise 2.1: simple interest

Suppose that 1,000 euros are deposited in the bank for three years and the bank remunerates the deposit at the end of the term, at a simple interest rate of 5% per year. How much interest will be earned and what will the total amount that can be withdrawn be?

Simple interest *is obtained by applying the interest rate (a) to the amount deposited, (b) in proportion to the time elapsed.*

In the example: 5 euros are earned for every 100 euros deposited, for each year elapsed. That is to say:

$$\text{Interest} = 1{,}000 \times 0.05 \times 3 = 150\ €$$
$$\text{Final amount} = 1{,}000 + 150 = 1{,}150\ €$$

Simple interest: generalization

- Notation:
  - $C_0 =$ original principal amount, which matures (is delivered) on the initial date 0.
  - $i_s =$ simple interest rate per year (usually).
  - $n =$ term, expressed in years (usually).
  - $FV \equiv V_n =$ final or future value (at the end of the term) of the original principal amount.
- A word of caution:

  Remember that the term must always be expressed in the same time units as the interest rate.

Simple interest: generalization

- ***Simple interest*** *is calculated by applying the interest rate on the initial amount, proportionally to the time elapsed:*

$$\text{Interest} = C_0 \cdot i_s \cdot n$$

It therefore indicates the reward obtained at the end of the period considered. Thus:

- **Final or future value:**

$$FV = C_0 + \text{Interest} = C_0 + C_0 \cdot i_s \cdot n = C_0 \cdot (1 + i_s \cdot n)$$

Simple interest: generalization

- Remember:

Basic valuation equation (capitalization) with simple interest:

$$V_n = FV = C_0 \cdot (1 + i_s \cdot n)$$

- ***Accumulation (or capitalization)*** refers to the process of obtaining final values.

Exercise 2.2: simple interest

Indicate the final amount and the interest obtained on a deposit of 500 euros for a term of eight months, if the financial institution pays the interest at the end of the term and applies a simple annual interest rate of 2.25%.

Compound interest

Exercise 2.3: compound interest

Assume that a deposit of 1,000 euros is made over a period of three years. Interest is paid annually to the same deposit account, at an interest rate of 5% per year, and no other contributions are made nor is the money withdrawn at the end of the term. What is the final amount accumulated?

In this case, in addition to the initial amount deposited, the interest paid each year will also generate interest in each of the successive years.

Exercise 2.3: solution

Deposit of 1,000 EUR at 5% per year. Term: 3 years.

Final value year 1: $V_1 = 1{,}000 \cdot 1.05 = 1{,}050$

Final value year 2: $V_2 = 1{,}050 \cdot 1.05 = 1{,}000 \cdot (1.05)^2 = 1{,}102.50$

Final value year 3: $V_3 = 1{,}102.50 \cdot 1.05 = 1{,}000 \cdot (1.05)^3 = 1{,}157.62$

Total interest: $1{,}157.62 - 1{,}000 = 157{,}62 = 150 + 7.62$

Interest generated in simple capitalization (5% annually on the amount invested over 3 years).

Interest generated by the interest earned at the end of each previous period.

- **Reinvestment effect of interest when calculating compound interest!**

Compound interest: generalization

- Notation:
  - $C_0 =$ original principal amount, which matures (it is due) on the initial date 0.
  - $i =$ compound interest rate annually (usually).
  - $n =$ term, expressed in years (usually).
  - $FV \equiv V_n =$ final or future value (due at the end of the term) of the original principal amount.

- Word of caution:

  Remember that the term must always be expressed in the same time units as the interest rate.

Compound interest: generalization

- ***Compound interest:*** *the interest generated in each period generates interest in successive periods.*

  In other words, the final amount is calculated taking into account the **reinvestment** of periodically generated interest.

- **Final or future value:** Generalizing the previous example:

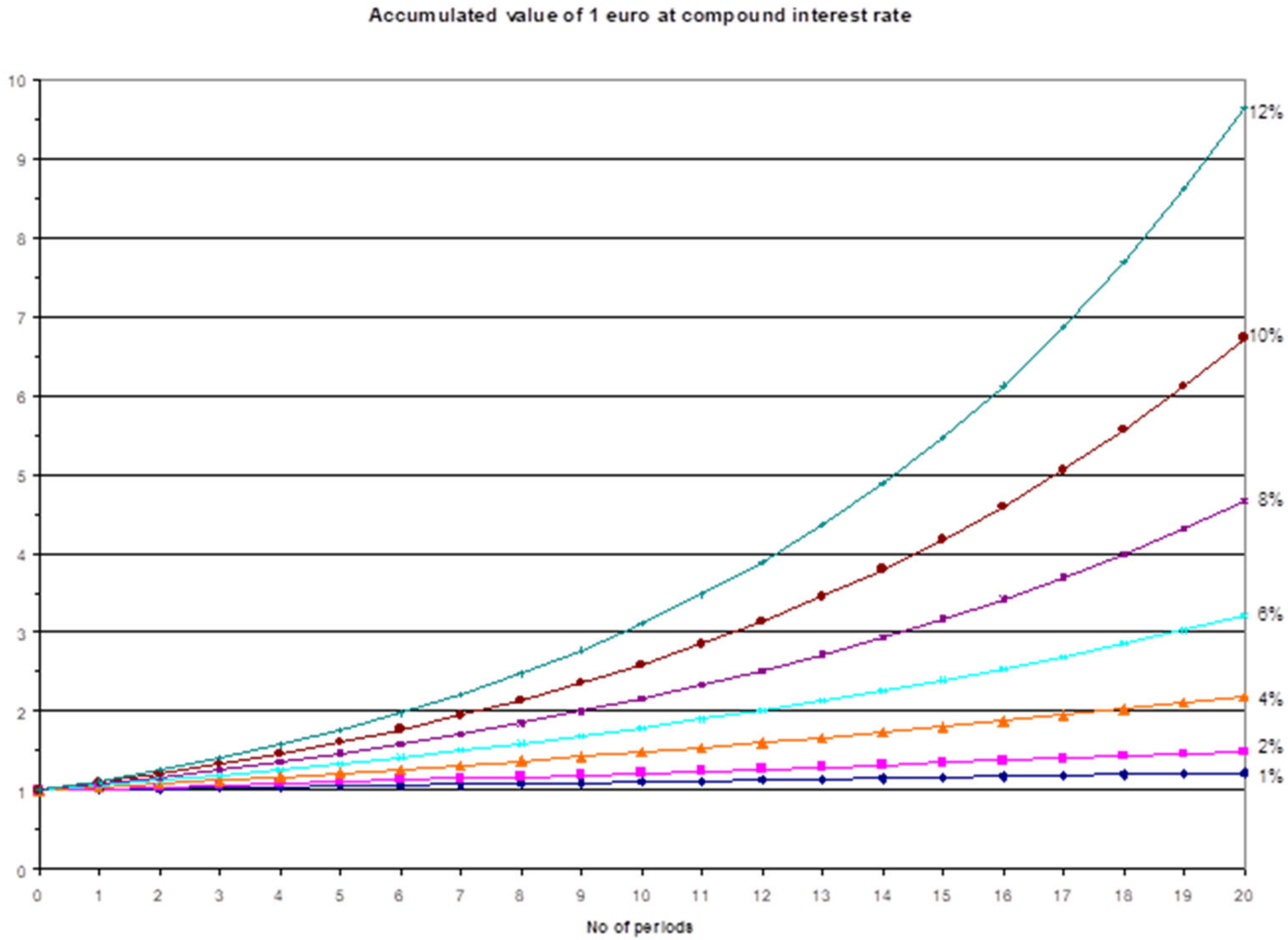
$$V_n = C_0 \cdot \underbrace{(1+i)^n}$$

Accumulation (or capitalization) **factor** associated with the term *n* with compound interest rate i.

  This is the **basic capitalization equation with compound interest.**

- In the rest of the subject we will exclusively use a compound interest rate in the valuation of financial transactions.

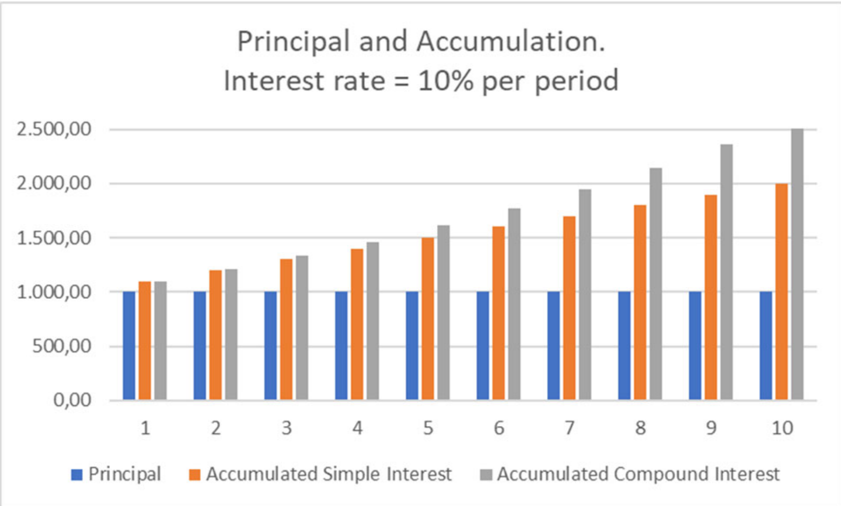
Exercise 2.4 : accumulation factors at compound interest



Comparison of simple and compound interest

Exercise 2.5: comparison simple/compound

Data: investment of 1,000 euros at an interest rate of 10% per year, for terms from one up to ten years.



Year	Amount invested	Final amount with simple interest	Final amount with compound interest
1	1,000.00	1,100.00	1,100.00
2	1,000.00	1,200.00	1,210.00
3	1,000.00	1,300.00	1,331.00
4	1,000.00	1,400.00	1,464.10
5	1,000.00	1,500.00	1,610.51
6	1,000.00	1,600.00	1,771.56
7	1,000.00	1,700.00	1,948.72
8	1,000.00	1,800.00	2,143.59
9	1,000.00	1,900.00	2,357.95
10	1,000.00	2,000.00	2,593.74

Interest rate in practice: APR and nominal rates

- This note briefly explains APR and nominal interest rates. They usually appear in the information on financial contracts provided by financial institutions to their clients.

1. **APR:** is the most appropriate tool to compare alternative investments or alternative financing transactions:
   1. It is an **annual compound interest rate.**
      - It is also called **the annual effective rate**, to distinguish it from the so-called nominal rate.
   2. In addition, the Spanish regulations on transparency of financial contracts use the term APR (for Annual Percentage Rate, or TAE in Spanish) for the measure of performance or cost that **includes not only interest but also other related expenses**, such as banking fees.

Interest rates in practice : TAE and nominal rate (cont.)

2. **In addition, in practice the so-called annual nominal interest rate or TIN** (following the Spanish acronym) is also provided:
    1. It is a simple annualization, proportionally over time, of the corresponding compound subannual effective rate.
        - Naming m (≥1) the frequency of interest payments in a year, the corresponding sub-annual compound rate to use in compound interest formulas is: TIN/m.
        - The relationship between the annual nominal rate, corresponding to a compound (effective) rate payable *m* times a year, and the annual effective rate (without considering expenses or fees) is:

$$1 + \mathrm{APR} = \left(1 + \frac{\mathrm{TIN}}{m}\right)^{m}$$

    2. The nominal rate (**TIN**) does not include attendant expenses, so it **is not suitable for comparing alternative investment or financing decisions**.

- APR and TIN are covered in greater depth in the more advanced subject of Financial Markets and Banking Operations.

2. Discount and present value

Exercise 2.6: present value at compound interest

Assume that a deposit of money is made for two years. Interest is paid quarterly to the same deposit account, at a compound quarterly interest rate of 1.125% (TIN = 4.5% = 1.125% x 4), and no other contributions are made or money withdrawn until the end of term.

If the total amount of money accumulated at the end of the second year is 10,936.25 euros, what was the amount of money invested at the beginning?

Exercise 2.6: solution

Data:

Quarterly effective interest rate, $i = 0.01125$

$n = 2\ years = 8$ quarters

By the basic capitalization equation:

$$V_n = C_0 \cdot (1+i)^n \rightarrow 10{,}936.25 = C_0 \cdot (1+0.01125)^8$$

Therefore the initial amount invested was:

$$C_0 = 10{,}936.25 \cdot (1+0.01125)^{-8} = 10{,}000$$

- This equation can be interpreted in the following way: it calculates the present value of a future amount of money, by removing the compound interest generated during the entire term.

Present value: generalization

- Naming $\mathrm{PV} \equiv V_0$ the **current or present value** of C_n euros that are due at the end of a period of *n* time units (usually measured in years) at a compound effective interest rate i, we have:

$$V_0 = C_n \cdot \underbrace{(1+i)^{-n}}$$

Discount factor associated to the period [0, *n*] with compound interest rate i

This is the **basic discounting equation with compound interest.**

- **Discount** refers to the calculation of current (present) values.

3. Multiple cash flows. Annuities

Valuation of multiple cash flows

- Most financial decisions involve multiple sums of money over time.
- It is simple **to calculate the present and final values of a set of** known cash flows:
  1. First, the individual values are calculated, at the same moment in time, of each one of the cash flows, using the present (or future) value formulas already seen.
  2. Second, all these calculated values are arithmetically added *at a common valuation moment*.
- <u>Remember</u>: For practical purposes, the most important thing is not to skip the first step.

It does not make any financial sense to directly add (or subtract) cash flows valued on different dates.

Exercise 2.7: final value of multiple capitals

Suppose that Mr Collins agrees to make three deposits of 100 euros each, on 15 February of each year, over three years. The first deposit is due on 15/02/25. How much money will Mr Collins have saved on 15/02/28 if the financial institution remunerates the deposits at an annual compound interest rate of 4%?

Notice that the question can be reformulated as the balance on the savings account after three years.

Exercise 2.7: solution

Each contribution of 100 euros can be treated as a separate deposit, so we would have 3 deposits:

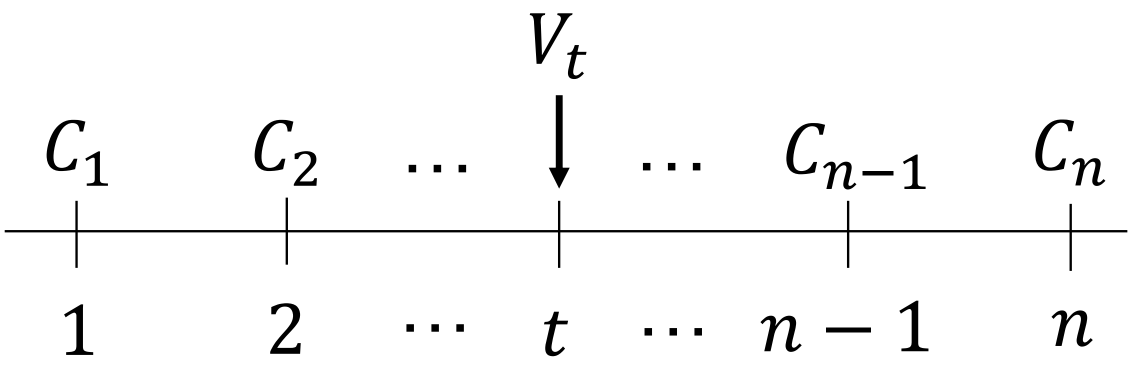
1. The first 100 euros would be deposited from 15/02/25 to 15/02/28 (three years);
2. the second 100 euros from 15/02/26 to 15/02/28 (two years);
3. and the third contribution of 100 euros from 15/02/27 to 15/02/28 (one year).

Therefore, the total final value is given by the sum of the final values of the three deposits:

$$V_{15.02.28} = 100 \cdot (1.04)^3 + 100 \cdot (1.04)^2 + 100 \cdot (1.04)^1$$
$$V_{15.02.28} = 112.49 + 108.16 + 104 = 324.65$$

Valuation of multiple known capitals: generalization

- Value at the moment t, of the n cash flows represented in the following time axis:



- It should be calculated as:

$$V_t = \sum_{s=1}^{n} C_s(1+i)^{t-s}$$

Where the exponent $(t-s)$ is positive (accumulation factor) if $t > s$ and it is negative (discount factor) if $t < s$.

Exercise 2.8

Three financial capitals (cash flows) have to be valued on 15/02/25 with the following amounts and maturities (indicated in the time subscript):

$$100_{15.01.25}, 150_{15.07.25} \text{ and } 200_{15.10.25}$$

using a monthly compound interest rate of 0.2%.

Obtain the value of the sum of all three cash flows on 15/02/25.

Property fundamental to valuation of a set of cash flows

- The **most important property of the valuation of a group of cash flows at compound interest** is the following:

  From the value of a set of capitals on a given date t*, it is possible to obtain the value on another date* t'*, capitalizing the first value if* $t' > t$*, or discounting it if* $t' < t$ *:*

$$V_{t'} = V_t \cdot (1+i)^{t'-t}$$

- This property is very useful both in theoretical developments and in practice, because it allows the different values on different dates of a given set of capitals to be easily calculated.

- The following exercise allows us to verify that this property is true, without having to prove it, as well as demonstrating its usefulness.

Exercise 2.9 (continued Exercise 2.8)

The following capitals are requested to be valued on 15/08/25:

$$100_{15.01.25}, 150_{15.07.25} \text{ and } 200_{15.10.25}$$

using a monthly compound interest rate of 0.2%.

Annuity formulas

Annuity formulas

- The **general valuation equation**

$$V_t = \sum_{s=1}^{n} C_s (1+i)^{t-s},$$

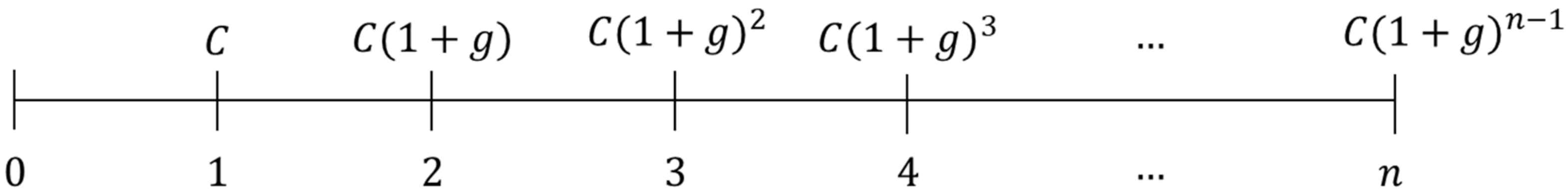
  is all we need to value a set of n (known) capitals, in any practical situation.

- However, if the number of cash flows is large enough, it may take some time to calculate it, unless a financial calculator or computer is used.

- Fortunately, there are a few **compact valuation formulas**, called annuity formulas, which, together with the fundamental property of valuation, allow a large number of capitals to be valued effortlessly, in various real-life practical situations.

A) Present value of cash flows following a geometric progression

- The most versatile formula is the one that allows you to calculate the **current value of n future cash flows that vary at a constant exchange rate g** where C denotes the amount of the first cash flow due at the end of the first period:



$$V_0 = C\frac{1-\left(\frac{1+g}{1+i}\right)^n}{i-g}$$

- The proof can be found in the final appendix.

  (Warning: If it so happens that $g = i$, a special formula must be used, which you will also find in the appendix.)

A) Present value…

- <u>Remember</u>:

  (i) The formula allows you to calculate the **value of the annuity one period before the maturity date of the first** future capital (t = 0);

  (ii) Should you want to value the capital on another date, use that formula together with the fundamental property of valuation;

  (iii) You must use the compound **interest rate** corresponding to a time period between two capitals (if the capitals are annual, you must use the annual rate; if they are monthly, the monthly rate, and so on).

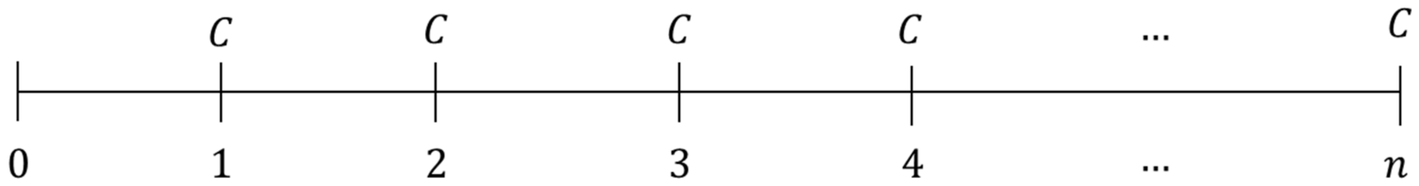
- Treatment of an <u>infinite number of cash flows</u>: The formula (with $g < i$) becomes:

$$V_0 = \frac{C}{i - g}$$

  - The proof is in the appendix.
  - This formula will be used in the dividend discount model for stock valuation in Unit 5.

B) Present value of constant cash-flows

- A particular example of this scenario, very common in practice, is that of constant cash flows (ordinary annuity):

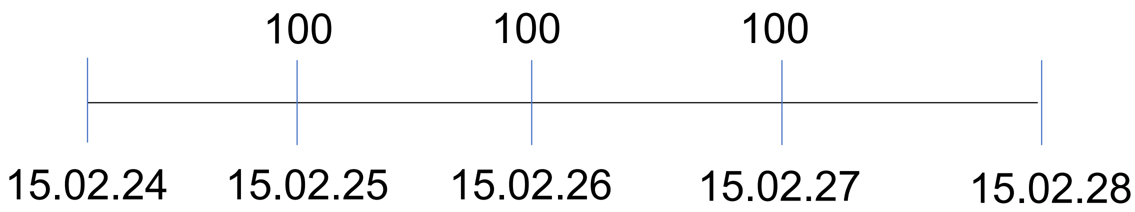


- To obtain the **present value of an ordinary annuity (present value of $\boldsymbol{n}$ constant future cash flows)**, you can use the previous formula, with $g = 0$.
- Remember: the three observations from the prior case are still valid.
- As for infinite number of future constant cash flows ($n \to \infty$): the formula used in the perpetual variable case can be employed, with $g = 0$. Therefore: $V_0 = \frac{C}{i}$

Exercise 2.10 (cont. Exercise 2.7):

Suppose that Mrs Collins agrees on a savings financial transaction consisting of making deposits of 100 euros on 15 February of each year for three years. The first contribution is made on 15/02/25. How much money will Mrs Collins have saved on 15/02/28 if the financial institution applies an annual compound interest rate of 4%?

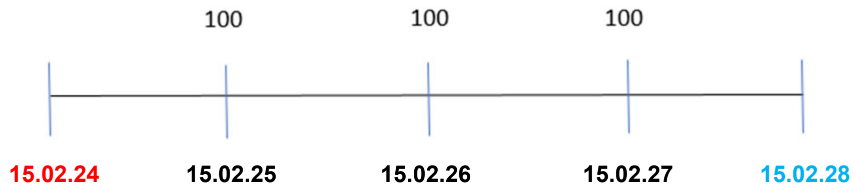
- The contributions made by Mrs Collins can be represented as:



- To obtain the desired value, we can combine the present value formula of an ordinary annuity, which allows us to value the capital contributed as of 15/02/24, with the fundamental property of valuation, to capitalize this value until 15/02/28 and thus obtain the value on the desired date.

Exercise 2.10: solution

Solution:



$$V_{15.02.28} = V_{15.02.24} \cdot (1+i)^4 = 100 \cdot \frac{1-(1.04)^{-3}}{0.04} \cdot (1.04)^4$$
$$= 324.65$$

- As expected, we have obtained the same result as with the general valuation equation used in Exercise 2.7 (check it!).
- The usefulness of working with these formulas becomes evident when dealing with a large number of cash flows.

Exercise 2.11:

Suppose that Mrs Sharp agrees to a savings financial transaction consisting of depositing 200 euros on day 15 of each month for five years. The first deposit is made on 15/03/24. How much money will Mrs Sharp have saved on 15/03/29 if a monthly compound interest rate of 0.30% (equivalent to an nominal annual rate of 3.60%) is applied?

Exercise 2.12:

Imagine that you arrange a savings operation consisting of quarterly deposits that grow by 1% each quarter. The first deposit is for 200 euros and is made on the settlement date, 15/02/25.

Knowing that the operation lasts five years and that the financial institution values these operations at a quarterly compound interest rate of 1.20% (equivalent to an annual nominal rate of 4.80%), what will the total amount saved at the end of the transaction (15/02/30) be?

Exercise 2.12: solution

Data: $C = 200$ $g = 0.01$ $i = 0.012 = \frac{0.048}{4}$

	C	$C(1+g)$	$C(1+g)^2$	$C(1+g)^3$	$C(1+g)^4$	...	$C(1+g)^{19}$	
15.11.24	15.02.25	15.05.25	15.08.25	15.11.25	15.02.26	...	15.11.29	15.02.30

Solution:

$$V_{15.02.30} = V_{15.11.24} \cdot (1+i)^{21} = C\frac{1-\left(\frac{1+g}{1+i}\right)^n}{i-g} \cdot (1+i)^{21} = 200\frac{1-\left(\frac{1.01}{1.012}\right)^{20}}{0.012-0.01} \cdot (1.012)^{21} = 4{,}983.53$$

Exercise 2.13:

Imagine that you arrange a savings operation consisting of monthly deposits that grow by 0.25% each month. The first deposit is of 100 euros and is made on the settlement date, 15/02/25. Knowing that the operation lasts ten years and that it is valued using a monthly compound interest rate of 0.2% (equivalent to an annual nominal rate of 2.40%), answer the following questions:

1) How much will the final deposit be?
2) What is the date on which the last deposit is due?
3) What will be the total amount saved at the end of the financial transaction (15/02/35)?

Exercise 2.14:

Imagine that you are considering investing in a property from which you expect to receive an annual income. You expect to receive 5,000 euros within a year and the following cash flows to grow at 3% per year in perpetuity.

If you use a compound (effective) interest rate of 5% per year, what is the present value of that property?

Exercise 2.15: savings for retirement

Suppose that Mr Cairns decides to carry out a savings operation to guarantee a supplement for his retirement. The savings operation requires monthly deposits over a period of twenty years. Knowing that a monthly compound interest rate of 0.2% is used (equivalent to an nominal annual rate of 2.40%), what would be the total capital saved at the end of twenty years, if:

a) The monthly deposits are constant throughout the entire financial transaction. The deposits are for 200 euros each month.

b) Monthly deposits grow by 0.10% each month, with the first deposit being 180 euros.

Exercise 2.16: savings for retirement

Suppose that Mr Cairns decides to carry out a savings operation to guarantee a supplement to his retirement. The savings operation means making monthly deposits over a period of twenty years. During the first ten years, the monthly deposits are a constant amount of 200 euros; and for the remaining ten years, they grow by 0.10% each month. The first deposit is due on the settlement date of the financial transaction (15 Feb 2024).

Knowing that a monthly compound interest rate of 0.2% (equivalent to a nominal annual rate of 2.40%) is used, what would be the total capital saved at the end of twenty years (on 15 Feb 2044)?

(Note: consider that the deposit due on 15 Feb 2034 is still 200 EUR and from that date onwards the deposits grow by 0.10% per month)

4. Applications of the analysis of the discounted cash flows

Applications

- Discount of cash flows has many practical applications in financial analysis.
- This section summarizes three of the most important ones, for illustrative purposes:

1. **Amortization of a loan at a fixed interest rate**

   Loans are one of the most frequent financial contracts.
   - They are analyzed in detail in the Financial Markets and Banking Operations subject.

2. **Calculation of the yield-to-maturity (YTM or IRR) of a bond**

   Bonds are one of the most common forms of investment in securities traded in financial markets.
   - This analysis will be completed in Unit 4 and treated in greater depth in the Financial Markets and Banking Operations subject.

3. **Net present value (NPV) of an investment project**

   One of the basic methods to evaluate the convenience of starting an investment by companies.
   - This and other methods are discussed in detail in the Corporate Finance subject.

4.1. Amortization of a level-payment fixed-rate loan

Loans

- A **loan** is a contract by which a person or a company, called *a borrower*, obtains a sum of money belonging to another person or company, called *a lender*, for a specific period of time.
  - The borrower (or *debtor*) has to return to the lender (or *creditor*) the capital lent (*principal*) with the interest generated over time, through a specific payment scheme.
- Most long-term loans are repaid through a series of **periodic payments**, called total installments, comprising the **interest** generated by the outstanding debt (the so-called *interest payment*) and the repayment or **amortization** of a part of the outstanding debt (*principal or capital repayment*):

$$Periodic\ payment\ = interest\ payment\ + principal\ repayment$$

Level-payment fixed-rate loan

- There are a variety of loans, characterized by different repayment schedules and modalities of the interest rate to be applied.
- However, the most traditional loan is:

  Level-payment, fixed-rate loan (or *préstamo francés* in Spanish), characterized by: *constant periodic payments and fixed interest rates*, throughout the life of the loan.
- In a level-payment fixed rate loan, it is very simple to calculate **the constant periodic amount** that the debtor must pay the lender. To do so, we use the formula for the present value of a constant annuity and make it equal to the amount borrowed (the principal). The present value of the annuity depends on the agreed interest rate and the term of the operation (or number of payments to be made), both of them set out in the contract.

Exercise 2.17: level-payment fixed-rate loan

An amount of 600 euros is borrowed to be repaid through six constant monthly payments. If a fixed compound monthly interest rate of 0.75% (equivalent to a nominal annual rate of 9%) is used, indicate the amount to be paid each month.

Solution:

- Data: principal = 600, $i = 0.0075$, duration = 1/2 years, number of monthly payments = 6
- The amount received (or principal) must be equal to the (present) value of all the outflows to be paid, all of which valued on the same date using the agreed rate:

$$600 = C\,\frac{1-(1+0.0075)^{-6}}{0.0075} \rightarrow C = 600/\left(\frac{1-(1+0.0075)^{-6}}{0.0075}\right) = 102.64$$

Obviously the figure obtained is greater than 100 euros per month, since you have to pay interest for the use of the money.

Exercise 2.17 (cont.)

Calculate the interest payment and the principal repayment per month, as well as the outstanding debt, at the end of each month.

- <u>Solution.</u> You can calculate what is requested, month by month, in the following way:

1. <u>Interest payment = previous outstanding debt × (monthly) interest rate</u>

   First month's $interest\ payment = 600 \cdot 0.0075 = 4.50$

2. <u>Principal repayment (debt reduction) = (monthly) installment – interest payment</u>

   First month's $principal\ repayment = 102.64 - 4.50 = 98.14$

3. <u>Outstanding debt at the end of the period (month) = previous outstanding debt – principal repayment</u>

   First month's $outstanding\ balance = 600 - 98.14 = 501.86$

Exercise 2.17 (cont.): amortization table

Repeating month by month, you can calculate the complete **amortization table:**

- As can be seen, in a level-payment fixed-rate loan, **although what is paid each period is fixed, the part paying interest decreases over time, while the part repaying the debt increases over time.**

Month	Full (periodic) payment	Interest payment	Principal repayment	Outstanding debt
0				€600.00
1	€102.64	€4.50	€98.14	€501.86
2	€102.64	€3.76	€98.88	€402.98
3	€102.64	€3.02	€99.62	€303.36
4	€102.64	€2.28	€100.37	€203.00
5	€102.64	€1.52	€101.12	€101.88
6	€102.64	€0.76	€101.88	€0.00

$\Sigma = 600$

Level-payment fixed-rate loan: generalization

- Notation:
  - C_0 = Principal or amount borrowed (nominal amount of loan), received on initial date 0
    - C_s = outstanding debt on date s (end of sth-period)
  - C = (constant) amount of the periodic payments. It is the sum of:
    - I_s = interest payment or amount of interest paid on date (end of period) s
    - A_s = principal repayment or amount of debt amortized on date (end of period) s
  - n = number of periodic payments
  - i = compound (effective) interest rate per period

- A word of caution:

  Remember that the interest rate used must always refer to the same time period as the frequency of payments. So, in the previous example, it should be a monthly rate. In other cases it may be a quarterly, semi-annual, annual, etc. rate.

Level-payment fixed-rate loan: generalization

- The amount of periodic payments is calculated by equating the principal of the loan, received at the beginning, to the present value of future outflows:

$$C_0 = C\,\frac{1-(1+i)^{-n}}{i}, \text{ thus: } C = C_0 \Big/ \left(\frac{1-(1+i)^{-n}}{i}\right)$$

- The rest of the variables in the amortization table can be calculated, period by period ($s = 1, 2, \ldots, n$), as follows:

  1. $I_s = C_{s-1} \cdot i$
  2. $A_s = C - I_s$
  3. $C_s = C_{s-1} - A_s$

Exercise 2.18:

A loan of 120,000 EUR is granted to be paid over 25 years through level monthly installments.

If the compound monthly interest rate applied by the financial institution is 0.375% (equivalent to a nominal annual interest rate of 4.50%), how much is the amount of the constant monthly installment that the borrower must pay?

4.2. Calculation of the yield to maturity (YTM or IRR) of a fixed-income security

Fixed income securities: bonds

A **fixed-income financial instrument (or security)** (also generically called a ***bond***, for simplicity) promises the investor, in exchange for paying an initial price, the receipt of certain cash flows in the future until its maturity date (or *amortization date*).

- The **objective** of the issuer (be it a public institution or a private company) is to obtain financing, provided by the investors who acquire the securities; and these become a *debt* for the issuer.
- Contrarily, for the **investors** who acquire these securities, they are an *asset* (the investors are lenders to the company).

- **Distinction** between fixed-income assets (bonds) and stocks:

  In both cases the issuer obtains financing; however, while in the case of stock shares the investors become owners of the company, recipients of fixed-income assets become creditors of the company.

Fixed-income securities: bonds

- There is a wide **variety** of fixed-income assets, with different features and names, which are covered in the subject of Financial Markets and Banking Operations.

  Examples include government bonds or treasury notes when considering public debt and the wide variety of corporate debt instruments issued by companies.

- Many bonds are ***marketable*** (tradeable in secondary markets) and, therefore, it is possible to buy and sell them by negotiating with other investors. They have *liquidity*.

Promised cash flows of coupon bonds

- Medium- and long-term bonds make periodic interest payments (traditionally called **coupons**) and pay back (amortize) the principal (or notional) borrowed upon the final maturity date.

- Is easy to calculate the **periodic payments**, when they are fixed-rate coupon bonds (i, effective rate per period):

  The bonds are issued for a subscription amount, called *nominal* (N), so:

  1. *Intermediate periodic* interest (coupon) payments $= N \cdot i$
  2. On the *final due date*, together with the last coupon payment, bonds pay back the nominal $= N \cdot i + N$

Exercise 2.19:

Consider the following bond issue:

- Nominal value of a bond: 1,000 euros
- Payment of annual coupons
- Annual coupon interest rate: 4%
- Duration: 5 years

Using this information, calculate the future cash flows received by an investor who purchases 100 of the bonds issued.

Exercise 2.19: solution

Each bond will pay:

- A coupon of $N \cdot i = 1{,}000 \cdot 0.04 = 40$, at the end of each year for the first four years.
- Additionally, at the end of the fifth year, the cash flow will be:

$$N \cdot i + N = 1{,}000 \cdot 0.04 + 1{,}000 = 1{,}040$$

Therefore, the investor will receive:

- $100 \times 40 = 4{,}000$ EUR during the first four years.
- $100 \times 1{,}040 = 104{,}000$ EUR in the fifth year.

Yield to maturity of a bond

A bond's yield to maturity (YTM)

- What is the **return** for the investor derived from purchasing a bond?

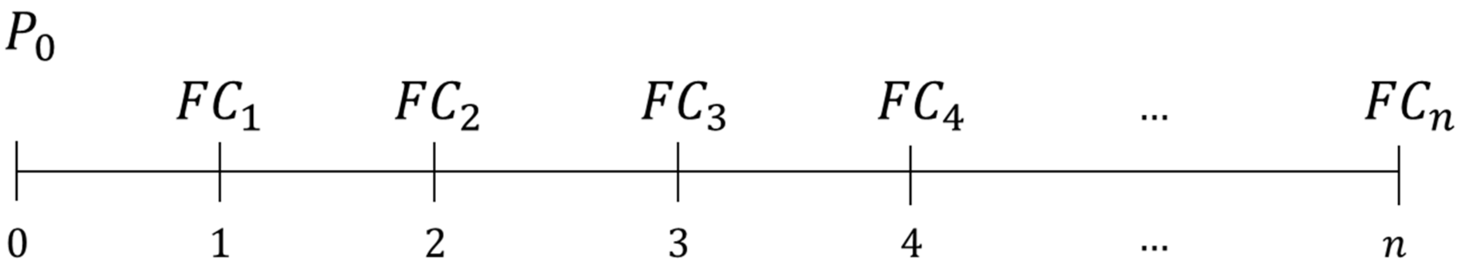
  It is possible to calculate that return with the tools already seen.

- ***yield to maturity* (or internal rate of return, IRR)** of a bond:

  The compound interest rate that equates the price paid for the bond to the present value of all promised future cash flows that will be received in exchange.

Calculation of yield to maturity (YTM)

- Notation :
  - P_0 = bond price, paid on the current date t = 0
  - FC_t = cash flows received from the bond, at each future date t
- Graphical representation:



- **Yield to maturity (YTM) equation:**

$$P_0 = \frac{FC_1}{(1 + \boldsymbol{YTM})^1} + \frac{FC_2}{(1 + \boldsymbol{YTM})^2} + \cdots + \frac{FC_n}{(1 + \boldsymbol{YTM})^n}$$

1. For the ***YTM*** obtained to be an effective *annual* compound rate, the terms of the exponents must be expressed in years.
2. This is solved by iteration, with an advanced calculator or computer, as shown in the following example (Exercise 2.20).

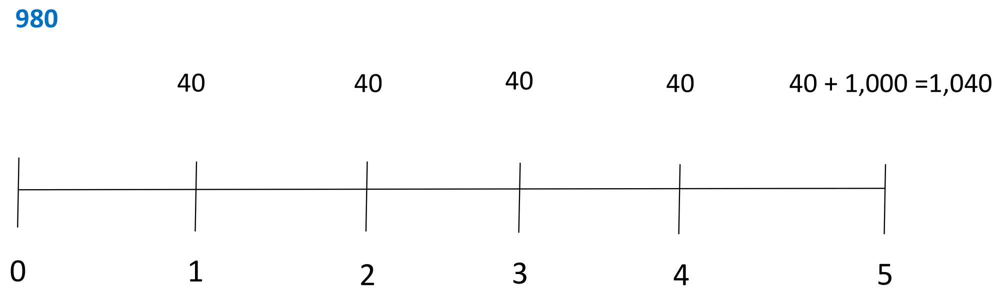
Exercise 2.20: calculation of the YTM

An investor purchases a bond with a face value of 1,000 euros for 980 euros, which pays annual coupons of 40 euros until it matures in five years.

What is the annual yield to maturity (or internal rate of return)?

Exercise 2.20: solution

- Graphical representation:



- Equation:

$$980 = \frac{40}{(1+YTM)^1} + \frac{40}{(1+YTM)^2} + \frac{40}{(1+YTM)^3} + \frac{40}{(1+YTM)^4} + \frac{1{,}040}{(1+YTM)^5}$$

Alternatively, using the present value of a constant annuity formula:

$$980 = 40 \cdot \frac{1-(1+YTM)^{-4}}{YTM} + 1{,}040 \cdot (1+YTM)^{-5}$$

- This must be solved with a computer. Solution: 4.455% per year.

Calculation of YTM (or IRR) with Excel: Exercise 2.20 (cont.)

- Below is **how to obtain the YTM using a spreadsheet** such as Excel, in this case for Exercise 2.20.

1. **First**, you must **enter the cash flows** in a column (or a row), using one cell for each date.
   - A minus sign must be used to indicate that the first cashflow introduced is a cash outflow.
2. **Second,** use the **=TIR() function** (Spanish version) or **IRR()** function (English version) to return the desired result.
   - Warning: If the coupons were not received annually but m times per year ($m > 1$), the sub-annual rate given by the =TIR() function would have to be converted to an annual rate, using the following formula: =(1+(=TIR()))^ m – 1

	A
1	-980
2	40
3	40
4	40
5	40
6	1040
7	=TIR(A1:A6)

= IRR (A1:A6)

Exercise 2.21:

An investor acquires, on the issuance date, a bond with a nominal value of 1,000 euros, which pays annual coupons of 40 euros.

What is the annual return to maturity if the price paid for that bond is the bond's face value?

Yield to maturity: basic idea 1

- **Two basic assumptions** are made when obtaining the YTM as a measure of the return of a bond:

1. **There is an inverse relationship between the price of a bond and its YTM.**
   - Indeed, given the future cash flows promised by a bond, the present value equation establishes an inverse relationship between the current price and the YTM.
   - The intuition is simple:

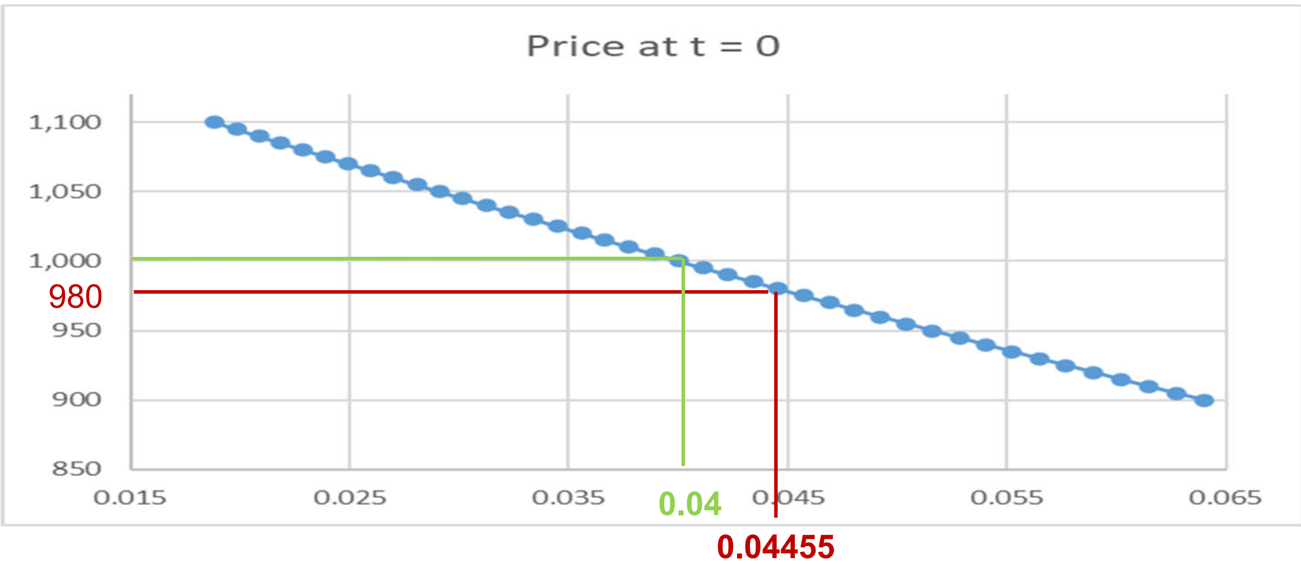
     Given the cash flows to be received on future dates, it will be the variation in the price of the security – given by the interaction between supply and demand in the market – that will determine the return for the investor derived from the purchase of the bond.

     Thus, if the price to be paid falls, the return rises ("we pay less now to obtain the same cash flows in the future, therefore the rate of return is higher") and if the price to be paid for the bond rises, the return falls ("We pay more now to obtain the same cash flows in the future").

Exercise 2.22: inverse relationship between price and YTM

As an example, we can see the expected return (YTM) that would be obtained from the purchase at different prices of a bond with a face value of 1,000 euros, annual coupon at 4% and five years until maturity.

The graph below highlights the values of the previous exercises 2.20 and 2.21.



Yield to maturity: basic idea 2

- The second basic idea underlying the YTM as a measure of the return of a bond is:

2. **For the YTM to be the return to maturity obtained on the investment made, the cash flows received must be reinvested at the same rate of return.**
   - Indeed, given the equation that allows obtaining the YTM:

$$P_0 = \frac{FC_1}{(1+\boldsymbol{YTM})^1} + \frac{FC_2}{(1+\boldsymbol{YTM})^2} + \cdots + \frac{FC_n}{(1+\boldsymbol{YTM})^n}$$

     Multiplying by $(1+\boldsymbol{YTM})^n$ both members of the previous equation we obtain:

$$P_0(1+\boldsymbol{YTM})^n = FC_1(1+\boldsymbol{YTM})^{n-1} + FC_2(1+\boldsymbol{YTM})^{n-2} + \cdots + FC_{n-1}(1+\boldsymbol{YTM})^1 + FC_n$$

   - The intuition is:

     All the cash flows are valued now at time *t* = *n*. Therefore, to obtain the YTM on the left side as a return on the investment made, the right side must be the final accumulated value received in exchange, which requires all the cash flows to be reinvested at that YTM as they are received.

Yield to maturity (YTM) as the bond's expected rate of return

The YTM (or IRR) as expected yield

- The yield to maturity:

a. It is used to measure the **return** of a bond **obtained** ex-post (under the assumption of reinvestment of the cash inflows).

b. However, as a basic criterion for making an *a priori investment decision*, it should be considered as a **predicted or expected return**, which ***will only be equal to the actual return obtained if a series of conditions are met***:

   1. First, it requires the issuer to fulfill their commitments (to pay the promised cash flows).

   2. Additionally, it requires for the investor:

      a. To hold the title until maturity; and
      b. To reinvest the cash inflows at the same rate of return (the YTM).

   - The consideration of all these issues and risks associated together with their management is a more complex question and will be addressed in the Markets Financial and Operations Banking subject.

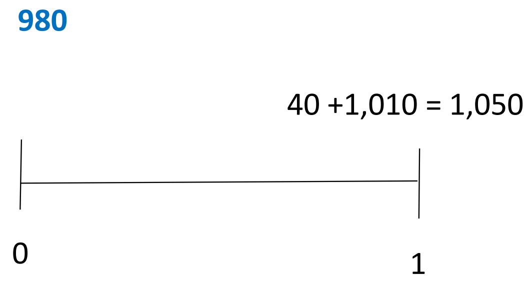
Exercise 2.20 (continued): actual return

Let us consider that the investor who purchased a bond with a face value of 1,000 euros for 980 euros, which pays annual coupons of 40 euros until its maturity date in five years, decides to sell it just after having received the first annual coupon, that is, just one year after purchasing it. The selling price in the market is 1,010 euros.

What is the <u>annual return actually obtained</u> on the investment?

Exercise 2.20 (continued): solution

- Graphical representation:



- IRR equation of the investment actually carried out:

$$980 = \frac{1{,}050}{(1+IRR)^1} \leftrightarrow IRR = \frac{1{,}050}{980} - 1 = 0.071428 \rightarrow 7.1428\%$$

- **Since the title has not been kept until maturity, the actual rate of return obtained is different from the initially *expected* IRR** (4.45%).

  In this example, the actual IRR is higher than the expected one; but if the selling price were lower, the actual rate of return would be less. (For example, if the selling price were 970 euros, the actual rate of return would be 3.06%; check it!)

Exercise 2.21 (continued)

Consider that the investor (A) who acquired a bond with a face value of 1,000 euros on the date of its issuance, which paid annual coupons of 40 euros until its maturity date, decides to sell it exactly after receiving the first coupon payment. The selling price is 1,010 euros.

With this data,

a. What is the annual return actually obtained on the investment?

b. What would the expected annual return be for whoever (investor B) buys the bond from investor A for 1,010 euros one year after it was issued?

4.3. Net present value

Net Present Value (NPV) of a project

- One type of decision that companies must make refers to which investment projects must be undertaken, many times choosing between alternative projects.
- There are various tools to analyze projects, which help in making investment decisions.
- One of the basic tools is the NPV:

  The **Net Present Value** quantifies the difference between the current value of the (estimated) future cash flows that a project promises to generate and the initial amount invested.

NPV calculation

- In accordance with the previous definition:

$$NPV = -FC_0 + \frac{FC_1}{(1+k)^1} + \frac{FC_2}{(1+k)^2} + \cdots + \frac{FC_n}{(1+k)^n}$$

  where:

  - FC_0 is the initial cost of the investment;
  - FC_s $(s = 1, 2, \ldots, n)$ are the expected future cash flows generated by the project; and
  - k is the appropriate discount rate, which is based on the level of risk of the project.

Use of NPV as a decision criterion

- As a decision rule, it is intended to let the manager know whether or not it is advisable to carry out a given investment project.
    - The basic idea is that *it should go ahead if the project's NPV is positive*, since it indicates an increase in wealth for company shareholders as a result of undertaking the investment.
    - The NPV also serves to compare alternative investment projects. We choose the one with the highest NPV.

Exercise 2.23: NPV of an investment project

We consider an investment project that involves an initial outlay of 200,000 euros and the relevant department of the company considers that it will have an economic life of three years. The following figure indicates the expected cash flows over the next three years:

Year	Expected cash flows
1	100,000
2	80,000
3	50,000

a. Considering that the company uses 7% per year as a discount rate, indicate whether or not it is advisable to carry out this investment project.

b. Were last year's cash flow 40,000 euros instead of 50,000, what would the NPV be?

Exercise 2.23: solution (a)

Solution for Question A:

k = 0.07

FC_0= 200,000 (initial outlay)

Year	Expected cash flows
1	100,000
2	80,000
3	50,000

$$NPV = -FC_0 + \frac{FC_1}{(1+k)^1} + \frac{FC_2}{(1+k)^2} + \cdots + \frac{FC_n}{(1+k)^n} =$$

$$= -200{,}000 + \frac{100{,}000}{(1+0.07)^1} + \frac{80{,}000}{(1+0.07)^2} + \frac{50{,}000}{(1+0.07)^3} =$$

$$= -200{,}000 + 100{,}000(1.07)^{-1} + 80{,}000(1.07)^{-2} + 50{,}000(1.07)^{-3}$$

$$= 4{,}147.94$$

Exercise 2.23: solution (b)

Solution for Question B:

k = 0.07

FC_0= 200,000 (initial outlay)

Year	Expected cash flows
1	100,000
2	80,000
3	40,000

$$NPV = -FC_0 + \frac{FC_1}{(1+k)^1} + \frac{FC_2}{(1+k)^2} + \cdots + \frac{FC_n}{(1+k)^n} =$$

$$= -200{,}000 + \frac{100{,}000}{(1+0.07)^1} + \frac{80{,}000}{(1+0.07)^2} + \frac{40{,}000}{(1+0.07)^3} =$$

$$= -200{,}000 + 100{,}000(1.08)^{-1} + 80{,}000(1.08)^{-2} + 40{,}000(1.08)^{-3}$$

$$= -4{,}015.04$$

- It is no longer advisable to carry out the project since NPV < 0.

NPV: appropriate discount rate

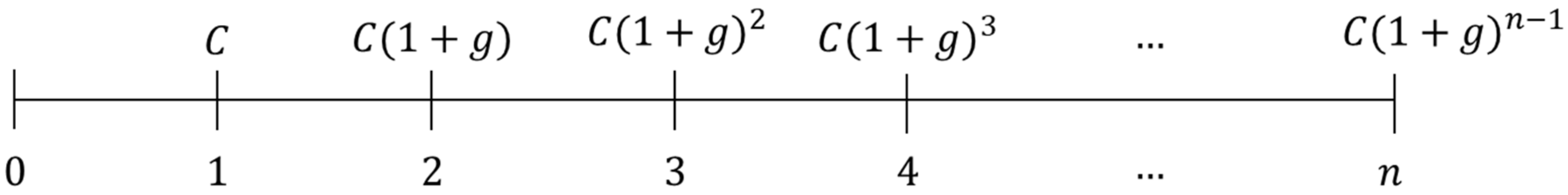
- In calculating the NPV, we take into account that one monetary unit today does not have the same value as one monetary unit at any other date. Therefore, we use a discount rate today to evaluate the cash flows an investment project will generate in the future; and also, to compare them with the initial cost of the investment.
- Financial economics theory has studied what the appropriate discount rate for such a project should be, which is based on its risk level. This discount rate is called *cost of capital*.
  - Details of estimating the cost of capital and other, practical issues related to the use of NPV are covered in the Corporate Finance subject.

Appendix: demonstrations and summary tables of annuity formulas

Appendix: demonstrations of annuity formulas

- First we will establish the proof of the most general formula and then treat the rest as individual cases.

1. **Present value of n variable future capitals at a constant exchange rate g** (as long as $g \neq i$):



This is calculated:

$$V_0 = C(1+i)^{-1} + C(1+g)(1+i)^{-2} + \cdots + C(1+g)^{n-1}(1+i)^{-n}$$

Notice that, starting from the first summand in the equation, i.e. $C(1+i)^{-1}$, each of the summands is equal to the previous one multiplied by $(1+g)(1+i)^{-1}$.

So the right side of the previous equation is the sum of a series of n numbers that vary in geometric progression of ratio $(1+g)(1+i)^{-1}$and the first term is $C(1+i)^{-1}$.

This sum can be calculated as:

$$V_0 = \frac{C(1+g)^{n-1}(1+i)^{-n}(1+g)(1+i)^{-1} - C(1+i)^{-1}}{(1+g)(1+i)^{-1} - 1}$$

$$= C \cdot \frac{(1+g)^n(1+i)^{-n} - 1}{1+g-(1+i)}$$

$$= C \cdot \frac{1-(1+i)^{-n}(1+g)^n}{i-g} \quad \textit{or, alternatively:} \quad C \cdot \frac{1-\left(\frac{1+g}{1+i}\right)^n}{i-g}$$

- If $\underline{g = i}$, the prior formula is not appropriate. To obtain a valid formula for this particular scenario, we must start from the initial present value equation of the previous general case:

$$V_0 = C(1+i)^{-1} + C(1+g)(1+i)^{-2} + \cdots + C(1+g)^{n-1}(1+i)^{-n}$$

Now we substitute g with i, thereby obtaining:

$$V_0 = C(1+i)^{-1} + C(1+i)(1+i)^{-2} + \cdots + C(1+i)^{n-1}(1+i)^{-n}$$
$$= C[(1+i)^{-1} + (1+i)^{-1} + \cdots + (1+i)^{-1}] = C \cdot n \cdot (1+i)^{-1}$$

2. Present value of infinite variable future cash flows at a constant exchange rate of g (with $g < i$):

This is the limiting case (when n becomes infinitely large) from case one, seen above:

$$V_0 = \lim_{n\to\infty} C \cdot \frac{1 - \left(\frac{1+g}{1+i}\right)^n}{i-g} = \frac{C}{i-g}$$

3. Present value of n constant cash flows:

This is a specific case, in which the cash flows do not change, based on case one. Therefore, we need only substitute $g = 0$ in the previous general formula, leaving us with:

$$V_0 = C \cdot \frac{1 - (1 + i)^{-n}}{i}$$

4. Present value of infinite constant cash flows:

This particular case, when $g = 0$, stems from case two:

$$V_0 = \frac{C}{i}$$

Appendix: summary of income tables

FORMULAS
Notation used in financial mathematics books

	present value at t = 0 (a period before the maturity of the first capital)
$g \neq i$ $(where\ q \equiv 1 + g)$	$V_0 = A(C,q)_{\overline{n}\vert i} = C \cdot \dfrac{1-(1+i)^{-n}q^n}{1+i-q}$
$g = i$	$V_0 = A(C,(1+i))_{\overline{n}\vert i} = C \cdot n \cdot (1+i)^{-1}$
$g = 0$ (constant annuity)	$V_0 = C \cdot a_{\overline{n}\vert i} = C \cdot \dfrac{1-(1+i)^{-n}}{i}$

FORMULAS for perpetual capitals ($n \to \infty$).
Notation used in financial mathematics books

	present value at t = 0 (a period before the maturity of the first capital)
$g < i$ $(where\ q \equiv 1 + g)$	$V_0 = A(C, q)_{\overline{\infty\|}i} = \dfrac{C}{1 + i - q}$
$g = 0$ (constant annuity)	$V_0 = C \cdot a_{\overline{\infty\|}i} = \dfrac{C}{i}$

Recommended bibliography

- The content of this topic can be found, in various versions, in several excellent textbooks.
- For more details, the following manual is recommended, which corresponds quite well to the content, level and approach of these notes, and from which some specific ideas have been taken:

  Finance Economics, 2nd ed (2009) by Z. Bodie, R. Merton and D. Cleeton , Pearson. Chapters 4, 6 and 8.

  Or the first edition, translated into Spanish: *Finance* , 1st ed. (2000), by Z. Bodie and R. Merton, Pearson.
- Also chapters 2, 3 and 4 from the book *Manual práctico de economía financiera* (2023), by M. Gutiérrez and J. Moreno, ed. Pirámide, are useful to complement these notes.

Unit 3

Arbitrage and relative valuation

Dept. Financial and Actuarial Economics

Universitat de València
Facultat d'Economia

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1. Absence of arbitrage and the law of one price (LOP)

Arbitrage and non-arbitrage

Remember:

- An arbitrage opportunity is an investment strategy that guarantees a positive payoff under certain conditions with no possibility of a future negative payoff and with no net investment. Therefore, an ***arbitrage*** *is a trading strategy that allows you to make money without any possibility of losing money in the future and without net investment today. Put a different way, it is a strategy implying zero outflows today and at least one inflow in the future.*
- The constant action of arbitrageurs, professionals dedicated to exploiting arbitrage opportunities as soon as they arise in the market, causes them to disappear quickly and any **set of asset prices is usually *free of arbitrages***.
- ***The absence of arbitrage (opportunities) entails relationships between asset prices that* are very relevant in practice.**

Absence of arbitrage and LOP: AAO $\Rightarrow$ LOP

Remember:

- ***Law of One Price****: Two assets (or portfolios) that are perfect equivalents must have the same price.*
- An implication of the absence of arbitrage, so important it deserves to be highlighted as *one of the fundamental results of financial theory,* is as follows:

  In the absence of arbitrage opportunities (AAO), the "Law of One Price" (LOP) must be met.

  - In other words, any violation of the LOP entails an opportunity for arbitrage.

Simplifying assumptions about markets

- In this unit we will see examples of relative valuation in the absence of arbitrage.
- For simplicity's sake, we assume financial markets exist in which **there are no frictions,** or imperfections, such as transaction costs and restrictions on the divisibility of assets.
  - At a theoretical level, although it is possible to extend the theory to a more realistic case including costs, it makes sense to start with a simplified analysis, where the underlying financial intuitions are more easily understood.
  - On a practical level, furthermore, relative valuation assuming an idealized world without transaction costs is a useful reference in the real world, taken as a starting point for institutional negotiators since they have lower costs than retail investors.

LOP for whom?
Practical importance of relative valuation

- As we will see, the LOP entails a variety of valuation relationships between many tradeable assets in financial markets.

- But, if arbitrages are usually exploited by the economic agents in the best position to do so, ie. arbitrageurs, whereas I as a retail investor trade for other reasons (investment or risk management), the question is: should I also be interested in relative valuation relationships?

  The answer is yes.

- First of all, **relative arbitrage valuation is useful for anyone** trading in the financial markets, because:

  If you trade at prices that allow others to arbitrage, whatever your trading objective is, you will be giving away "free meals"; that is, "you will be fighting the investment battle with one hand tied behind your back".

- On the other hand, the general idea of using *equivalent* assets as an aid to relative valuation can also be useful in contexts where arbitrage is not possible:
  - In fact, **relative valuation *by comparison*** is used to ***estimate* the fair value** of assets that are not traded, or are traded in illiquid markets, taking as a basis for reference the market prices of other assets (or portfolios) of *comparable* traded assets.

Exercise 3.1: price of a share in two markets

Unilever shares are traded both in London (LSE) and Amsterdam (Euronext).

If its price in London is 39.715 pounds per share, when the exchange rate is 1.2298 euros per pound, what should its price in euros be in Amsterdam?

Solution:

- The price in pounds (in London) is 39.715 pounds, that is, 48.842 euros at the current exchange rate ($39.715 \times 1.2298 = 48.842$).

  Then the price should be 48.842 euros in Amsterdam, by the LOP, to match the price in London converted to euros.

Exercise 3.2: stock arbitrage

Suppose that Unilever shares could be traded without restriction or market friction at a price per share of 39.715 pounds in London and 45.842 euros in Amsterdam, when the exchange rate is 1.2298 euros per pound.

Determine what arbitrage opportunity would be available to a eurozone arbitrageur (that is, using the euro as their usual unit of account). Calculate the profit they would obtain from the arbitrage, for each share traded.

Exercise 3.2: solution

- The arbitrageur could simultaneously buy shares in Amsterdam and sell them in London to make a asssured profit in euros of:

$$(39.715 \text{ pounds} \times 1.2298 \text{ euros per pound}) - 45.842 \text{ euros}$$
$$= €3 \text{ per share}$$

  - As eurozone arbitrageurs bought shares in Amsterdam and sold them in London, they would cause the buying price (in euros) to rise and the selling price (in pounds) to fall until stock prices in both markets were equivalent to the prevailing exchange rate.

Exercise 3.2, part 2 (advanced): arbitrage in pounds

Consider the initial situation set out in Exercise 3.1.

Determine what arbitrage opportunity exists for an arbitrageur whose usual unit of account is the pound sterling. Calculate the profit they would make for each share traded.

The search for *arbitrage* strategies in practice and their dangers

- Firstly, as already explained, pure arbitrage strategies rarely occur.
- So, as a **first lesson:**

  If you see different prices for *seemingly* identical assets, be careful.

  - Following the advice of Bodie, Merton and Cleeton (2009, p. 201), the first thing you should suspect is that:

  (i) something may be interfering with the normal trade of the competitive market;

  For example, impediments to negotiation or delays in the dissemination of information.

  (ii) or there is some *apparently* insignificant difference, which may not have been detected, between the assets.

  For example, liquidity differences.

The search for *arbitrage* strategies in practice and their dangers

- Secondly, in practice, certain investors carry out other trading strategies which they sometimes label *arbitrage*, but which strictly speaking are not arbitrages because there is some likelihood of losing money (and sometimes a lot!).

- Below we will discuss an example of this type.

  The general **lesson** to be drawn from the example is clear:

  Beware of speculation disguised as arbitrage!

The search for *arbitrage* strategies in practice and their dangers

- One example are strategies called **statistical arbitrage** (abbreviated *stat. arb.* in English), which involve simultaneously trading in assets that have deviated from some statistically detected relationship in the past.
    - It includes ***convergence arbitrage,*** that rests on the discrepancy between observed price differentials of a pair of related assets and the historical average differentials.

      The idea is quite simple: investors think that spreads tend to revert to their historical average when they move too far from it, so the expected return to the mean could be exploited to make a profit by taking opposing positions in the assets.

The search for *arbitrage* strategies in practice and their dangers

Real-life case: Long-Term Capital Management (LTCM)

- The case of LTCM is studied in advanced risk management courses around the world for its many and important lessons fundamentally pertaining to extreme events, risk quantification, leverage and liquidity risk (Stulz, 2003).

- Here we will focus only on the issue of convergence (or differential) arbitrage.

- Since 1994, LTCM managed a *hedge fund* which during its first four years of life obtained returns above 20%, mainly carrying out convergence arbitrage strategies in bond markets.

  In particular, the managers took opposite positions in US Treasury bonds with different maturities and differences in liquidity.

- When Russia defaulted on its debt in August 1998, the markets reacted in unison in a process of *flight to quality/liquidity*, which caused very large losses in the market value of the hedge fund.
- Additionally, investor withdrawals and daily losses led to asset sales, which in turn caused further losses in the fund's value, as its large asset sales caused the market to sink further. The fund lost $1.85 billion during August 1998 and $1 billion a week during early September 1998.
- Finally, the Federal Reserve Bank of New York organized a LCTM bailout to prevent its collapse endangering the financial system.

2. LOP applications

Application 1: exchange rates and triangular arbitrage

- The LOP entails several non-arbitrage relationships in the foreign exchange markets.
- One of them is:

  For any three currencies that are freely negotiable, we need only know the exchange rates between any pair of them to work out the third one.

Exercise 3.3: exchange rates

Suppose the euro price of the pound sterling is €1.15 per pound and the price of pounds in US dollars is $1.22 per pound. What should the price of the dollar in euros be?

Solution: €0.94

- To see this, think about the fact that there is an indirect method equivalent to buying dollars in euros, going through the pound market:
  - Buy pounds with euros first and then use the pounds to buy dollars:
    - Since for 1.15 euros you get a pound and for 1 pound you get 1.22 dollars, $1.22 costs you €1.15.
    - That is, each dollar through this indirect route costs €0.94 (= 1.15/1.22).
- So, by the LOP, the direct purchase of dollars in the foreign exchange market should cost the same as through the indirect route, so the price of a dollar should be €0.94 (or, to be more precise, €0.942623).

Exercise 3.3 (cont.): triangular arbitrage

- If the price of the dollar were not €0.94, then the LOP would be violated and there would be an arbitrage opportunity, which would not last long.

  Suppose that, in the previous situation, the price of the dollar was €1.1 in the foreign exchange market. What arbitrage opportunity would be available?

Exercise 3.3 (cont.): solution

<u>Solution</u>:

- You could instantly convert €1.15 into €1.342, by doing the following:
  - Convert €1.15 into 1 pound, the pound into \$1.22 and finally the \$1.22 into €1.342, selling them at the exchange rate of 1.1 euros per dollar ($1.22 \times 1.1 = 1.342$).
- With this, you would instantly earn 0.192 euros (= 1.342 – 1.15) for every 1.15 euros invested (or, alternatively, for every 1.22 dollars sold).
  - That is equivalent to earning 0.192/1.22 (= 0.157377) euros for each dollar sold, that is, a profit equal to the difference between the current price of the dollar and what it should be according to the LOP (0.157377 = 1.1 – 0.942623).
- That profit is a *safe profit*, that is, without risk.
- Arbitrageurs would increase the scale of this arbitrage as much as possible, until it would no longer be possible to carry it out due to the variation in the exchange rates.

Application 2: forward price determination

- Relative valuation based on the absence of arbitrage is also used in the valuation of **derivative instruments**, which are, by definition, those *financial instruments whose value is derived from the price of an underlying asset.*

- In particular, the LOP serves to **determine the forward price** of an asset, relatively, given the spot price of the underlying asset and the prevailing interest rate.

  The relationship varies in details, depending on the pattern of income obtained from the ownership of the specific underlying asset.

- Now let's look at a simple case, for illustrative purposes:

  The forward price of a stock that does not pay dividends during the period until delivery in the forward contract must equal the spot price of the stock plus the accrued interest until the delivery date.

  (A case with dividends will be studied in Unit 7.)

Forward price

- Remember that the *spot price* of the stock is simply its price on the stock market on the day in question. We speak of a *spot* price to highlight that it is a price for immediate delivery and so distinguish it from the *forward price* for future delivery.
- Mathematically, the above relationship can be written:

$$F = S \cdot (1 + i)^T$$

where:

F = forward price of the stock, *on the date we are determining it* (usually the current date);

S = cash price of the share, on the same date (today);

i = annual effective compound interest rate; and

T = period until the date of delivery of the forward contract, measured in years.

- Additionally, in the demonstration, we will use S_T to refer to the spot price of the stock prevailing *on the delivery date* of the forward contract.

Forward price: demo

- To demonstrate the above relationship, let us compare two ways of acquiring the share of stock:
  - (A) a direct route: taking a long position in a forward contract; and
  - (B) an indirect one: buying the stock and borrowing money.

A. Direct route: taking a long position in a forward contract:
   - Provides a net value on the delivery date equal to the difference between the value of the share you receive and the price you pay to receive it (the agreed forward price):

$$\textit{Net final value} = S_T - F$$

   - Notice that you don't have to pay anything to start with.

Forward price: demo

B. Indirect route: buying the share in cash and borrowing $F/(1+i)^T$ euros at rate i for period T.

- It provides a final net value equal to the value of the share held in the portfolio minus the payment that must be made to repay the loan, which consists of what was asked for along with the interest generated:

$$\textit{Net final value} = S_T - F$$

- This negotiation strategy has an initial cost equal to what was spent on the purchase of the share minus the income from the requested loan:

$$\text{Net initial cost (price)} = S - F/(1+i)^T$$

Forward price: demo

- Thus, the two trading strategies provide the same net final value, $S_T - F$, therefore they are completely equivalent.

- So, for the LOP, their respective initial prices (initial costs) must match:

$$0 = S - F/(1+i)^T$$

Thus:

$$F = S(1 + i)^T$$

Exercise 3.4: forward price

The current cash price of a share is 20 euros and the compound effective interest rate is 8% per year. The stock is not expected to pay any dividends over the next three months.

What should be the forward price of the stock for delivery in three months?

Exercise 3.5 (advanced): arbitrage with forward contracts

In the situation presented in Exercise 3.2 above, suppose that you could enter into a forward contract for delivery within three months, at a delivery price of 24 euros.

a) What position should you take?

b) What arbitrage opportunity would be available?

Application 3: put-call parity

- Options give the right to buy or sell an asset in the future at a price set in advance. They are another example of a financial derivative instrument that is valued using the absence of arbitrage oportunities hypothesis.
- There are a wide variety of valuation relationships between option prices and their underlying asset prices, based on the absence of arbitrage.
- Here we will see the simplest relationship based on the LOP, for illustrative purposes: the **so-called put-call parity relationship**, in cases where the underlying asset is a stock that does not pay dividends.
- First of all, it is worth briefly reviewing what **financial options are**.

Call and put options

- A **call option** is a contract that gives *the right (but not the obligation) to buy an* (underlying) *asset* at a fixed future date, at a delivery price set in advance (called *the strike price*). Acquiring this right has a cost that is paid at the beginning, called an option price or *premium*.
- A **put option** is a contract that gives *the right (but not the obligation) to sell an asset* (underlying) at a fixed future date, at a delivery price set in advance (called the strike *price)*. Acquiring this right has a cost that is paid at the beginning, called an option price or *premium*.

(This type of options, whose right can only be exercised on a final date at a fixed price, are denominated *European* options, to distinguish them from other types of options with other characteristics. More details in Unit 7.)

Put-call parity

- Due to the LOP, the valuation relationship called **put-call parity** is, for the simplest of cases, the following:

  The premium of a call option on a stock that does not pay dividends during the life of the option must coincide with the premium of a put option, with the same delivery date and exercise price, adding the stock price and subtracting it the discounted exercise price from the date of delivery of the options.

Put-call parity

- Mathematically, this valuation relationship can be written:

$$C = P + S - \frac{E}{(1+i)^T}$$

where:

C = premium (price) for the right to buy the stock, at a delivery (or strike) price E, after a period of time T;

P = premium (price) for the right to sell the stock, at the same delivery (or strike) price E, after the same period of time T;

S = spot price of the stock;

E = strike price to buy or sell the stock on the delivery date;

i = annual effective compound interest rate; and

T = period until the exercise date of the right, or delivery date, measured in years.

- Additionally, in the demonstration, we will use S_T to refer to the spot price of the stock prevailing *on the exercise date* of both options.

Put-call parity: demo

- To demonstrate the parity relationship, we will compare buying the call option with the adquisition of an equivalent portfolio, trading with a put option, the stock, and borrowing money:

A. Acquisition of the right to buy (purchase of the *call* option):

- Final net value:
  - **If $S_T > E$**, then the right to buy is exercised:

    Final net value = $S_T - E$
  - **If $S_T \leq E$**, then the right to buy the stock is worthless and therefore not exercised and therefore expires:

    Final net value = 0
- Initial cost (price) paid = C

Put-call parity: demo

B. Equivalent portfolio: Buy a put option, buy the stock and borrow an amount of $E/(1+i)^T$ euros:

- Final net value:
  - **If $\boldsymbol{S_T} \geq \boldsymbol{E}$**, then the right to sell the stock is not exercised as it is worthless and the right expires. You end up with a share in your portfolio worth S_T and you have to pay back the loan (plus interest):

    Final net value = $S_T - E$

  - **If $\boldsymbol{S_T} < \boldsymbol{E}$**, then the right to sell the stock is exercised. The share delivered in the exercise is the share purchased at the beginning and eventually there are no shares in the portfolio to consider. In addition, the loan must be repaid plus interest. That is to say:

    Final net value = $E - E = 0$

- Initial net cost (price) paid = $P + S - E/(1+i)^T$

Put-call parity: demo

- Thus, since the two trading strategies provide the same net final value, in any circumstance that arises, they are completely equivalent.
- Therefore, according to the LOP, their respective initial prices (costs) must match:

$$C = P + S - \frac{E}{(1+i)^T}$$

Exercise 3.6: put-call parity relationship and the relative valuation of a call option

The current price of a share that will not pay dividends for the next three months is €100 and the prevailing interest rate is 5% (annual effective compound rate).

If the price of a put option on the stock with strike price of €100 and delivery in three months is currently €3.37, what should the price of a call option with both the same delivery date and strike price as the put option be?

Exercise 3.7 (advanced): arbitrage with options

In the situation presented in Exercise 3.6, suppose that the call option could be negotiated in the market at a price of €6.60.

What arbitrage opportunity would be available? Detail the entire arbitrage strategy and the profit you would make from it.

Exercise 3.8: *put-call* parity relationship and relative valuation of a put option

The current price of a share that will not pay dividends for the next three months is €90 and the prevailing interest rate is 5% (annual effective compound rate).

If the price of a call option on the stock, with a strike price of €100 and delivery date in three months, is currently €0.91, what should the price of a put option with the same delivery date and strike price be?

3. Synthetic assets and financial engineering

Synthetic assets

- In relative valuation based on LOP, we have seen several examples of trading strategies or portfolios that are perfect substitutes for other assets.
- Examples:
  - In the foreign exchange market, the purchase of a currency is equivalent to the indirect purchase through two other exchange rates.
  - In the forward market, taking a long position in a forward contract on a stock is equivalent to buying the stock in the spot market along with the possibility of financing it at the current risk-free interest rate.
  - In the options market, taking a long position in a call option is equivalent to the creation of a portfolio made up of a put option, the underlying asset and the possibility of financing it at the prevailing risk-free interest rate.

Synthetic assets

- These examples of trading strategies, or portfolios, illustrate the possibility of ***synthesizing*** the pattern of future cash flows generated by a financial instrument through trading (taking either long or short positions) with other financial instruments.
- Such strategies are called *synthetic instruments* or *replicating portfolios*.

Exercise 3.9: synthetic creation of a put option

Synthesize a put option on a stock which does not pay dividends and has a delivery date of T and an exercise (or strike) price of E by creating a replicating portfolio.

That replicating portfolio should be made up of a call option, the underlying stock asset and the possibility of investing or financing at the risk-free interest rate of i(annual effective compound rate).

Exercise 3.9: solution

- *Put-call* parity ratio gives us the clue to the synthetic portfolio. We know that:

$$C = P + S - \frac{E}{(1+i)^T}$$

- So:

$$P = C - S + \frac{E}{(1+i)^T}$$

- Which gives us a clue as to how to synthesize the put option:

  Buy a call option (take a long position), sell (short position) the underlying stock and invest an amount of $E/(1+i)^T$ euros at the rate i during the period T.

Exercise 3.10 (advanced): synthetic creation of the risk-free asset with risky assets

Synthesize the possibility of obtaining E euros with certainty at the end of T years, trading with a stock (without dividends) and call and put options on that stock as the underlying asset.

Synthetic instruments and financial engineering

- The concept of a synthetic instrument, or replicating portfolio, is central to what is called *financial engineering*.

  Financial engineering: *a combination of quantitative techniques with financial principles to conceive and create innovative financial products and strategies.*

- In addition to the usefulness already shown in the valuation of assets, financial engineering methods are used in a variety of other practical problems.
  - Examples:

    cash flow management, risk management, creation of innovative financial products, etc.

Creation of new instruments and products (1)

- Replicating portfolios of existing financial instruments are only a part of the multitude of synthetic instruments that can be created by combining the entire variety of financial instruments available, in static and dynamic trading strategies.

- Part of the business of financial intermediaries (especially investment banking) consists of creating new instruments and products characterized by having cash flow patterns different from those existing already in the market and that cover an investment need of their clients.

  - Investment products with non-standard, usually complex, payment patterns are called ***structured products***.

  - Example: A hybrid investment product, between fixed income and equity, whose profitability depends on the evolution of the stock market, but which has a guaranteed return of the invested capital in the event that the stock market falls.

    (There are products of this type offered by banks, under different commercial names.)

    Example:

    https://www.dbs.com.sg/treasures/investments/product-suite/structured-investments#:~:text=Structured%20products%20are%20financial%20instruments,or%20a%20mix%20of%20these.

Exercise 3.11: synthetic creation of the right to receive stock dividends

As an example of synthetic creation of a new financial instrument, let us think about the possibility of collecting dividends from a company, over a period of time, without being subject to the risk of possible falls in the price of the corresponding share.

- Investing in the stock is the direct way to guarantee the collection of dividends, but it is subject to the risk derived from the share price falling during the investment period:

  Losses could ultimately be incurred, even by investing in a share that pays high dividends, if the sale price of the share at the end of the investment term was low enough so that the capital loss offsets the dividend gain.

Exercise 3.11 (cont.)

- Now, if you buy the stock and, simultaneously, take a short position in a forward contract and borrow cash equal to the discounted forward price, a synthetic payment pattern consisting of *exclusively receiving* the stock's dividends is guaranteed, until the delivery date of the forward contract.

- In fact, let us go step by step:

  1. The purchase of the stock in cash and the simultaneous short position in the forward contract guarantees the collection of dividends as long as the stock is held in our portfolio, plus a final sale amount of the stock already set in advance.
  2. That combination eventually allows you to have no stock in your portfolio after its delivery according the short position in the forward contract at the maturity date.
  3. Finally, the payment to be made to cancel out the requested loan will be exactly offset by the inflow corresponding to the forward price agreed in the forward contract on the delivery date.
  4. Therefore, the only net cash flow left is the dividend income (from step 1), as intended.

Creation of new instruments and products (2)

- When new instruments or products are created, the issuing company can **manage the risk** associated with the resulting position, just *replicating* its payment pattern with the assets available in the markets.
- The **company's goal**, in that case, is to issue the synthetic product at a price that allows it to obtain a *profit compared to the price of the replicating strategy* that covers its structure of cash flows.
- Notice that the **company's client** could also theoretically synthesize it on their own, but generally will not *for two reasons*:
    1. First, once again, because financial institutions have access to markets under better conditions than their clients, especially with lower **transaction costs**.
    2. Second, a **complex** payment pattern may require quantitative and financial knowledge, available only to highly qualified specialized professionals.
        - For example, when dynamic strategies with complex derivative assets are required.

Creation of new instruments and products (2)

Example of issue the synthetic product at a price that allows it to obtain a *profit compared to the price of the replicating strategy* that covers its structure of cash flows.

You are an options dealer who deals in non-publicly traded options. One of your clients wants to purchase a one-year European call option on HAL Computer Systems stock with a strike price of $20. Another dealer is willing to write a one-year European put option on HAL stock with a strike price of $20, and sell you the put option for a price of $3.50 per share. If HAL pays no dividends and is currently trading for $18 per share, and if the risk-free interest rate is 6%, what is the lowest price you can charge for the option and guarantee yourself a profit?

Answer:

Using put-call parity, we can replicate the payoff of the one-year call option with a strike price of $20 by holding the following portfolio:

- Buy the one-year put option with a strike price of $20 from the dealer;
- buy the stock; and
- take out a loan of $20.

Creation of new instruments and products (2)

With this combination, we have the following **final payoff** depending on the final price of HAL stock in one year, $S_T = S_1$

	Final HAL stock price	
	$S_1 < \$20$	$S_1 > \$20$
Buy put option	$20 – S_1$	0
Buy stock	S_1	S_1
Take out loan	– 20	– 20
Portfolio	0	$S_1 – 20$
Sell call option	0	$– (S_1 – 20)$
Total Payoff	0	0

Note that the final payoff of the portfolio matches the payoff of a call option. Therefore, we can sell the call option to our client and have future payoff of zero no matter what happens. Doing so is worthwhile as long as we can sell the call option for more than the cost of the portfolio, which is:

$$S + p - E(1+i)^{-T} = \$18 + \$3.50 - \$20(1.06)^{-1} = \$2.632$$

Final summary

Final summary

1. A fundamental implication of the absence of arbitrage is the compliance with the Law of One Price (LOP).
2. The LOP establishes that two assets or portfolios that are perfect equivalents must have the same price.
3. The LOP allows for relative valuation, which is very useful in many practical situations.
4. The idea of creating synthetic instruments has important practical applications in valuation (through the LOP) and in other equally important financial topics, such as risk management, as shown in the field of financial engineering.

Recommended bibliography

- The content of this topic can be found, in various excellent textbooks.
- For more details, the following manual is recommended, which corresponds quite well to the content, level and approach of these notes, and from which some specific ideas have been taken:

  Finance Economics, 2nd ed. (2009) by Z. Bodie, R. Merton and D. Cleeton, Pearson. Chapters 7, 8, 14 and 15.

  - Or the same chapters in the first edition, translated into Spanish: *Finanzas*, 1ª ed. (2000), de Z. Bodie y R. Merton, Pearson.

Unit 4

Term structure of interest rates and bond valuation

Dept. Financial and Actuarial Economics

VNIVERSITAT DE VALÈNCIA

Facultat d'Economia

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1. Zero coupon bonds and the term structure of interest rates

Zero-coupon bonds

- A **zero-coupon bond** is a fixed-income asset that, in exchange for the price paid for it, promises to pay a single known amount of money on its maturity date. (It is called a zero-coupon bond because it does not make any intermediate periodic interest payments, or *coupons*.)
- Therefore, the interest (reward) earned on the investment is the difference between the amount received on its maturity date and the price paid today for the bond.

Graphically:

- Where:
  - $P_0(n)$ = price of a zero-coupon bond with maturity n, paid at time t = 0;
  - CF_n = payment promised for the bond, on its maturity date; and
  - n = term until the maturity of the bond.

Discount zero-coupon bonds

- Frequently, **zero-coupon bonds are known as *pure discount* bonds**:

  This means that the cash flow obtained at their maturity date is the nominal *(N)* or the bond's face value, $CF_n = N$, *and the price paid is just its discounted value.*

  - Example: A U.S. T-Bill or a Spanish *Letra del Tesoro*. Both are zero-coupon bonds since they pay their holders in a single cash flow. In the Spanish *Letra del Tesoro* that cash flow is €1,000 (its face value) at the maturity date (currently, 3, 6, 9 or 12 months).

Yield to maturity of a zero-coupon bond

- **Yield to maturity of a zero-coupon bond** can be obtained as *the compound interest rate that equals its current price in the market to the present value of the future cash flow promised at maturity.*
- Just as we did in Unit 2 for coupon bonds:

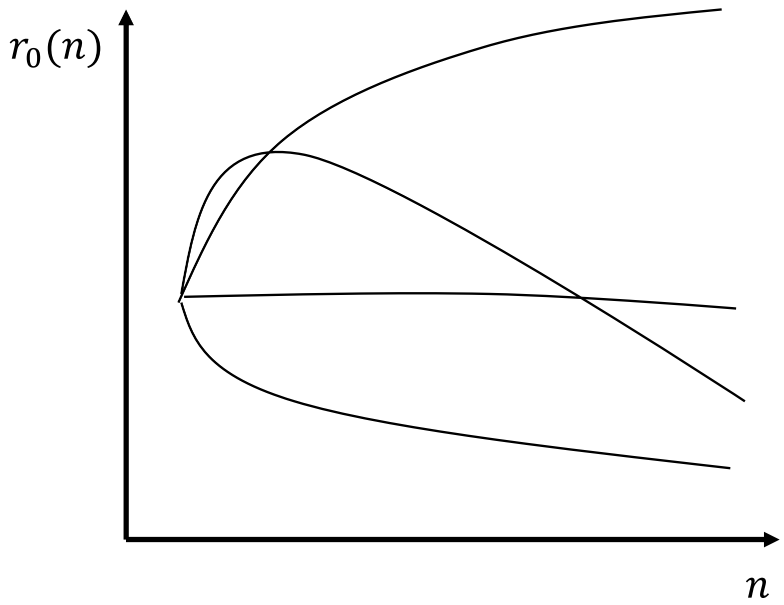
$$P_0(n) = \frac{CF_n}{\left(1 + \boldsymbol{r}(\boldsymbol{n})\right)^n}$$

where:

- $\boldsymbol{r}(\boldsymbol{n}) =$ yield to maturity of a zero coupon bond with maturity n. It is assumed to be an annual effective interest rate (with n also measured in years).

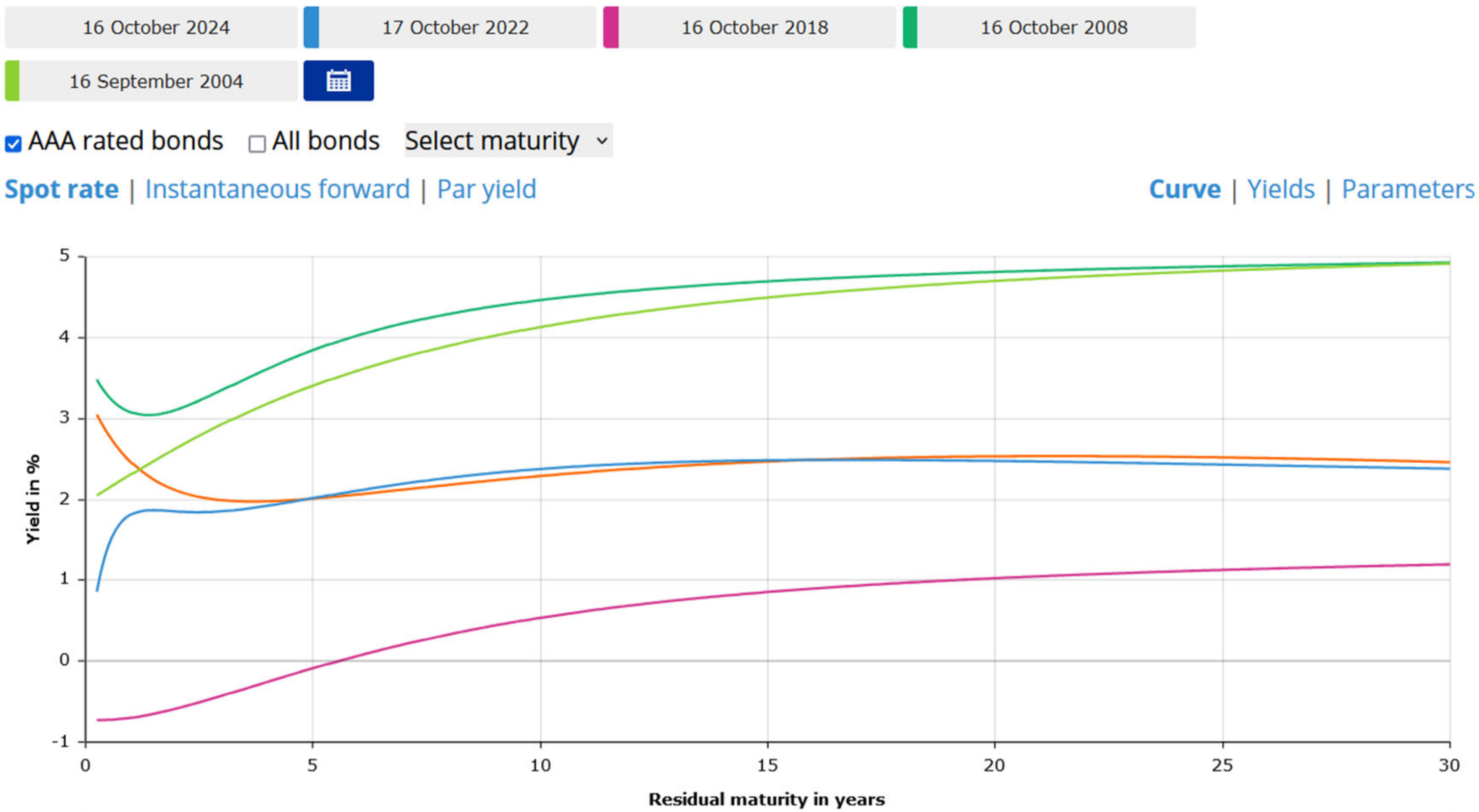
Term structure of interest rates (TSIR)

- The relationship between zero-coupon bond interest rates and different maturities is called **term structure of interest rates, or TSIR**.
- At a given date, the graphical representation of the TSIR, or interest rate curve, can take on different shapes: increasing (with a positive slope), decreasing (a negative slope), approximately flat or even hump-shaped.



Yield curves in the eurozone

- The ECB publishes yield curves:



Source: https://www.ecb.europa.eu/stats/financial_markets_and_interest_rates/euro_area_yield_curves/html/index.en.html

Calculation of TSIR

- From the prices of zero-coupon bonds of different terms, you can obtain the **TSIR**:

$$P_0(n) = \frac{CF_n}{\left(1 + r(n)\right)^n} \rightarrow r(n) = \left(\frac{CF_n}{P_0(n)}\right)^{1/n} - 1$$

- You can also get the **discount factors**, $v(n)$, for each term n:

$$v(n) = \frac{1}{\left(1 + r(n)\right)^n} = \frac{P_0(n)}{FC_n}$$

- **This is equivalent to knowing the TSIR or the discount factors.**

  As we will see, both have the basic components to value any contract or financial asset that promises known cash flows in the future.

Exercise 4.1: calculation of the TSIR

Given the prices of the following zero-coupon bonds and their term until maturity, obtain the TSIR and represent it graphically. Also calculate the discount factors for each term.

Time to maturity (in years)	Current price	Face value
1	980.39	1,000
2	475.91	500
3	9,195.99	10,000
4	4,428.09	5,000
5	850.84	1,000
8	746.74	1,000
10	6,801.96	10,000

Exercise 4.1: solution

- From the data in the table, the interest rates and discount factors for each of the terms can be calculated:

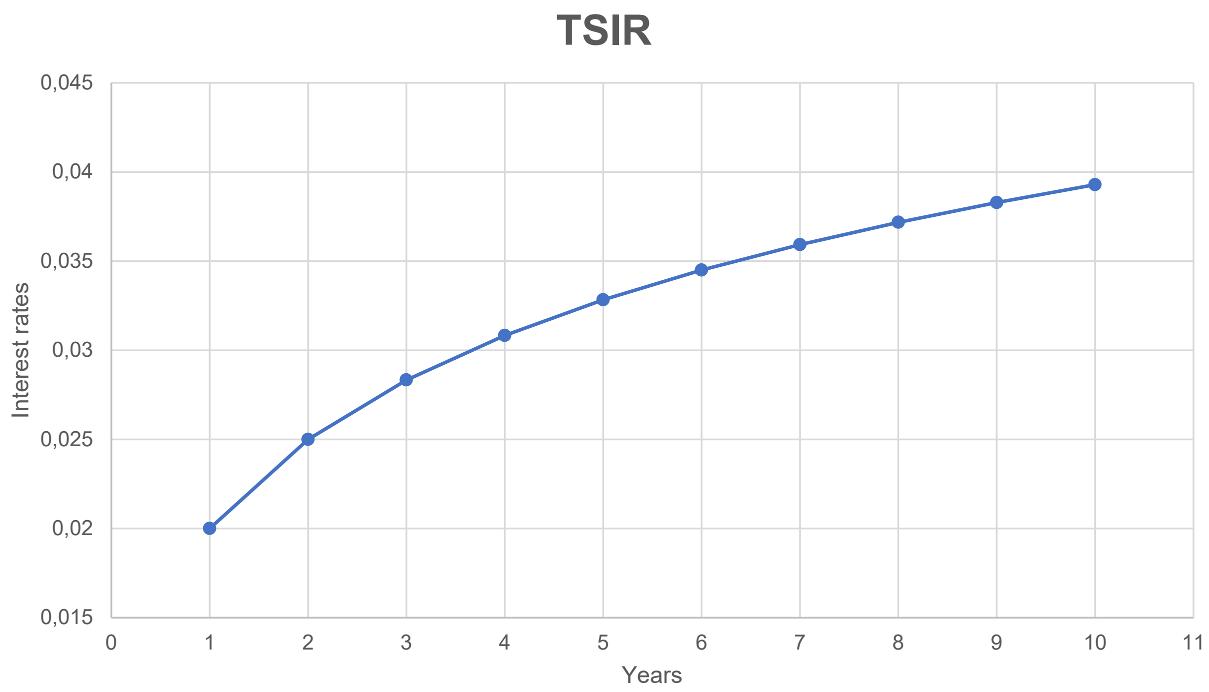
$$r(n) = \left(\frac{CF_n}{P_0(n)}\right)^{1/n} - 1 \quad \text{and} \quad v(n) = \frac{1}{\left(1+r(n)\right)^n} = \frac{P_0(n)}{CF_n}$$

Time to maturity (years)	Current price	Face value	Interest rates	Discount factors
1	980.39	1,000	**0.0200**	0.98039
2	475.91	500	**0.0250**	0.95182
3	9,195.99	10,000	**0.0283**	0.919599
4	4,428.09	5,000	**0.0308**	0.885618
5	850.84	1,000	**0.0328**	0.85084
8	746.74	1,000	**0.0372**	0.74674
10	6,801.96	10,000	**0.0393**	0.680196

Upward sloping curve

Exercise 4.1: solution (cont.)

- Graphic representation of the TSIR:



Importance of the TSIR

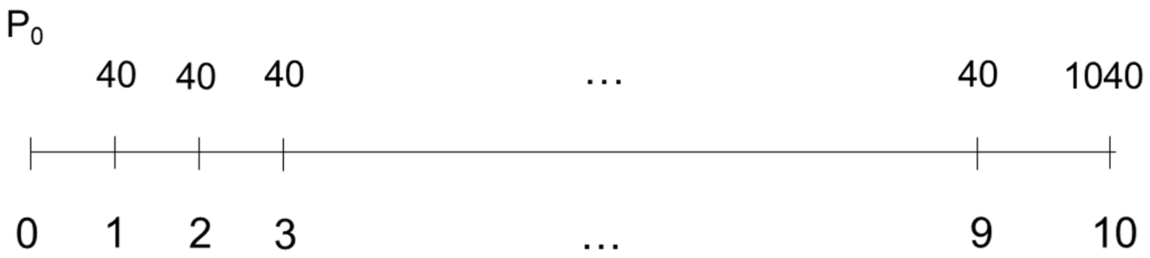
- Yield curves play a fundamental role in the analysis and management of fixed-income assets and portfolios.
- In particular, in the next section we will show that the TSIR constitutes an essential pillar in the valuation of financial instruments that promise to pay known cash flows, through their application to the valuation of bonds with coupons.
- Later we will review the main explanations provided by financial theory for the shape and behaviour of the TSIR.
- In the Financial Markets and Banking Operations subject, other applications and further details regarding the TSIR are covered.

2. LOP and coupon bond valuation

Synthetic creation of a coupon bond

Any coupon bond can be replicated with a zero-coupon bond portfolio.

Example: A 10-year bond with nominal €1,000 and 4% coupon:

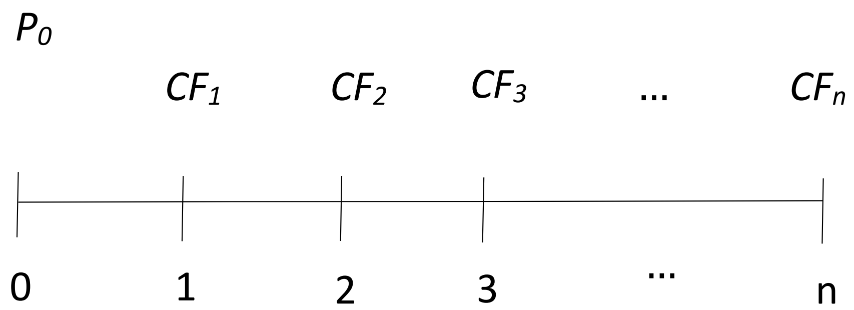


It can be created synthetically with a portfolio of 10 zero-coupon bonds:

The first 9 zero-coupon bonds would have a nominal €40 (the annual bond coupon with periodic coupons) and the last zero-coupon bond would have a nominal of €1,040 (coinciding with the nominal bond with coupons plus its last periodic coupon).

Synthetic creation of a coupon bond: generalization

- Let’s consider any generic **coupon bond**:

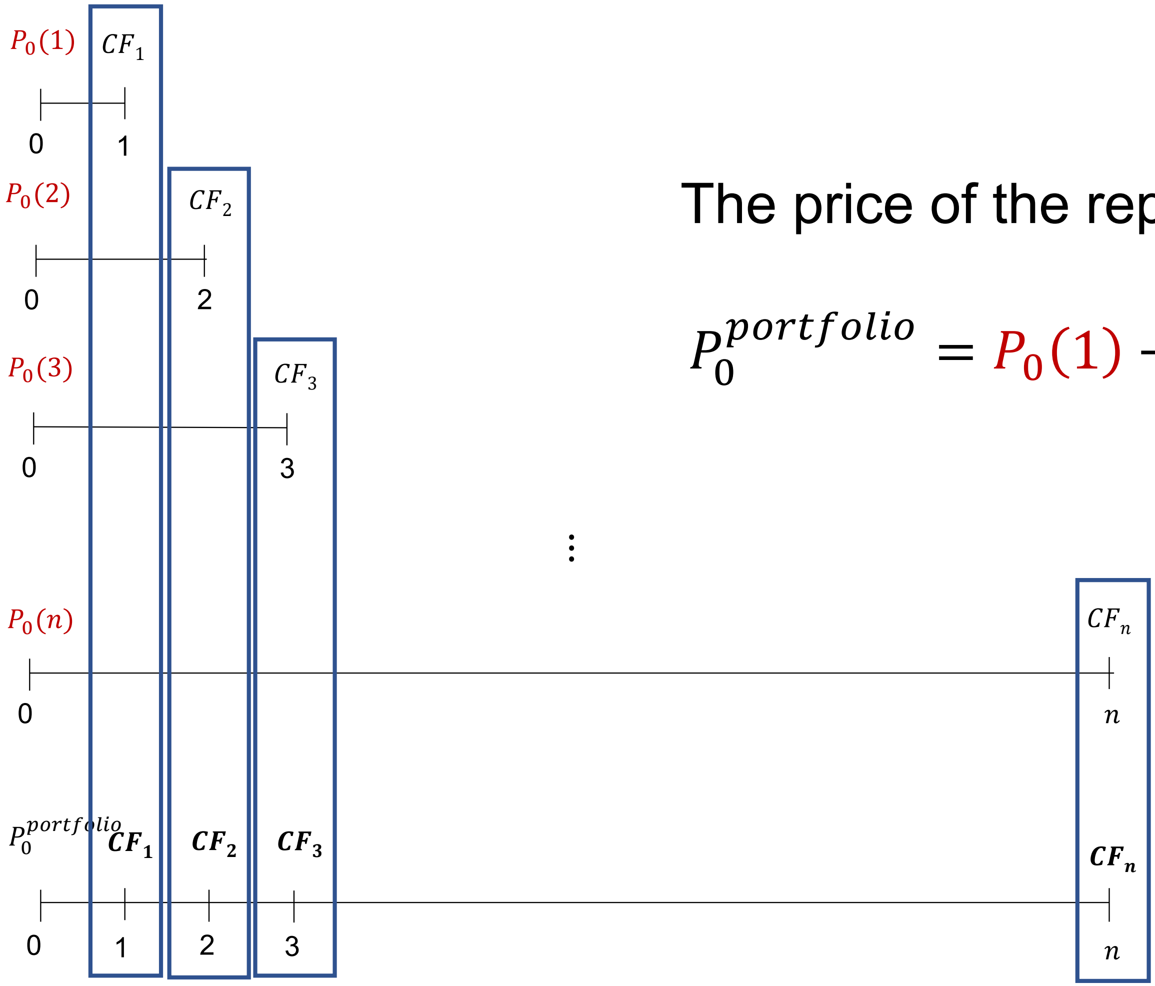


- You can **create it synthetically with n zero-coupon bonds**, each with a promised (nominal) final payment equal to CF_s, for $s = 1,2,3, \dots, n$, and price $P_0(s)$, respectively.

- The **price (or total cost) of the replicating portfolio** will be the sum spent on the purchase of each of the zero-coupon bonds:

$$P_0^{portfolio} = \sum_{s=1}^{n} P_0(s)$$

Synthetic creation: graphic representation



The price of the replicating portfolio is:

$$P_0^{portfolio} = P_0(1) + P_0(2) + P_0(3) + \cdots + P_0(n)$$

LOP and valuation of a coupon bond

- **From LOP**, *the price of a coupon bond must be equal to the cost of the replicating portfolio:*

$$P_0 = P_0^{portfolio}$$

If the bond were traded at a different price, an arbitrage opportunity would arise: buying the cheapest alternative and selling (or issuing) the most expensive; this would today generate a profit equal to the price difference of both equivalent alternatives.

- That is, it must be fulfilled:

$$P_0 = P_0(1) + P_0(2) + \cdots + P_0(n)$$

- **Using the TSIR**, i.e., the yields on zero-coupon bonds:

$$P_0 = \frac{CF_1}{(1+\boldsymbol{r}(\mathbf{1}))^1} + \frac{CF_2}{(1+\boldsymbol{r}(\mathbf{2}))^2} + \cdots + \frac{CF_n}{(1+\boldsymbol{r}(\boldsymbol{n}))^n}$$

Or equivalently, using the discount factors:

$$P_0 = CF_1 \cdot v(1) + CF_2 \cdot v(2) + \cdots + CF_n \cdot v(n)$$

Exercise 4.2: valuation of a coupon bond

Suppose that we find today the following zero-coupon bonds in the financial market, with maturities on 1, 2, 3, 4 and 5 years, respectively, valued at the following prices:

Name of the zero coupon bond	Maturity (in years)	Current price	Face value
B_1	1	980.39	1,000
B_2	2	475.91	500
B_3	3	9,195.99	10,000
B_4	4	4,428.09	5,000
B_5	5	850.84	1,000

What should the value of a bond of 1,000 euros be today, with coupon payments of 50 euros per year (annual coupon rate = 5%) and maturity after five years?

Exercise 4.2: solution

- In Exercise 4.1 we got the TSIR for these five zero-coupon bonds (and the associated discount factors).

Maturity (years)	Interest rates	Discount factors
1	**0.0200**	0.98039
2	**0.0250**	0.95182
3	**0.0283**	0.91960
4	**0.0308**	0.88562
5	**0.0328**	0.85084

- Therefore, using the TSIR, the value of the bond is:

$$P_0 = \frac{50}{(1+\mathbf{0.02})^1} + \frac{50}{(1+\mathbf{0.025})^2} + \frac{50}{(1+\mathbf{0.0283})^3} + \frac{50}{(1+\mathbf{0.0308})^4} + \frac{1.050}{(1+\mathbf{0.0328})^5} = 1{,}080.25$$

- Equivalently, using the discount factors:

$$P_0 = 50 \cdot 0.98039 + 50 \cdot 0.95182 + 50 \cdot 0.91960 + 50 \cdot 0.88562 + 1{,}050 \cdot 0.85084 = 1{,}080.25$$

Exercise 4.3 (advanced): arbitrage with bonds

Assume it were possible to buy the coupon bond from Exercise 4.2 in the market, paying a price of €1,070.

What arbitrage strategy would be available?

- Solution:
- The only price that does not generate an arbitrage opportunity is: $P_0^{portfolio} = 1{,}080.25$, previously obtained.
  - Since $1{,}070 < P_0^{portfolio}$, we should buy the coupon bond and sell the appropriate units of the five zero-coupon bonds that make up the replicating portfolio. This strategy entails a profit of $1{,}080.25 - 1{,}070 = €10{,}25$ today.

Exercise 4.3: solution (cont.)

Below, we detail the replicating portfolio (assuming all bonds are divisible). In this example, the replicating portfolio consists of:

- $\frac{50}{1,000} = 0.05\ units\ of\ zero\ coupon\ bond\ B_1$
- $\frac{50}{500} = 0.10\ units\ of\ zero\ coupon\ bond\ B_2$
- $\frac{50}{10,000} = 0.005\ units\ of\ zero\ coupon\ bond\ B_3$
- $\frac{50}{5,000} = 0.01\ units\ of\ zero\ coupon\ bond\ B_4$
- $\frac{1,050}{1,000} = 1.05\ units\ of\ zero\ coupon\ bond\ B_5$

The cost (or price) of the replicating portfolio can be calculated simply by adding up the cost of each item in the portfolio:

$$P_0^{portfolio} = 0.05 \cdot 980.39 + 0.1 \cdot 475.91 + 0.005 \cdot 9,195.99 + 0.01 \cdot 4,428.09 + 1.05 \cdot 850.84 = 1,080.25$$

$$P_0^{portfolio} = 49.02 + 47.59 + 45.98 + 44.28 + 893.38 = 1,080.25$$

The following table (on the next slide) summarizes the arbitrage strategy.

Exercise 4.3: solution (cont.)

Arbitrage strategy if the market price of the coupon bond is 1,070:

	Cash flows					
	t=0	t=1	t=2	t=3	t=4	t=5
Buy 1 coupon bond	-1,070	+50	+50	+50	+50	+1,050
Sell the replicating portfolio:						
Sell 0.05 bonds B_1	+49.02	-50				
Sell 0.1 bonds B_2	+47.59		-50			
Sell 0.005 bonds B_3	+45.98			-50		
Sell 0.01 bonds B_4	+44.28				-50	
Sell 1.05 bonds B_5	+893.38					-1,050
Net profit	1,080.25 -1,070 = **+10.25**	0	0	0	0	0

Exercise 4.4 (advanced):

Suppose it was possible to buy the coupon bond from Example 4.2 at a price of 1,085 euros.

What arbitrage strategy would be available?

Exercise 4.5

Assume you find the following zero coupon bonds in the financial market, with maturities after 1, 2 and 3 years, respectively, valued at the following prices:

Time to maturity (in years)	Price of zero-coupon bond	Face value
1	970.87	1,000
2	933.51	1,000
3	889.00	1,000

a) What is the TSIR obtained from the information provided by the zero-coupon bonds?

b) What should be the value today of a coupon bond with a face value of 500 euros, with annual coupon payments of 22 euros (coupon rate = 4.4% annually) and maturity after 3 years?

c) Assume that the previous coupon bond is correctly valued in the market. What is its yield to maturity (YTM)? Relate this YTM to the TSIR obtained in the first question.

Exercise 4.6 (advanced):

Suppose it was possible to trade the coupon bond from Exercise 4.5 in the market at a price of 495.25 euros.

What arbitrage strategy would be available?

Exercise 4.7

Suppose a flat TSIR, so the yield on zero-coupon bonds is equal to 3%, for all maturities. Imagine the same market as in Exercise 4.5, with three zero-coupon bonds of nominal 1,000 euros and maturity at the end of years one, two and three, respectively.

With this information, please answer the following questions:

a) What would the prices of those three zero-coupon bonds be?

b) If there is a coupon bond with a nominal value of 500 euros, payment of annual coupons at 3% and maturity at 3 years, what should be its non-arbitrage price?

c) Assuming that the coupon bond was being traded on the market at that price, what would its YTM be? Could we find the solution to this question without making any calculations?

Exercise 4.8 (a): YTM and spot rate relationship from TSIR

Suppose you can find in the financial market the following zero-coupon bonds, with maturities after 1, 2 and 3 years, respectively, valued at the following prices:

Time to maturity (in years)	Price zero-coupon bond	Face value
1	980.39	1,000
2	942.60	1,000
3	889.00	1,000

a) Obtain the TSIR.

b) Consider the following three coupon bonds A, B and C, each with a maturity of three years, nominal €1,000 and annual coupons of 2%, 4% and 6%, respectively. What should their value be today?

c) Suppose coupon bonds A, B and C are correctly valued in the market. What would their YTM be? Relate these YTM values to the TSIR obtained in the first question.

Exercise 4.8 (b): YTM and spot rate relationship from TSIR

Suppose you can find in the financial market the following zero coupon bonds, with maturities after 1, 2 and 3 years, respectively, valued at the following prices:

Time to maturity (in years)	Price zero-coupon bond	Face value
1	961.54	1,000
2	924.56	1,000
3	889.00	1,000

a) Obtain the TSIR.

b) Consider the following three coupon bonds A, B and C, each with a maturity of three years, nominal €1,000 and annual coupons of 2%, 4% and 6%, respectively. What should their value be today?

c) Suppose coupon bonds A, B and C are correctly valued in the market. What would their YTM be? Relate these YTM values to the TSIR obtained in the first question.

Exercise 4.8 (c): YTM and spot rate relationship from TSIR

Suppose you can find in the financial market the following zero coupon bonds, with maturities after 1, 2 and 3 years, respectively, valued at the following prices:

Time to maturity (in years)	Price zero-coupon bond	Face value
1	961.54	1,000
2	942.60	1,000
3	942.32	1,000

a) Obtain the TSIR.

b) Consider the following three coupon bonds A, B and C, each with a maturity of three years, nominal €1,000 and annual coupons of 2%, 4% and 6%, respectively. What should their value be today?

c) Suppose coupon bonds A, B and C are correctly valued in the market. What would their YTM be? Relate these YTM values to the TSIR obtained in the first question.

3. Theories of the term structure of interest rates

Theories of the TSIR

The TSIR on a given date provides a snapshot of the promised return (interest rate) of zero-coupon bonds for different terms (or periods until maturity). It is natural to ask what determines the shape of what we call the *zero curve* (or TSIR). Why is it sometimes sloping down, sometimes sloping up (as in Exercise 4.1), sometimes flat, and sometimes partly sloping up and then down (i.e. with a hump)?

A number of different theories have been proposed to try to explain the general shape of the TSIR and its evolution. The most important ones are:

- expectation hypothesis (or expectation theory);
- liquidity preference hypothesis (or liquidity preference theory); and
- segmentation hypothesis (or market segmentation theory).

A. Expectations hypothesis (or expectation theory)

This is the simplest one. It conjectures that spot rates are determined by expectations of what rates will be in the future. Thus, current long-term interest rates are a combination of current short-term rates and expected short-term interest rates in the future.

To visualize this process, suppose that, as in Exercise 4.1, the spot rate curve slopes upward, with rates increasing for longer maturities. The two-yearly rate (0.025) is greater than the one-yearly rate (0.02). According to this theory, it is argued that this is so because the market believes the one-yearly rate will most likely go up next year.

Thus, the expectations are inherent in the current spot rate structure. The current spot rate curve implies an expectation about what the spot rate curve will be next year. For example, if currently the TSIR is flat, according to this theory it will be flat next year as well.

B. Liquidity preference theory

This theory highlights the role of investors' **preferences**.

The liquidity preference explanation asserts that investors usually prefer short-term bonds over long-term bonds. The reason for this preference is because the short-term bonds are less sensitive to changes in market interest rates. To compensate for higher interest risk, long-term bonds offer higher yields – according to this theory. Therefore there is a liquidity premium embedded in the yield offered for a particular bond and the standard theory assumes that the premium increases with the term (with the maturity date of the bond).

An increasing liquidity premium over the term, whatever the cause, implies an increasing yield curve on average. However, on any given date, the observed ETTI can take on many forms, because it is also determined by underlying expectations.

C. Segmentation hypothesis (or market segmentation theory)

The market segmentation explanation of the term structure argues that the market for fixed-income securities is segmented by maturity dates.

It assumes that investors know the maturity date that they prefer, based on their projected need for future funds or their risk preference. Thus, a group of investors competing for long-term bonds is different from a group of investors competing for short-term bonds. Hence there need be no relation between the prices (defined by the interest rates) of these two types of instruments; therefore, short and long rates can move around independently.

Taken to an extreme, it suggests that all points of the TSIR are mutually independent and each one is determined by the forces of supply and demand in its own market.

Recommended bibliography

- The content of this topic can be found in various several excellent textbooks.
- For more details, the following manuals are recommended:

Finance Economics, 2nd ed. (2009) by Z. Bodie, R. Merton and D. Cleeton, Pearson. Chapter 8.

- Or the same chapter in the first edition, translated into Spanish: *Finanzas*, 1ª ed. (2000), de Z. Bodie y R. Merton, Pearson.

Manual práctico de economia financiera (2023), de M. Gutiérrez y J. Moreno, Pirámide. Chapters 4 and 5.

Recommended bibliography

Bodie, Z.; Merton, R.C. and Cleeton (2009): *Finance*. Chapter 8.

Gutiérrez, M. y J. Moreno (2023): *Manual práctico de economía financiera*. Chapter 5.