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Weak solutions of anisotropic (and crystalline) inverse mean curvature flow as limits of p-capacitary potentials

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ABSTRACT

We construct weak solutions of the anisotropic inverse mean curvature flow (A-IMCF) under very mild assumptions both on the anisotropy (which is simply a norm in $\mathbb{R}^N$ with no ellipticity nor smoothness requirements, in order to include the crystalline case) and on the initial data. By means of an approximation procedure introduced by Moser, our solutions are limits of anisotropic p-harmonic functions or p-capacitary functions (after a change of variable), and we get uniqueness both for the approximating solutions (i.e., uniqueness of p-capacitary functions) and the limiting ones. Our notion of weak solution still recovers variational and geometric definitions similar to those introduced by Huisken-Ilmanen, but requires to work within the broader setting of BV-functions. Despite of this, we still reach classical results like the continuity and exponential growth of perimeter, as well as outward minimizing properties of the sublevel sets. Moreover, by assuming the extra regularity given by an interior rolling ball condition (where a sliding Wulff shape plays the role of a ball), the solutions are shown to be continuous and satisfy Harnack inequalities. Finally, examples of explicit solutions are built.

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1. Introduction and statement of main results

1.1. Moser's approach to inverse mean curvature flow (IMCF)

For $p > 1$, let u be a positive p-harmonic function or p-capacitary potential, that is, a solution to

$$\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) = 0. \tag{1.1}$$

After the change of variable

$$v = (1-p)\log u, \tag{1.2}$$

the new function v satisfies

$$\operatorname{div}\big(|\nabla v|^{p-2}\nabla v\big) = |\nabla v|^p. \tag{1.3}$$

R. Moser developed in [35,36] the striking idea of taking limits as $p \searrow 1$ of solutions v_p to (1.3) leading to weak solutions of the degenerate elliptic equation:

$$\operatorname{div}\left(\frac{\nabla v}{|\nabla v|}\right) = |\nabla v|, \tag{1.4}$$

which has its own independent interest since, if v is smooth and $\nabla v \neq 0$, then the level sets of v evolve according to the IMCF, that is, each point moves in the unit outward normal direction with velocity given by the inverse $1/H$ of the mean curvature.

In particular, Moser constructs weak solutions of (1.4) by using explicit barriers to first solve (1.1) with suitable Dirichlet boundary conditions. Unlike (1.4), the p-Laplace equation (1.1) has the advantage of being the Euler-Lagrange equation of the variational problem of minimizing the p-capacity.

Roughly speaking, the overall goal of this paper is to reproduce Moser's successful technique within an anisotropic framework by imposing the mildest possible requirements both for the anisotropy (in order to include the crystalline case) and the domains where the solutions are defined. Moving beyond traditional Euclidean isotropic settings is essential to allow for situations where attributes vary with direction, as this additional flexibility will be decisive for applications to crystal development or noise reduction.

Having this in mind, as a starting point, in [12] we studied existence and regularity properties of solutions to the anisotropic version of (1.1).

1.2. Anisotropic implementation of Moser's technique

Let $N \geq 2$ and $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz-continuous boundary. Hereafter, we will use the notation Ω^e for the exterior domain $\mathbb{R}^N \setminus \overline{\Omega}$. We can formally define the boundary value problem:

$$\begin{cases} \operatorname{div}\big(F^{p-1}(\nabla u)\partial F(\nabla u)\big) = 0 & \text{in } \Omega^e, \\ u = 1 & \text{on } \partial\Omega, \\ u = 0 & \text{as } |x| \to \infty, \end{cases} \tag{1.5}$$

where F is a norm on $\mathbb{R}^N$. Here the equality on the boundary is understood in the sense of traces, and ∂F stands for an element of the subdifferential (for the sake of clarity, we keep this formal notation to present our main results, see subsections 4.1 and 3.1 for precise definitions).

As before, the transformation (1.2), leads to

$$\begin{cases} \operatorname{div}\big(F^{p-1}(\nabla v)\partial F(\nabla v)\big) = F^p(\nabla v) & \text{in } \Omega^e, \\ v = 0 & \text{on } \partial\Omega, \\ v \to \infty & \text{as } |x| \to \infty. \end{cases} \tag{1.6}$$

As Moser's scheme suggests, the natural candidate for a definition of anisotropic or crystalline IMCF (A-IMCF) should be the limit of (1.6) as $p \searrow 1$:

$$\begin{cases} \operatorname{div}\big(\partial F(Dv)\big) = F(Dv) & \text{in } \Omega^e, \\ v = 0 & \text{on } \partial\Omega, \\ v \to \infty & \text{as } |x| \to \infty, \end{cases} \tag{1.7}$$

where Dv represents the distributional derivative of a BV-function, which makes sense as a Radon measure (for a comprehensive introduction to this framework, see subsection 2.4).

Indeed, our main result confirms the above heuristics.

Theorem 1.1 *(Existence & uniqueness of weak solutions of A-IMCF). Let $1 < p < N$ and consider $\Omega \subset \mathbb{R}^N$ a bounded domain with Lipschitz boundary. If u_p solves (1.5) and we set $v_p = (1-p)\log u_p$, there exists a sequence $p_k \searrow 1$ and a function $v \in BV_{\text{loc}}(\Omega^e) \cap L^\infty_{\text{loc}}(\Omega^e)$ such that $v_{p_k} \to v$ in $L^q_{\text{loc}}(\Omega^e)$ for every $1 \leq q < \frac{N}{N-1}$ and the limit has the following properties:*

(a) *v is the unique weak solution of (1.7).*
(b) *The discontinuity or singular set of v has measure zero with respect to the $(N-1)$-dimensional Hausdorff measure.*

(c) *We can find constants* $0 < r_1 < r_2$ *with* $\mathcal{W}_{r_1} \subset \Omega \subset \mathcal{W}_{r_2}$ *so that*

$$v_{r_1} \leq v \leq v_{r_2}, \quad \textit{where} \;\; v_{r_i} = (N-1)\log\left(F^\circ\left(\frac{\cdot}{r_i}\right)\right), \quad i = 1, 2.$$

Here F° *denotes the dual of the norm* F *and* $\mathcal{W}_r$ *its corresponding ball (Wulff shape) of radius* r*.*

Observe that, by translation invariance, we can suppose that $0 \in \Omega$ and, since Ω is bounded, there exist $0 < r_1 < r_2$ such that $\mathcal{W}_{r_1} \subset \Omega \subset \mathcal{W}_{r_2}$.

To the best of our knowledge, the most general existence and uniqueness result in this spirit can be found in [19]. The authors obtain a weak solution $v \in \mathrm{Lip}_{\mathrm{loc}}(\Omega^{\mathrm{e}})$ in the case that Ω has smooth boundary and for norms $F \in C^\infty(\mathbb{R}^N \setminus \{0\})$ with a strictly positive definite Hessian matrix.

Here, we broaden the existing literature's findings in a number of ways. In fact, we just require the anisotropy to be a norm, without making any additional commitments about smoothness or uniform ellipticity. Specifically, we permit norms whose dual unit balls have corners and/or flat pieces, including crystalline situations as the ℓ_∞ or ℓ_1 norms in $\mathbb{R}^N$. Moreover, about the regularity of Ω, we just ask for Lipschitz continuity of the boundary, instead of the traditional C^∞ much stronger constraint.

In short, we deal with an unfriendly setting because we will not be able to produce any argument that involves gradient or second derivatives of the functions v nor principal curvatures (even in weak sense) of $\partial\Omega$. To overcome these additional technical difficulties, the proof of the existence in Theorem 1.1 strongly relies on the ideas developed in [4] to study a Dirichlet problem coming from the *total variation flow*, and the subsequent adaptation of them to study IMCF with a damping term on bounded [32] and unbounded [33] Euclidean domains.

More precisely, we start by introducing a notion of weak solution to (1.7) (see Definition 3.1), where the equation makes sense as an equality for measures and which relies on the existence of a bounded vector field $\mathbf{z}$, whose divergence is a Radon measure that plays the role of the left hand side of (1.4), even if ∇v may vanish. To overcome this, we exploit intensively Anzellotti's theory of pairings [6] and the total variation adapted to anisotropies by [1]. Let us highlight that measure divergence fields have already shown their key role in physics as models of capillarity for perfectly wetting fluids [28] and in continuum mechanics [40].

While we managed to follow the existence proof of [32,33] with suitable adaptations and correction of several issues, the comparison principles there leading to uniqueness fail in the case that F is not strictly convex. Luckily, we manage to combine a trick inspired by the uniqueness proof by Huisken and Ilmanen [23, Theorem 2.2] with the use of truncation functions to achieve comparison results (and hence uniqueness) for the A-IMCF (1.7). We stress that our uniqueness claim and proof differs quite from that in [23] because of our different notion of weak solution.

Furthermore, these new insights also work for the approximating problems (1.5) and (1.6), even for generic boundary conditions, meaning that as a by-product of the techniques developed here we attain an upgrade of our existence and regularity results in [12]. With more detail,

Corollary 1.2. *Let $1 < p < \infty$, $\Omega \subset \mathbb{R}^N$ a bounded domain with Lipschitz boundary and $\varphi \in W^{1-\frac{1}{p},p}(\partial\Omega)$. Consider either $D = \Omega$ or $D = \Omega^e$, then the weak solution of*

$$\begin{cases} \operatorname{div}\big(F^{p-1}(\nabla u)\partial F(\nabla u)\big) = 0 & \text{in } D \\ u = \varphi & \text{on } \partial\Omega, \\ u \to 0 \ \text{as } |x| \to \infty & \text{if } D = \Omega^e, \quad p < N \end{cases} \tag{1.8}$$

constructed in [12] is unique. In particular, if $\varphi \equiv 1$, then uniqueness of minimizers in $\{u \in L^{pN/(N-p)}(\mathbb{R}^N) : \nabla u \in L^p(\mathbb{R}^N), u \geq 1 \text{ in } \Omega\}$ of the anisotropic p-capacity functional

$$\mathrm{Cap}_p^F(\overline{\Omega}) := \inf\left\{\int_{\mathbb{R}^N} F^p(\nabla u)\,\mathrm{d}x : u \in C_c^\infty(\mathbb{R}^N), u \geq 1 \text{ in } \overline{\Omega}\right\} \tag{1.9}$$

is guaranteed.

Recall that, in physics, those minimizers are known as p-capacitary potentials and are related to thermal or electric conductivity problems in heterogeneous media, where properties of the environment are allowed to depend upon the direction.

1.3. Further regularity and Harnack inequalities

In [12] we proved that solutions to (1.5) are Lipschitz continuous provided that the domain is regular enough in the following sense:

Definition 1.3 *(Uniform interior ball condition).* Let $r > 0$. We say that Ω satisfies the $\mathcal{W}_r$-condition if, for any $x \in \partial\Omega$, there exists $y \in \mathbb{R}^N$ such that

$$\mathcal{W}_r + y \subseteq \overline{\Omega} \quad \text{and} \quad x \in \partial\left(\mathcal{W}_r + y\right).$$

Accordingly, it is natural to expect also extra regularity for the solutions of (1.7) under the same circumstances. Indeed,

Theorem 1.4. *Let $r > 0$ and suppose that Ω satisfies the $\mathcal{W}_r$-condition. Then there exists $v \in BV_{\mathrm{loc}}(\Omega^e) \cap C^0(\Omega^e)$ unique solution of (1.7). Moreover, one can find a constant $\mathcal{C} > 0$ depending only on the dimension N, such that, if $x_0 \in \Omega^e$ and $\rho > 0$ satisfy $\mathcal{W}_{2\rho}(x_0) \subset \Omega^e$, then*

$$\mathrm{osc}_{\mathcal{W}_\rho(x_0)} v \le \mathcal{C},$$

where $\mathrm{osc}_B v := \sup_B v - \inf_B v$ *is the oscillation of* v *on a subset* B.

The proof follows the strategy in [36] of constructing explicit barrier functions leading to continuity via comparison arguments and getting a Harnack inequality of the type

$$\sup_{\mathcal{W}_\rho(x_0)} u_p \le e^{\frac{\mathcal{C}}{p-1}} \inf_{\mathcal{W}_\rho(x_0)} u_p$$

for solutions to (1.5), which after the typical transformation (1.2) and passing to the limit as $p \searrow 1$, yields the claimed estimate for v.

1.4. "Classical" geometric properties of solutions

On the other hand, despite of our milder regularity assumptions and our a priori different notion of solutions, the output of the developments here does not differ as much as could be expected from those by Huisken-Ilmanen, since our solution v has also a similar variational nature (see Proposition 8.2) and its corresponding sublevel sets

$$E_t = \{x : v(x) < t\} \quad \text{and} \quad G_t = \{x : v(x) \le t\} \tag{1.10}$$

also minimize a suitable functional (cf. Proposition 8.3). Additionally, we recover the continuity of the anisotropic version of the perimeter, exponential growth and minimizing hull properties stated in [23].

Theorem 1.5 *(Geometric properties of solutions of A-IMCF). The sublevel sets in* (1.10) *satisfy the following properties:*

(a) G_τ *and* E_t *are sets of finite perimeter for every* $t > \tau \ge 0$.
(b) *The anisotropic perimeter* P_F *of* G_t *and* E_t *coincide for all* $t > 0$.
(c) $e^{-t} P_F(E_t)$ *is constant for* $t > 0$.
(d) E_t *is outward* F*-minimizing for any* $t > 0$*, that is, it minimizes the anisotropic perimeter when compared with all bounded subsets that contain it, while enclosing* Ω.
(e) G_t *is strictly outward* F*-minimizing for any* $t \ge 0$*. Moreover, among the anisotropic perimeter minimising envelopes of* Ω*,* G_0 *is the one that maximises the Lebesgue measure.*

We actually recover these *classical* properties (see [23, Property 1.4 and Lemma 1.6]) under much weaker regularity assumptions. This makes the proof quite more convoluted since our functions are not continuous and, therefore, claims that were immediate in [23], require a more careful study here, including the use of notions like the measure theoretic interior or reduced boundary, not present in the original developments.

For the proof of (a) we follow [33, Section 4] and fill some gaps of the arguments there, while adapting them to the anisotropic environment. For the rest of items, we argue in the spirit of [23] but modifying appropriately some notions to better suit our purposes (see section 8 for precise definitions and further geometric properties of solutions).

1.5. Relevance of IMCF

Given a smooth manifold M of dimension $N-1$, a classical solution of the IMCF is a family of hypersurfaces $X : M \times [0,T) \to \mathbb{R}^N$ satisfying

$$\partial_t X(p,t) = \frac{\nu}{H}(p,t), \qquad p \in M, \quad 0 < t < T, \tag{1.11}$$

where ν is the unit outward normal to the evolving hypersurfaces $M_t = X(M,t)$ and $H = \operatorname{div} \nu$ is the corresponding mean curvature, which is assumed to be positive everywhere so that the above equation is well-defined.

If the initial condition is smooth, star-shaped and mean convex (with $H > 0$), then Gerhardt [21] and Urbas [41] obtained independently that the solution exists for all times and the rescaled hypersurfaces $e^{-\frac{t}{N-1}} M_t$ converge to a round sphere as $t \to \infty$. The anisotropic version of this result for smooth anisotropy and initial condition was proved in [43].

However, without geometric conditions on the initial data, singularities may develop; in fact, for the case $N = 3$ it is proved in [38] that singularities happen only if $\inf_{M_t} H \to 0$ during the flow. Because of this, Huisken and Ilmanen [23] introduced the notion of weak solution to (1.11) using the aforementioned level-set approach. Later on, they showed higher regularity properties in [24]: every weak solution to the flow is smooth after the first instant t_0 where a level set ∂E_{t_0} becomes star-shaped.

The reason why this flow attracts so much attention is its well established effectiveness to generate interesting inequalities which are geometric and physically meaningful. In a nutshell, it was applied by Huisken-Ilmanen [23] to prove the Riemannian Penrose inequality in asymptotically flat 3-manifolds, from which one gets a lower bound for the total energy of a universe at a space-like infinity as a function of the largest black hole in the system. Later on, Brendle–Hung–Wang [10] derived a Minkowski type inequality for Anti-de Sitter Schwarzschild manifolds by means of IMCF, which leads to a Gibbons–Penrose's inequality in Schwarzschild spacetime [11]. Further development of such a powerful tool in anisotropic situations should lead to new useful inequalities (see e.g. anisotropic Minkowski inequalities in [43]).

1.6. Structure of the paper

The paper is organized as follows. We first introduce in section 2 the basic background material about anisotropies, functions of bounded variation, divergence-measure vector fields, pairing of measures and anisotropic total variation; in particular, the latter requires to include the technical details of the proofs that lead us to get Cauchy-Schwarz type inequalities (Lemma 2.11) under milder assumptions than those in the previous literature. In section 3 we give our notion of solution of A-IMCF (see Definition 3.1) and deduce the first consequences, that will be crucial for the rest of the paper, like the nullity of the singular set with respect of the Hausdorff measure (Lemma 3.3). Next, we prove a comparison result (Theorem 3.6) for sub- and supersolutions of A-IMCF, leading to uniqueness of solutions. We start with Moser's strategy in section 4, where we define the corresponding approximating problems (before and after the change of variable) and obtain suitable comparison theorems for them; this leads us to attain Corollary 1.2. In turn, the proofs of Theorem 1.1 and Theorem 1.4 are carried out in sections 5 and 6, respectively. The next natural step is to worry about the geometric properties of the sublevel sets of the solutions of A-IMCF, which is the content of sections 7 and 8; these altogether provide a proof of Theorem 1.5. Finally, we work out explicit examples of solutions: the expected evolution of Wulff shapes, a rectangle whose vertices do not grow linearly, as well as a concrete case where *fattening* occurs.

2. Background material: anisotropies and BV functions

2.1. Anisotropies

A function $F : \mathbb{R}^N \to [0, +\infty[$ is said to be an *anisotropy* if it is convex, positively 1-homogeneous (i.e., $F(\lambda x) = \lambda F(x)$ for all $\lambda > 0$ and all $x \in \mathbb{R}^N$) and coercive. We will always consider additionally that F is even, that is, a norm. In particular, as all norms in $\mathbb{R}^N$ are equivalent, there exist constants $0 < c \leq C < \infty$ such that

$$c|\xi| \leq F(\xi) \leq C|\xi| \tag{2.1}$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^N$.

We define the dual or polar function $F^\circ : \mathbb{R}^N \to [0, +\infty[$ of F by

$$F^\circ(\xi) := \sup\{\xi \cdot \xi^\star \,:\, \xi^\star \in \mathbb{R}^N,\, F(\xi^\star) \leq 1\} = \sup\left\{\frac{\xi \cdot \xi^\star}{F(\xi^\star)} \,:\, \xi^\star \in \mathbb{R}^N\right\}$$

for every $\xi \in \mathbb{R}^N$. It can be verified that F° is convex, lower semi-continuous and 1-positively homogeneous. Moreover, (2.1) leads to

$$\frac{1}{C}|\xi| \leq F^\circ(\xi) \leq \frac{1}{c}|\xi|. \tag{2.2}$$

From the definition of F° one gets a Cauchy-Schwarz type inequality of the form

$$x \cdot \xi \leq F^\circ(x)F(\xi) \qquad \text{for all } \; x, \xi \in \mathbb{R}^N. \tag{2.3}$$

The Wulff shape $\mathcal{W}_r$ of F (centred at 0) is defined by

$$\mathcal{W}_r := \{\xi^\star \in \mathbb{R}^N \,:\, F^\circ(\xi^\star) < r\}.$$

As we are dealing with even anisotropies, $\mathcal{W}_r$ is a centrally symmetric convex body. We say that F is *crystalline* if, furthermore, $\mathcal{W}_r$ is a convex polytope. More generally, the Wulff shape centred at x_0 of radius r is given by

$$x_0 + \mathcal{W}_r = \mathcal{W}_r(x_0) = \{x \in \mathbb{R}^N \,:\, F^\circ(x - x_0) = r\}.$$

Since both F and F° are norms, it follows that they are Lipschitz functions. Therefore, there exists ∇F and ∇F° a.e. in $\mathbb{R}^N$. Hence, arguing on points where $\nabla F^\circ(x)$ is well-defined, the following generalization of [42, Proposition 5.3] holds (with the same proof):

Lemma 2.1. *Let F be any norm in $\mathbb{R}^N$, then for a.e. $x \in \mathbb{R}^N \setminus \{0\}$,*

(i) $F(\nabla F^\circ(x)) = 1$.
(ii) $\dfrac{x}{F^\circ(x)} \in \partial F(\nabla F^\circ(x))$.

2.2. Approximate continuity and differentiability of L^1 functions

Throughout the paper, set $N \geq 2$ a fixed integer, $\mathcal{L}^N$ and $\mathcal{H}^{N-1}$ will denote the Lebesgue measure and $(N-1)$-dimensional Hausdorff measure in $\mathbb{R}^N$, respectively. $|E|$ means the $\mathcal{L}^N$-Lebesgue measure of a subset $E \subset \mathbb{R}^N$ and $⨍_E = \frac{1}{|E|}\int_E$ the average integral. Moreover, the integrals on the boundary of a set are computed with respect to $d\mathcal{H}^{N-1}$.

Hereafter, U stands for any open subset of $\mathbb{R}^N$ with Lipschitz continuous boundary and, as usual, $B_r(x)$ represents the Euclidean ball in $\mathbb{R}^N$ centred at x with radius r. Additionally, we consider the two half balls determined by a given direction ν:

$$B_r(x,\nu)^\pm := \big\{y \in B_r(x) \,:\, \pm(y - x)\cdot \nu > 0\big\},$$

where $\cdot$ denotes the Euclidean inner product, and we will use $|u|$ for the Euclidean norm of a function u.

Definition 2.2 *(Approximate discontinuity and jump sets).* Let $u \in L^1_{\mathrm{loc}}(U)$.

• The approximate discontinuity set $S_u \subset U$ is defined in terms of its complement, i.e., $x \notin S_u$ if there exists a unique $\tilde{u}(x) \in \mathbb{R}$ such that

$$\lim_{r\searrow 0} ⨍_{B_r(x)} \big|u(y) - \tilde{u}(x)\big|\, dy = 0.$$

• u is approximately continuous at x if $x \notin S_u$ and $\tilde{u}(x) = u(x)$, that is, if x is a Lebesgue point of u.

• We now distinguish jump points, where the nature of the discontinuity is more specific: $x \in J_u$ is an approximate jump point of u if there exist $u^+(x) \neq u^-(x) \in \mathbb{R}$ and $\nu_u(x) \in \mathbb{S}^{N-1}$ such that

$$\lim_{r \searrow 0} \fint_{B_r(x,\nu_u(x))^\pm} \left|u(y) - u^\pm(x)\right| dy = 0.$$

By Lebesgue's differentiation theorem, one can show (cf. [3, Proposition 3.64 (a)]) that $\tilde{u}$ exists a.e. in U, and coincides with u; in particular, $|S_u| = 0$.

Now take a family of mollifiers $\left\{\rho_\varepsilon(x) = \frac{1}{\varepsilon^N}\rho(x/\varepsilon)\right\}_{\varepsilon>0}$, where $\rho \in C_c^\infty(\mathbb{R}^N)$ is a non-negative, even function, with $\operatorname{supp}\rho \subset B_1(0)$ and $\int_{\mathbb{R}^N} \rho = 1$. By [3, Propositions 3.64 (b) and 3.69 (b)], we have the following result:

Lemma 2.3 *(Convergence to the precise representative). For any $u \in L^1_{\text{loc}}(U)$, the mollified functions $u * \rho_\varepsilon \in C^\infty(U)$ pointwise converge as $\varepsilon \searrow 0$ out of $S_u \setminus J_u$ to the precise representative $u^* : U \setminus (S_u \setminus J_u) \to \mathbb{R}$ given by*

$$u^*(x) := \begin{cases} \tilde{u}(x), & \text{if } x \in U \setminus S_u \\ \dfrac{u^+(x) + u^-(x)}{2}, & \text{if } x \in J_u. \end{cases}$$

There is also a notion of approximate differentiability.

Definition 2.4. $u \in L^1_{\text{loc}}(U)$ is said to be approximately differentiable at $x \in U \setminus S_u$ if there exists $\nabla u(x) \in \mathbb{R}^N$, called approximate gradient of u at x, such that

$$\lim_{r \searrow 0} \frac{1}{r} \fint_{B_r(x)} \left|u(y) - \tilde{u}(x) - \nabla u(x) \cdot (y - x)\right| dy = 0.$$

2.3. Subdifferential of a functional and F^p

Let X be a reflexive Banach space with dual X', and denote by $\langle \cdot, \cdot \rangle$ the pairing between X' and X. A multivalued operator on X $\mathcal{A} : X \to 2^{X'}$ is said to be *monotone* if

$$\left\langle \tilde{\xi} - \tilde{\eta}, \xi - \eta \right\rangle \geq 0 \qquad \text{for every} \quad (\xi, \tilde{\xi}), (\eta, \tilde{\eta}) \in \operatorname{graph}(\mathcal{A}).$$

Let $\Phi : X \to \mathbb{R} \cup \{\infty\}$ be lower semicontinuous, proper and convex; its subdifferential $\partial\Phi : X \to 2^{X'}$ is a multivalued operator given as follows:

$$\delta \in \partial\Phi(\zeta) \quad \Longleftrightarrow \quad \Phi(\eta) - \Phi(\zeta) \geq \left\langle \delta, \eta - \zeta \right\rangle \quad \text{for all } \eta \in X.$$

This is a well-known example of a monotone operator from X to X'.

In particular, we are interested in ∂F^p, which can be written in terms of ∂F by means of the chain rule for subdifferentials (see [8, Corollary 16.72]):

$$\mathbf{z} \in \partial F^p(\xi) \quad \Longleftrightarrow \quad \mathbf{z} = pF^{p-1}(\xi)\tilde{\mathbf{z}}, \quad \text{with} \quad \tilde{\mathbf{z}} \in \partial F(\xi). \tag{2.4}$$

This is indeed very useful because there is a well-known characterization of the subdifferential for 1-homogeneous functions:

$$\tilde{\mathbf{z}} \in \partial F(\nabla v) \quad \text{if, and only if,} \quad \tilde{\mathbf{z}} \cdot \nabla v = F(\nabla v) \quad \text{and} \quad F^\circ(\tilde{\mathbf{z}}) \leq 1. \tag{2.5}$$

2.4. Functions of bounded variation

For a detailed study of the space of functions of bounded variation, we refer to [3]. Let us denote by $\mathfrak{M}(U; \mathbb{R}^N)$ the space of all $\mathbb{R}^N$-valued Radon measures, which is isomorphic to the product space $\mathfrak{M}(U)^N$.

Definition 2.5. We say that $u \in L^1(U)$ is a function of bounded variation, and write $u \in \mathrm{BV}(U)$, if its distributional derivative Du, defined by

$$\langle Du, \varphi \rangle := -\int_U u \, \mathrm{div}\varphi \, dx \quad \text{for all} \quad \varphi \in C_c^1(U; \mathbb{R}^N),$$

belongs to $\mathfrak{M}(U; \mathbb{R}^N)$. Alternatively, one can give the following equivalent characterization: $u \in BV(U)$ if, and only if, $u \in L^1(U)$ and the total variation of Du is finite, meaning that

$$|Du|(U) = \int_U d|Du| := \sup \left\{ \int_U u \, \mathrm{div}\varphi \, : \, \varphi \in C_c^1(U; \mathbb{R}^N), \, \|\varphi\|_\infty \leq 1 \right\} < \infty.$$

Unlike Sobolev spaces, $BV(U)$ includes characteristic functions of sufficiently regular sets. The next result gathers some properties of BV-functions that will be useful in the sequel:

Lemma 2.6. (a) *[3, Theorem 3.78] Every $u \in BV(U)$ is approximately differentiable $\mathcal{L}^N$-a.e. in U and S_u is countably $\mathcal{H}^{N-1}$-rectifiable; that is,*

$$S_u = (S_u \setminus J_u) \cup \bigcup_{i \in \mathbb{N}} K_i, \quad \textit{where} \quad \mathcal{H}^{N-1}(S_u \setminus J_u) = 0,$$

and each K_i is a compact subset of a $(n-1)$-dimensional Lipschitz manifold.

(b) *$BV(U)$ is a Banach space when it is endowed with the norm*

$$\|u\|_{BV(U)} := \|u\|_{L^1(U)} + |Du|(U).$$

2.5. Decomposition of the measure Du

Given $\mu, \tilde{\mu}$ two positive measures, we say that $\tilde{\mu}$ is absolutely continuous with respect to μ, and write $\tilde{\mu} \ll \mu$, if, given a measurable set E, $\mu(E) = 0$ implies $\tilde{\mu}(E) = 0$. In this case, $\frac{\tilde{\mu}}{\mu}$ means the Radon–Nikodým derivative of $\tilde{\mu}$ with respect to μ (called density when μ is a Lebesgue measure). In turn, $\mu \llcorner E$ is the restriction of μ to the set E.

Notice that Lemma 2.6 (a) implies that it is possible to define for $\mathcal{H}^{N-1}$-a.e. $x \in J_u$ a unit normal vector field to J_u; in fact, $\nu_u(x)$ is an orientation of J_u, and coincides with $\frac{Du}{|Du|}(x)$. Moreover, by Lemma 2.3, we get that the mollified functions $u * \rho_\varepsilon$, with $u \in BV(U)$, converge to the precise representative u^* for $\mathcal{H}^{N-1}$-a.e. point in the domain of u.

We may decompose Du in three mutually singular measures (see e.g. [3, Definition 1.24]), as follows (see [3, Section 3.9]):

$$Du = \nabla u \, \mathcal{L}^N + D^s u, \text{ with } \begin{cases} D^s u = D^j u + D^c u, \\ D^j u = (u^+ - u^-) \, \nu_u \, \mathcal{H}^{N-1} \llcorner J_u, \\ D^c u = D^s u \llcorner (U \setminus S_u). \end{cases} \tag{2.6}$$

Here $D^s u$ is the singular part of Du with respect to Lebesgue measure $\mathcal{L}^N$, and for $\mathcal{L}^N$-a.e. $x \in U$ the vector $\nabla u(x)$ is the approximate gradient of u at x (cf. [3, Theorem 3.83]). $D^j u$ and $D^c u$ are measures known as the jump and Cantor part of Du, respectively. Sometimes, one calls $D^d u = \nabla u \, \mathcal{L}^N + D^c u$ the diffuse part.

2.6. Divergence-measure fields and weak normal traces

Here we revisit the integration by parts formula with weak regularity assumptions for the domains of integration and the functions involved. With this aim, consider the space of vector fields whose divergence in the sense of distributions is a bounded Radon measure, i.e.,

$$\mathcal{DM}^\infty(U) = \{\mathbf{z} \in L^\infty(U; \mathbb{R}^N) \, : \, \operatorname{div} \mathbf{z} \in \mathfrak{M}(U; \mathbb{R}^N) \text{ and is finite}\},$$

where $\operatorname{div} \mathbf{z}$ acts as a measure as follows:

$$\int_U \varphi \, d(\operatorname{div} \mathbf{z}) = -\int_U \mathbf{z} \cdot \nabla \varphi \, dx \quad \text{for all } \varphi \in C_c^\infty(U).$$

Let $B \subset U$ be any bounded open set with Lipschitz continuous boundary and ν the corresponding outward unit normal on ∂B. Now it makes sense to define as in [6, Theorem 1.2] a weak trace of the normal component of $\mathbf{z} \in \mathcal{DM}^\infty_{\mathrm{loc}}(U)$ on ∂B as the linear operator

$$[\,\cdot\,, \nu] : \mathcal{DM}^\infty(B) \to L^\infty(\partial B) \quad \text{such that} \quad \big\| [\mathbf{z}, \nu] \big\|_{L^\infty(\partial B)} \le \|\mathbf{z}\|_\infty,$$

which coincides with $\mathbf{z}(x) \cdot \nu(x)$ for all $x \in \partial B$ when $\mathbf{z}$ is smooth enough.

Here we gather the main properties of divergence-measure fields:

Lemma 2.7. *For $\mathbf{z} \in \mathcal{DM}^\infty(U)$ the following hold:*

(a) *[18, Proposition 3.1]* $|\operatorname{div} \mathbf{z}| \ll \mathcal{H}^{N-1}$.

(b) *[2, Proposition 3.4], [22, Lemma 2.4]* $\operatorname{div} \mathbf{z}$ *can be decomposed as*

$$\operatorname{div} \mathbf{z} = \operatorname{div}^a \mathbf{z} + \operatorname{div}^j \mathbf{z} + \operatorname{div}^c \mathbf{z}, \text{ with } \begin{cases} \operatorname{div}^a \mathbf{z} \ll \mathcal{L}^N, \\ \operatorname{div}^c \mathbf{z}(B) = 0, \text{if } \mathcal{H}^{N-1}(B) < \infty, \end{cases}$$

and

$$\operatorname{div}^j \mathbf{z} = \big([\mathbf{z}, \nu_u]^+ - [\mathbf{z}, \nu_u]^-\big)\mathcal{H}^{N-1} \llcorner J_u. \tag{2.7}$$

2.7. Pairing measures

Let $U \subset \mathbb{R}^N$ be any open set.

Definition 2.8 *(Anzellotti's pairing).* Given the pair $\mathbf{z} \in \mathcal{DM}^\infty_{\mathrm{loc}}(U)$ and $u \in BV_{\mathrm{loc}}(U) \cap L^\infty_{\mathrm{loc}}(U)$, we define $(z, Du) \in \mathcal{D}'(U)$ by means of

$$\big\langle (\mathbf{z}, Du), \varphi \big\rangle := -\int_U u^* \varphi \, d(\operatorname{div} \mathbf{z}) - \int_U u\, \mathbf{z} \cdot \nabla \varphi \, dx \quad \text{for all } \varphi \in C_0^\infty(U).$$

This is a well-defined Radon measure on U, and $(\mathbf{z}, Du) \ll |Du|$ (see [14, Lemma 5.1]). Moreover, we can write (see [18, Theorem 3.2])

$$(\mathbf{z}, Du) = (\mathbf{z}, Du)^a + (\mathbf{z}, Du)^j + (\mathbf{z}, Du)^c, \quad \text{with } (\mathbf{z}, Du)^a = \mathbf{z} \cdot \nabla u \, \mathcal{L}^N,$$

and a further characterization for the jump part (cf. [17, Theorem 4.12]) is

$$(\mathbf{z}, Du)^j = \frac{[\mathbf{z}, \nu_u]^+ + [\mathbf{z}, \nu_u]^-}{2} (u^+ - u^-)\mathcal{H}^{N-1} \llcorner J_u. \tag{2.8}$$

Lemma 2.9 *(Generalized Green's formula and product rules). For a vector field $\mathbf{z} \in \mathcal{DM}^\infty_{\mathrm{loc}}(U)$ and $u \in BV_{\mathrm{loc}}(U) \cap L^\infty_{\mathrm{loc}}(U)$, it holds*

$$\int_\Omega u^* d(\operatorname{div} \mathbf{z}) + \int_\Omega d(\mathbf{z}, Du) = \int_{\partial\Omega} [\mathbf{z}, \nu] u \, d\mathcal{H}^{N-1}, \tag{2.9}$$

for any open and bounded $\Omega \subset U$ with Lipschitz continuous boundary. Moreover, we get the product rules:

(a) $\operatorname{div}(u\mathbf{z}) = u^* \operatorname{div} \mathbf{z} + (\mathbf{z}, Du)$ *as measures,*

(b) *[17, Proposition 4.9] For $\mathbf{z} \in \mathcal{DM}^\infty_{\mathrm{loc}}(U)$, and $u, v \in BV_{\mathrm{loc}}(U) \cap L^\infty_{\mathrm{loc}}(U)$,*

$$(v\mathbf{z}, Du) = v^*(\mathbf{z}, Du) + \frac{(u^+ - u^-)(v^+ - v^-)}{4} \operatorname{div} \mathbf{z} \llcorner (J_u \cap J_v). \tag{2.10}$$

Hereafter, unless otherwise specified, we will always identify any BV function v with its precise representative v^*, without explicit use of $*$.

2.8. Anisotropic total variation

For an anisotropy F, an open set U and $u \in BV(U)$, the F-total variation or anisotropic total variation $F(Du) = |Du|_F \in \mathfrak{M}(U)$ of Du is given by

$$\begin{aligned} |Du|_F &:= F(\nu_u)|Du| \\ &= F(\nabla u)\mathcal{L}^N + F(\nu_u)|u^+ - u^-|\, \mathcal{H}^{N-1} \llcorner J_u + F(\nu_u)|D^c u|. \end{aligned} \tag{2.11}$$

We will use the notation $|Dv|_F(U) = \displaystyle\int_U dF(Dv)$.

Here are the main properties:

Lemma 2.10. *Let $U \subset \mathbb{R}^N$ be a bounded open set with Lipschitz continuous boundary and $u \in BV(U)$. Then,*

(a) *the F-total variation of Du in an open bounded set $A \subset U$ with Lipschitz continuous boundary can be computed as follows (cf. [1])*

$$|Du|_F(A) = \sup\left\{ \int_A u \operatorname{div} \mathbf{z} \,:\, \mathbf{z} \in C_c^1(A;\mathbb{R}^N),\ F^\circ(\mathbf{z}) \le 1 \ \textit{a.e. in } A \right\}. \tag{2.12}$$

(b) $\displaystyle\int_U dF(Du)$ *is $L^1(U)$-lower semicontinuous on $BV(U)$.*

Notice that (a) is proven in [34, p.185] while (b) is obtained in [1, Theorem 5.1]. On the other hand, we note that if $\operatorname{div} \mathbf{z} \in L^2(U)$, the following result has been shown in [13, Proposition 4.1] for item (a) and (b) can be found in [34, proof of Theorem 7]. However, we need to work out the technical details to be able to obtain the same conclusions under milder assumptions.

Lemma 2.11. *Let U be an open set with bounded Lipschitz boundary, and $\mathbf{z} \in \mathcal{DM}^\infty_{\mathrm{loc}}(U)$. Then the following Cauchy-Schwarz type inequalities hold:*

(a) $\big|(\mathbf{z}, Du)\big| \le \|F^\circ(\mathbf{z})\|_{L^\infty(U)} F(Du)$, *for all $u \in BV_{\mathrm{loc}}(U) \cap L^\infty_{\mathrm{loc}}(U)$.*

(b) $|[\mathbf{z}, \nu]| \le \|F^\circ(\mathbf{z})\|_{L^\infty(U)} F(\nu)$, *$\mathcal{H}^{N-1}$-a.e. on ∂U.*

Proof. In order to prove (a), we first extend u to all of $\mathbb{R}^N$; let $g \in W^{1,1}(\mathbb{R}^N \setminus U)$ such that $g = u$ at ∂U (see, for example, [29, Theorem 18.18]). Then, let $\overline{u} := u\chi_U + g\chi_{\mathbb{R}^N \setminus U}$ and consider $u_n := \overline{u} \star \rho_n \in C^\infty(\mathbb{R}^N)$, with $\{\rho_n\}_{n\in\mathbb{N}}$ a sequence of mollifiers. We have that $u_n \to u$ in $L^1(U)$ and, by Lemma 2.3, $u_n \to u^*$ $\mathcal{H}^{N-1}$-a.e. as $n \to \infty$. Moreover, by [3, Proposition 3.7],

$$\int_A |\nabla u_n| \xrightarrow{n\to\infty} |Du|(A) \quad \text{for every open set } A \Subset U \text{ with } |Du|(\partial A) = 0.$$

Then, by Resethnyak's continuity theorem ([3, Theorem 2.39]) and (2.11),

$$\int_A F(\nabla u_n) \xrightarrow{n\to\infty} |Du|_{_F}(A) \quad \text{for every open set } A \Subset U \text{ with } |Du|(\partial A) = 0.$$

Therefore, given an open set $A \Subset U$ such that $|Du|(\partial A) = 0$ and $\varphi \in C_0^\infty(A)$, by the definition of Anzellotti's pairing,

$$\begin{aligned}\langle(\mathbf{z}, Du), \varphi\rangle &= -\int_A u\mathbf{z}\cdot\nabla\varphi - \int_A u^*\varphi\, d(\operatorname{div}\mathbf{z}) \\ &= -\lim_{n\to\infty}\left(\int_A u_n\mathbf{z}\cdot\nabla\varphi + \int_A u_n\varphi\, d(\operatorname{div}\mathbf{z})\right) \\ &= \lim_{n\to\infty}\int_A \varphi\,\mathbf{z}\cdot\nabla u_n \le \|F^\circ(\mathbf{z})\|_{L^\infty(U)}\|\varphi\|_{L^\infty(A)}\lim_{n\to\infty}\int_A F(\nabla u_n) \\ &= \|F^\circ(\mathbf{z})\|_{L^\infty(U)}\|\varphi\|_{L^\infty(A)}|Du|_{_F}(A),\end{aligned}$$

which implies (a).

Before proving (b), we need the following approximation result:

Claim: Given $\mathbf{z} \in \mathcal{DM}^\infty_{\mathrm{loc}}(U)$ there exists $\mathbf{z}_n \in C^\infty(\overline{U};\mathbb{R}^N)$ such that

$$\mathbf{z}_n \to \mathbf{z} \text{ in } L^1_{\mathrm{loc}}(U;\mathbb{R}^N), \quad \operatorname{div}\mathbf{z}_n \stackrel{*}{\rightharpoonup} \operatorname{div}\mathbf{z} \text{ in } \mathfrak{M}(U).$$

The claim is proved as the corresponding result for vector fields with divergence in $L^p(U)$; see [26, Lemma 2.3] (cf. [5, Lemma 3.6]).

Take a partition of unity of $\overline{U}$, $\{\theta_j\}_{j=1}^k \subset C_0^\infty(\mathbb{R}^N)$, $0 \le \theta_j \le 1$, satisfying the following: if $\operatorname{supp}\theta_j \cap \partial U \neq \emptyset$, then there is a bounded open cone C_j with vertex 0 such that $x + C_j \cap \overline{U} = \emptyset$ for all $x \in \partial U \cap \operatorname{supp}\theta_j$ and there exists $r > 0$ such that $x - C_j \subseteq U$ for all $x \in \partial U \cap (\operatorname{supp}\theta_j + B_r(0))$. Moreover, for each $j \in \{1, \ldots, k\}$, take a mollifier $\rho_j \in C^\infty(\mathbb{R}^N)$ with the property that, if $\operatorname{supp}\theta_j \cap \partial U \neq \emptyset$, then $\operatorname{supp}\rho_j \subset C_j$.

Letting $\rho_{j,n}(x) := n^N\rho_j(nx)$, we define

$$\mathbf{z}_n := \sum_{j=1}^k \rho_{j,n} \star (\theta_j\overline{\mathbf{z}}) \in C^\infty(\mathbb{R}^N;\mathbb{R}^N), \quad \text{where} \quad \overline{\mathbf{z}} = \begin{cases} z, & \text{in } U \\ 0, & \text{in } \mathbb{R}^N \setminus U.\end{cases}$$

Note that, if $x \in \partial U \cap \operatorname{supp}\theta_j$, then $x - C_j \subset U$ and $\operatorname{supp}\rho_j \subset C_j$ thus

$$(\rho_{j,n} \star (\theta_j\overline{\mathbf{z}}))(x) = \int_{x-C_j} \rho_{j,n}(x-y)\theta_j(y)\overline{\mathbf{z}}(y)dy = \int_U \rho_{j,n}(x-y)\theta_j(y)\mathbf{z}(y)dy.$$

By construction, $\mathbf{z}_n \to \mathbf{z}$ in $L^1_{\rm loc}(U;\mathbb{R}^N)$. On the other hand, for any $j = 1,\dots,k$,

$$\operatorname{div}(\theta_j \overline{\mathbf{z}}) = \overline{\operatorname{div}(\theta_j \mathbf{z})} + [\theta_j \overline{\mathbf{z}}, \nu]\mathcal{H}^{N-1} \llcorner \partial U, \tag{2.13}$$

where $\overline{\operatorname{div}(\theta_j \mathbf{z})}(E) = \operatorname{div}(\theta_j \mathbf{z})(E \cap U)$ for any $E \subset \mathbb{R}^N$. Now, for $x \in \mathbb{R}^N$,

$$\rho_{j,n} \star \left([\theta_j \overline{\mathbf{z}}, \nu]\mathcal{H}^{N-1} \llcorner \partial U\right)(x) = \int_{\partial U \cap \operatorname{supp}\, \theta_j} \rho_{j,n}(x-y)[\theta_j \overline{\mathbf{z}}, \nu](y) d\mathcal{H}^{N-1}(y),$$

thus $\rho_{j,n} \star \left([\theta_j \overline{\mathbf{z}}, \nu]\mathcal{H}^{N-1} \llcorner \partial U\right) = 0$ if $\partial U \cap \operatorname{supp}\, \theta_j = \emptyset$. Otherwise, one gets $y + C_j \subseteq \mathbb{R}^N \setminus \overline{U}$ for all $y \in \partial U \cap \operatorname{supp} \theta_j$ and $\operatorname{supp} \rho_j \subset C_j$. In this case, if $y \in \partial U \cap \operatorname{supp}\, \theta_j$, $\rho_{j,n}(x-y) \neq 0$ only if $x \in y + \operatorname{supp}\, \rho_j \subset \mathbb{R}^N \setminus \overline{U}$ so

$$\int_{\partial U \cap \operatorname{supp}\, \theta_j} \rho_{j,n}(x-y)[\theta_j \overline{\mathbf{z}}, \nu](y) d\mathcal{H}^{N-1}(y) = 0 \quad \text{for every } x \in \overline{U},$$

that is, $\rho_{j,n} \star \left([\theta_j \overline{\mathbf{z}}, \nu]\mathcal{H}^{N-1} \llcorner \partial U\right) = 0$ in $\overline{U}$. Hence, using (2.13), we can write

$$(\operatorname{div}(\rho_{j,n} \star (\theta_j \overline{\mathbf{z}}))) \llcorner U = (\rho_{j,n} \star \operatorname{div}(\theta_j \overline{\mathbf{z}})) \llcorner U = \left(\rho_{j,n} \star \overline{\operatorname{div}(\theta_j \mathbf{z})}\right) \llcorner U.$$

Consequently,

$$\operatorname{div}(\mathbf{z}_n) \llcorner U = \sum_{j=1}^k \operatorname{div}\, (\rho_{j,n} \star (\theta_j \overline{\mathbf{z}})) \llcorner U = \sum_{j=1}^k \left(\rho_{j,n} \star \overline{\operatorname{div}(\theta_j \mathbf{z})}\right) \llcorner U.$$

Letting $n \to \infty$, since $\left(\overline{\operatorname{div}(\theta_j \mathbf{z})}\right) \llcorner U = \operatorname{div}(\theta_j \mathbf{z})$, we finally obtain the claim.

Therefore, by the definition of the normal trace $[\mathbf{z}, \nu]$ (cf. [2, Section 3]) and the classical Green's formula, we obtain, for any $0 \le \Psi \in C^\infty(U) \cap W^{1,1}(U)$ with bounded support,

$$\begin{aligned}
\int_{\partial U} \Psi\ \mathbf{z}_n \cdot \nu &= \int_{\partial U} \Psi\ \sum_{j=1}^k (\rho_{j,n} \star (\theta_j \overline{\mathbf{z}})) \cdot \nu \\
&= \int_U \Psi\ \sum_{j=1}^k \rho_{j,n} \star \overline{\operatorname{div}(\theta_j \mathbf{z})} + \int_U \nabla \Psi \cdot \sum_{j=1}^k (\rho_{j,n} \star (\theta_j \mathbf{z})) \\
&\xrightarrow{n\to\infty} \int_U \Psi\ d(\operatorname{div} \mathbf{z}) + \int_U \nabla \Psi \cdot \mathbf{z} = \int_{\partial U} \Psi\ [\mathbf{z}, \nu].
\end{aligned}$$

Then, by approximation we have that

$$\lim_{n\to\infty} \int_{\partial U} \Psi\ \mathbf{z}_n \cdot \nu = \int_{\partial U} \Psi\ [\mathbf{z}, \nu]$$

for every $\Psi \in W^{1,1}(U)$. Consequently, for any such Ψ and by (2.3),

$$
\begin{aligned}
\left|\int_{\partial U}\Psi\,[\mathbf{z},\nu]\right| &\leq \lim_{n\to\infty}\sum_{j=1}^{k}\int_{\partial U}\Psi(x)\left(\int_U \rho_{j,n}(x-y)\theta_j(y)\,|\mathbf{z}(y)\cdot\nu(x)|\,\mathrm{d}y\right)\mathrm{d}x \\
&\leq \lim_{n\to\infty}\sum_{j=1}^{k}\int_{\partial U}\Psi(x)F(\nu(x))\left(\int_U \rho_{j,n}(x-y)\theta_j(y)F^\circ(\mathbf{z}(y))\mathrm{d}y\right)\mathrm{d}x \\
&\leq \|F^\circ(\mathbf{z})\|_{L^\infty(U)}\int_{\partial U}\Psi F(\nu).
\end{aligned}
$$

This last inequality directly gives (b). □

3. Definition and uniqueness of solutions for A-IMCF

3.1. Definition of solutions and first properties

As in [32,33] we introduce the following notion of solution for (1.7):

Definition 3.1. A non-negative $v \in BV_{\mathrm{loc}}(\Omega^e)\cap L^\infty_{\mathrm{loc}}(\Omega^e)$ is said to be a weak solution of (1.7) if there is a vector field $\mathbf{z}\in\mathcal{DM}^\infty_{\mathrm{loc}}(\Omega^e)$, with $F^\circ(\mathbf{z})\leq 1$, satisfying

$$
\operatorname{div}\mathbf{z} = (\mathbf{z}, Dv) = F(Dv) \quad \text{as measures in } \ \Omega^e, \tag{3.1}
$$

together with the boundary conditions

$$
\begin{cases} v = 0 & \mathcal{H}^{N-1}\text{-a.e. on } \ \partial\Omega, \\ v(x)\to\infty & \text{as } |x|\to\infty. \end{cases} \tag{3.2}
$$

Remark 3.2. In the terminology by Moser (see e.g. [37, Definition 1.1]), the behaviour at infinity in (3.2) means that all our solutions are *proper*, which geometrically implies that solutions stay bounded at finite times, and hence the corresponding sublevel sets are precompact. Thus we do not need to add the latter as an extra assumption as in [23]. As pointed out in [27], there are some manifolds that do not admit a proper solution, even in more regular settings. We stress that the lack of properness leads to non-uniqueness of solutions (cf. [36, Proposition 2.4]).

Lemma 3.3. *A weak solution v of (1.7) in the sense of Definition 3.1 has no jump part, i.e., it satisfies $D^j v = 0$.*

Proof. Let $\mu\in\mathfrak{M}(\Omega^e)$ be such that $\mu \ll \mathcal{H}^{N-1}\llcorner J_v$ and let us denote by $\theta^{J_v}[\mu]$ the Radon-Nikodým derivative of μ with respect to $\mathcal{H}^{N-1}\llcorner J_v$. Now by (3.1) and (2.8) we can write

$$\begin{aligned}\theta^{J_v}[F(Dv)] = \theta^{J_v}[(\mathbf{z}, Dv)] &= \frac{[\mathbf{z},\nu_v]^+ + [\mathbf{z},\nu_v]^-}{2}(v^+ - v^-)\\ &= \Big(\frac{\theta^{J_v}[\operatorname{div}\mathbf{z}]}{2} + [\mathbf{z},\nu_v]^-\Big)(v^+ - v^-),\end{aligned}$$

which follows from (2.7). Using now that, by (3.1), $\operatorname{div}\mathbf{z} = F(Dv)$ as measures, we get by means of 2.11 (b)

$$\begin{aligned}\theta^{J_v}[F(Dv)]\left(1 - \frac{v^+ - v^-}{2}\right) &= [\mathbf{z},\nu_v]^-(v^+ - v^-)\\ &\leq \|F^\circ(\mathbf{z})\|_{L^\infty(\Omega^e)} F(\nu_v)|v^+ - v^-|\\ &\leq F(\nu_v)|v^+ - v^-| = \theta^{J_v}[F(Dv)],\end{aligned}$$

where we applied $F^\circ(\mathbf{z}) \leq 1$ and (2.11). This leads to $v^+ \geq v^-$ $\mathcal{H}^{N-1}$-a.e. Redoing the above argument, but substituting $[\mathbf{z},\nu_v]^-$ instead of $[\mathbf{z},\nu_v]^+$ by means of (2.8), one can reach the reverse inequality. Thus $v^+ = v^-$ $\mathcal{H}^{N-1}$-a.e., and by (2.6) the claim follows. □

In [22], BV functions with null singular set with respect to the Hausdorff measure $\mathcal{H}^{N-1}$ were named as Diffuse BV functions and denoted by DBV. We recall some of their properties, which will be useful in the sequel.

Lemma 3.4. *Let $U \subset \mathbb{R}^N$ be an open set, $v \in DBV_{\rm loc}(U)$ and $f : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function. Then, the following chain rules hold:*

(a) *$D(f \circ v) = f'(v)Dv$.*
(b) *For any $\mathbf{z} \in \mathcal{DM}^\infty_{\rm loc}(U)$, if f is non-decreasing, we have*
  (b1) *$(\mathbf{z}, D(f \circ v)) = f'(v)(\mathbf{z}, Dv)$.*
  (b2) *If additionally $(\mathbf{z}, Dv) = F(Dv)$, then*

$$(\mathbf{z}, D(f \circ v)) = F(D(f \circ v)).$$

Proof. (a) follows from the standard chain rule formula for BV functions [3, Theorem 3.99]. In order to show (b1), we proceed as follows:

$$\begin{aligned}(\mathbf{z}, D(f \circ v)) =& \frac{(\mathbf{z}, D(f \circ v))}{|D(f \circ v)|}|D(f \circ v)| = \frac{(\mathbf{z}, Dv)}{|Dv|}|D(f \circ v)|\\ =& f'(v)\frac{(\mathbf{z}, Dv)}{|Dv|}|Dv| = f'(v)(\mathbf{z}, Dv),\end{aligned}$$

where the second equality is due to [17, Proposition 4.5 (iii)] and the third one comes from (a). Now, we obtain (b2) by combining the previous items in the statement with (2.11). Indeed,

$$
\begin{aligned}
(\mathbf{z}, D(f\circ v)) =& f'(v)(\mathbf{z}, Dv) = f'(v)F(Dv)\\
=& f'(v)F(\nu_v)|Dv| = F(\nu_v)|D(f\circ v)| = |F(D(f\circ v))|.
\end{aligned}
$$

Here we have used that $\nu_{f\circ v} = \frac{D(f\circ v)}{|D(f\circ v)|} = \nu_v$, $D(f\circ v)$-a.e. $\square$

Corollary 3.5. *Let v be a weak solution of (3.1) with associated vector field $\mathbf{z}$. Then the measure $\operatorname{div}\mathbf{z} = F(Dv)$ vanishes when restricted to the level sets, that is,*

$$\operatorname{div}\mathbf{z}\llcorner\{v=t\} = 0 \quad \textit{for all} \quad t \geq 0.$$

Proof. As in [33, Remark 4.4] one proves that $Dv\llcorner\{v=t\}=0$ for any $v\in DBV_{\rm loc}(\Omega^e)\cap L^\infty_{\rm loc}(\Omega^e)$, and then uses the anisotropic total variation and equation (3.1). $\square$

3.2. Uniqueness of the anisotropic IMCF

Theorem 3.6. *Suppose that u_1 and u_2 are a non-negative weak supersolution and a non-negative weak subsolution of (1.7) in the sense that u_1, $u_2 \in DBV_{\rm loc}(\Omega^e)\cap L^\infty_{\rm loc}(\Omega^e)$ and there exist vector fields $\mathbf{z}_i \in \mathcal{DM}^\infty_{\rm loc}(\Omega^e)$, with $F^\circ(\mathbf{z}_i)\leq 1$, $i=1,2$, satisfying*

$$\operatorname{div}(\mathbf{z}_1) \leq (\mathbf{z}_1, Du_1) = F(Du_1) \quad \textit{as measures in} \ \ \Omega^e,$$

and

$$\operatorname{div}(\mathbf{z}_2) \geq (\mathbf{z}_2, Du_2) = F(Du_2) \quad \textit{as measures in} \ \ \Omega^e,$$

together with boundary conditions

$$\begin{cases} u_i = \varphi_i & \mathcal{H}^{N-1}\textit{-a.e. on } \partial\Omega,\\ u_i(x)\to\infty & \textit{as } |x|\to\infty \end{cases},$$

with $\varphi_i\in L^1(\partial\Omega)$, such that $\varphi_2\leq\varphi_1$ $\mathcal{H}^{N-1}$-a.e. Then $u_2\leq u_1$ a.e. in Ω^e.

Proof. Let $T^h:\mathbb{R}\to(-\infty,h]$ be the truncation operator defined by

$$T^h(s) := \begin{cases} s & \text{if } s\leq h,\\ h & \text{if } s>h. \end{cases} \tag{3.3}$$

Suppose by contradiction that $u_2>u_1$ holds on a set of positive measure. Then, since $u_1\geq 0$, there exists $h>0$ such that $T^h(u_2)>u_1$ on a set of positive measure. Therefore, we can find $\varepsilon>0$ such that $l := \operatorname{ess\,sup}\left\{T^h(u_2) - \frac{u_1}{1-\varepsilon}\right\} - \varepsilon > 0$ (note that $l\leq h$). Define $\overline{u}_1 := \frac{u_1}{1-\varepsilon}$ and $\overline{u}_2 := T^h(u_2) - l$. We have

$$\operatorname{div}(\mathbf{z}_1) \leq (1-\varepsilon)F(D\overline{u}_1), \tag{3.4}$$

$$\operatorname{div}(\mathbf{z}_2) \geq F(Du_2), \tag{3.5}$$

$$\operatorname{ess\,sup}\{\overline{u}_2 - \overline{u}_1\} = \varepsilon, \tag{3.6}$$

and we note that $\overline{u}_2(x) \geq \overline{u}_1(x)$ implies $u_2(x) > u_1(x)$.

Hereafter, we use the notation $u_+ := \max\{u, 0\}$ for the positive part of a function u. Since $\overline{u}_1 \to \infty$ as $|x| \to \infty$ and $\overline{u}_2 \leq h$, we have $(\overline{u}_2 - \overline{u}_1)_+ = 0$ on ∂B_R for R large enough. Moreover, $(\overline{u}_2 - \overline{u}_1)_+ \leq (u_2 - u_1)_+ = 0$ on $\partial\Omega$. Therefore, if we integrate $(\overline{u}_2 - \overline{u}_1)_+$ over $B_R \setminus \overline{\Omega}$ with respect to the measures in (3.5) and (3.4), take the difference between them and integrate by parts, we get

$$\begin{aligned} &-\int_{B_R\setminus\overline{\Omega}} d(\mathbf{z}_2 - \mathbf{z}_1, D(\overline{u}_2 - \overline{u}_1)_+) \\ &\geq \int_{B_R\setminus\overline{\Omega}} (\overline{u}_2 - \overline{u}_1)_+ \, d(F(Du_2) - F(D\overline{u}_1)) + \varepsilon \int_{B_R\setminus\overline{\Omega}} (\overline{u}_2 - \overline{u}_1)_+ \, dF(D\overline{u}_1). \end{aligned} \tag{3.7}$$

We observe that, by Lemma 3.4 (b2), $(\mathbf{z}_i, D\overline{u}_i) = F(D\overline{u}_i)$, $i = 1, 2$. Therefore, since $F^\circ(\mathbf{z}_i) \leq 1$, $i = 1, 2$, by (2.3) and Lemma 3.4 (b1) the first integral on the left hand side is nonpositive:

$$\int_{B_R\setminus\overline{\Omega}} d(\mathbf{z}_2 - \mathbf{z}_1, D(\overline{u}_2 - \overline{u}_1)_+) = \int_{B_R\setminus\overline{\Omega}} \chi_{\{\overline{u}_2 - \overline{u}_1 \geq 0\}} \, d(\mathbf{z}_2 - \mathbf{z}_1, D(\overline{u}_2 - \overline{u}_1)) \geq 0.$$

On the other hand, multiplying the inequality of measures $\operatorname{div}(\mathbf{z}_1) \leq F(D\overline{u}_1)$ by $\frac{1}{2}((\overline{u}_2 - \overline{u}_1)_+)^2$, integrating by parts and using again Lemma 3.4 (b1), we obtain

$$\int_{B_R\setminus\overline{\Omega}} (\overline{u}_2 - \overline{u}_1)_+ \, d(F(D\overline{u}_2) - F(D\overline{u}_1)) \geq -\int_{B_R\setminus\overline{\Omega}} \frac{((\overline{u}_2 - \overline{u}_1)_+)^2}{2} \, dF(D\overline{u}_1). \tag{3.8}$$

Therefore, since

$$\int_{B_R\setminus\overline{\Omega}} (\overline{u}_2 - \overline{u}_1)_+ \, d(F(Du_2) - F(D\overline{u}_1)) \geq \int_{B_R\setminus\overline{\Omega}} (\overline{u}_2 - \overline{u}_1)_+ \, d(F(D\overline{u}_2) - F(D\overline{u}_1)),$$

(3.8) together with (3.7) yield

$$\int_{B_R\setminus\overline{\Omega}} (\overline{u}_2 - \overline{u}_1)_+ \left(\varepsilon - \frac{(\overline{u}_2 - \overline{u}_1)_+}{2} \right) dF(D\overline{u}_1) \leq 0.$$

Now, letting $R \to \infty$ we have

$$\int_{\Omega^e} (\overline{u}_2 - \overline{u}_1)_+ \left(\varepsilon - \frac{(\overline{u}_2 - \overline{u}_1)_+}{2} \right) dF(D\overline{u}_1) \leq 0.$$

Thus, by (3.6),

$$\int_{\Omega^e} (\overline{u}_2 - \overline{u}_1)_+ \, dF(D\overline{u}_1) = 0.$$

This, in turn, yields

$$\int_{\Omega^e} (\overline{u}_2 - \overline{u}_1)_+ \, d|D\overline{u}_1| = 0.$$

Moreover, from (3.7), we get

$$\int_{\Omega^e} (\overline{u}_2 - \overline{u}_1)_+ \, d|D\overline{u}_2| \le \int_{\Omega^e} (\overline{u}_2 - \overline{u}_1)_+ \, d|Du_2| = 0.$$

Finally,

$$\begin{aligned}\frac{1}{2}|D((\overline{u}_2 - \overline{u}_1)_+)^2|(\Omega^e) &= \int_{\Omega^e} (\overline{u}_2 - \overline{u}_1)_+ \, d|D(\overline{u}_2 - \overline{u}_1)| \\ &\le \int_{\Omega^e} (\overline{u}_2 - \overline{u}_1)_+ d|D\overline{u}_1| + \int_{\Omega^e} (\overline{u}_2 - \overline{u}_1)_+ \, d|D\overline{u}_2| = 0,\end{aligned}$$

which implies that $(\overline{u}_2 - \overline{u}_1)_+$ is equivalent to a constant in Ω^e. Since $\overline{u}_2 - \overline{u}_1 \to 0$ as $|x| \to \infty$, such a constant is zero. This is in contradiction with (3.6). □

4. Definition and uniqueness of solutions of the approximating problems

4.1. Anisotropic p-Laplace equation and change of variable

Definition 4.1. A function $u \in L^{pN/(N-p)}(\Omega^e)$ such that $\nabla u \in L^p(\Omega^e)$ is said to be a weak solution of the problem (1.5) if its trace on $\partial\Omega$ is equal to 1, $\lim_{|x|\to\infty} u(x) = 0$ and there exists $\mathbf{z} \in \partial F(\nabla u)$ such that

$$\operatorname{div}\big(F^{p-1}(\nabla u)\mathbf{z}\big) = 0$$

in the weak sense, meaning that

$$\int_{\Omega^e} F^{p-1}(\nabla u)\mathbf{z} \cdot \nabla\varphi = 0 \quad \text{for every} \quad \varphi \in W_0^{1,p}(\Omega^e). \tag{4.1}$$

In [12] we proved an existence theorem for weak solutions of the above problem:

Theorem 4.2. *If $1 < p < N$ and $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz-continuous boundary, there exists a weak solution u of the problem (1.5) satisfying the estimates*

$$(F^\circ)^{\frac{p-N}{p-1}}\left(\frac{x}{r_1}\right) \le u(x) \le (F^\circ)^{\frac{p-N}{p-1}}\left(\frac{x}{r_2}\right) \quad a.e. \text{ in } \Omega^e \tag{4.2}$$

for any constants $0 < r_1 < r_2$ such that $\mathcal{W}_{r_1} \subseteq \Omega \subseteq \mathcal{W}_{r_2}$.

We will see that the transformation (1.2) gives a weak solution to the problem (1.6) in the following sense:

Definition 4.3. We say that a function $v \in W^{1,p}_{\mathrm{loc}}(\Omega^e)$ is a *weak solution* of (1.6) if its trace on $\partial\Omega$ is 0, $\lim\limits_{|x|\to\infty} v(x) = \infty$ and there exists $\tilde{\mathbf{z}} \in \partial F(\nabla v)$ such that

$$\operatorname{div}\big(F^{p-1}(\nabla v)\tilde{\mathbf{z}}\big) = F^p(\nabla v) \quad \text{in the weak sense,}$$

that is,

$$-\int_{\Omega^e} F^{p-1}(\nabla v)\tilde{\mathbf{z}}\cdot\nabla\varphi = \int_{\Omega^e} F^p(\nabla v)\varphi, \tag{4.3}$$

for every $\varphi \in L^\infty(\Omega^e)$ with compact support whose distributional gradient $\nabla\varphi$ belongs to $L^p(\Omega^e;\mathbb{R}^N)$.

Proposition 4.4. *If u is a weak solution of* (1.5)*, then v given by* (1.2) *is a weak solution of* (1.6)*.*

Proof. Notice that

$$\mathbf{z} \in \partial F(\nabla u) \quad \text{if and only if} \quad \tilde{\mathbf{z}} = -\mathbf{z} \in \partial F(\nabla v). \tag{4.4}$$

Indeed, by 1-homogeneity of F, we have

$$\tilde{\mathbf{z}}\cdot\nabla v = (p-1)\mathbf{z}\cdot\frac{\nabla u}{u} = \frac{p-1}{u}F(\nabla u) = F\Big(\frac{1-p}{u}\nabla u\Big) = F(\nabla v).$$

Because of the lower bound in (4.2), for each open and bounded set $B \subset \Omega^e$, we can find a constant $C = C(p, r_1, N, B) > 0$ so that $u \geq C$ on B. As $r \mapsto 1/r^{p-1}$ is Lipschitz-continuous away from zero, a well-known result by Stampacchia [39] ensures that $1/u^{p-1} \in W^{1,p}(B)$, and hence for any $\varphi \in W^{1,p}_0(B)\cap L^\infty(B)$ it holds that

$$\Big(\frac{p-1}{u}\Big)^{p-1}\varphi \in W^{1,p}_0(B)\cap L^\infty(B)$$

can be used as a test function in (4.1). Accordingly, for $\mathbf{z} \in \partial F(\nabla u)$ we have

$$0 = \int_B F^{p-1}\Big(\frac{p-1}{u}\nabla u\Big)\mathbf{z}\cdot\nabla\varphi - \int_B F^p\Big(\frac{p-1}{u}\nabla u\Big)\varphi,$$

where we applied the homogeneity of F and that $\nabla u\cdot\mathbf{z} = F(\nabla u)$. Now, as $\tilde{\mathbf{z}} = -\mathbf{z} \in \partial F(\nabla v)$, this can be rewritten as

$$0 = -\int_B F^{p-1}(\nabla v)\tilde{\mathbf{z}}\cdot\nabla\varphi - \int_B F^p(\nabla v)\varphi,$$

as desired. $\square$

As an immediate consequence of Theorem 4.2, we conclude:

Corollary 4.5. *If $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz-continuous boundary, there exists a weak solution v of the problem (1.6). Moreover, v satisfies the estimates*

$$(N-p)\log\left(F^\circ\left(\frac{x}{r_2}\right)\right) \le v(x) \le (N-p)\log\left(F^\circ\left(\frac{x}{r_1}\right)\right) \quad a.e.\ in\ \Omega^e, \tag{4.5}$$

for any constants $0 < r_1 < r_2$ such that $\mathcal{W}_{r_1} \subseteq \Omega \subseteq \mathcal{W}_{r_2}$.

4.2. Comparison theorems for sub- and supersolutions

Consider the following generalization of (1.6) with generic Dirichlet boundary conditions:

$$\begin{cases} \operatorname{div}\big(F^{p-1}(\nabla v)\partial F(\nabla v)\big) = F^p(\nabla v) & \text{in } \Omega^e \\ v = \varphi & \text{on } \partial\Omega \\ v \to \infty & \text{as } |x| \to \infty \end{cases}. \tag{4.6}$$

Theorem 4.6. *Suppose that v_1 and v_2 are a non-negative weak supersolution and a non-negative weak subsolution of (4.6) with boundary values $\varphi_1 \ge \varphi_2 \in W^{1-\frac{1}{p},p}(\partial\Omega)$, respectively, on $\partial\Omega$, i.e., $v_i \in W^{1,p}_{\mathrm{loc}}(\Omega^e)$, its trace on $\partial\Omega$ is equal to φ_i, $\lim\limits_{|x|\to\infty} v_i(x) = \infty$ and there exist $\mathbf{z}_i \in \partial F(\nabla u_i)$, for $i = 1, 2$ such that*

$$\operatorname{div}\big(F^{p-1}(\nabla v_1)\mathbf{z}_1\big) \le F^p(\nabla v_1) \quad \textit{in the weak sense,}$$

and

$$\operatorname{div}\big(F^{p-1}(\nabla v_2)\mathbf{z}_2\big) \ge F^p(\nabla v_2) \quad \textit{in the weak sense.}$$

Then $v_1 \ge v_2$ a.e. in Ω^e.

Proof. The proof follows the ideas of the proof of Theorem 3.6, suitably adapted to this more regular case. Let $T^h : \mathbb{R} \to (-\infty, h]$ be the truncation operator defined by (3.3).

We argue by contradiction. Suppose that $v_2 > v_1$ holds on a set of positive measure. Then, since $v_1 \ge 0$, there exists $h > 0$ such that $T^h(v_2) > v_1$ on a set of positive measure. Therefore, we can find $\varepsilon > 0$ such that $l := \operatorname{ess\,sup}\left\{T^h(v_2) - \frac{v_1}{1-\varepsilon}\right\} - \frac{\varepsilon}{p} > 0$ (note that $l \le h$). Define $\overline{v}_1 := \frac{v_1}{1-\varepsilon}$ and $\overline{v}_2 := T^h(v_2) - l$. We have

$$\operatorname{div}\big(F^{p-1}(\nabla\overline{v}_1)\mathbf{z}_1\big) \le (1-\varepsilon)F^p(\nabla\overline{v}_1) \quad \text{in the weak sense,} \tag{4.7}$$

$$\operatorname{div}\big(F^{p-1}(\nabla\overline{v}_2)\mathbf{z}_2\big) \ge F^p(\nabla\overline{v}_2) \quad \text{in the weak sense,} \tag{4.8}$$

$$\operatorname{ess\,sup}\{\overline{v}_2 - \overline{v}_1\} = \frac{\varepsilon}{p}. \tag{4.9}$$

Since $v_1 \to \infty$ as $|x| \to \infty$, we have $\overline{v}_1 > h$ on ∂B_R for all R large enough. Therefore, since $\overline{v}_2 < h$, $(\overline{v}_2 - \overline{v}_1)_+ = 0$ on ∂B_R for all R large enough. Moreover, $(\overline{v}_2 - \overline{v}_1)_+ \leq (v_2 - v_1)_+ = 0$ on $\partial\Omega$. Accordingly, if we multiply (4.8) and (4.7) by $(\overline{v}_2 - \overline{v}_1)_+$ over $B_R \setminus \overline{\Omega}$, take the difference between them and integrate by parts, we get

$$
\begin{aligned}
&-\int_{B_R\setminus\overline{\Omega}} \left(F^{p-1}(\nabla\overline{v}_2)\mathbf{z}_2 - F^{p-1}(\nabla\overline{v}_1)\mathbf{z}_1\right)\cdot\nabla(\overline{v}_2 - \overline{v}_1)_+ \\
&\geq \int_{B_R\setminus\overline{\Omega}} (F^p(\nabla\overline{v}_2) - F^p(\nabla\overline{v}_1))(\overline{v}_2 - \overline{v}_1)_+ + \varepsilon\int_{B_R\setminus\overline{\Omega}} F^p(\nabla\overline{u}_1)(\overline{v}_2 - \overline{v}_1)_+.
\end{aligned}
\tag{4.10}
$$

We recall that by (2.4) it holds $pF^{p-1}(\nabla\overline{v}_i)\mathbf{z}_i \in \partial F^p(\nabla\overline{v}_i)$, $i = 1, 2$, by the chain rule for subdifferentials (2.4). Therefore, by the monotonicity of ∂F^p, the integral on the left hand side is nonpositive. On the other hand, multiplying the inequality

$$
\operatorname{div}\left(F^{p-1}(\nabla\overline{v}_1)\mathbf{z}_1\right) \leq F^p(\nabla\overline{v}_1)
$$

by $\frac{p}{2}((\overline{v}_2 - \overline{v}_1)_+)^2$ and integrating by parts, we obtain

$$
\int_{B_R\setminus\overline{\Omega}} (F^p(\nabla\overline{v}_2) - F^p(\nabla\overline{v}_1))(\overline{v}_2 - \overline{v}_1)_+ \geq -\int_{B_R\setminus\overline{\Omega}} F^p(\nabla\overline{v}_1)\frac{p((\overline{v}_2 - \overline{v}_1)_+)^2}{2}. \tag{4.11}
$$

Therefore, (4.11) together with (4.10) yield

$$
\int_{B_R\setminus\overline{\Omega}} (\overline{v}_2 - \overline{v}_1)_+ \left(\varepsilon - \frac{p(\overline{v}_2 - \overline{v}_1)_+}{2}\right) F^p(\nabla\overline{v}_1) \leq 0.
$$

Now, letting $R \to \infty$ we have

$$
\int_{\Omega^e} (\overline{v}_2 - \overline{v}_1)_+ \left(\varepsilon - \frac{p(\overline{v}_2 - \overline{v}_1)_+}{2}\right) F^p(\nabla\overline{v}_1) \leq 0.
$$

Thus, by (4.9),

$$
(\overline{v}_2 - \overline{v}_1)_+ F(\nabla\overline{v}_1) = 0 \quad \text{a.e. in } \Omega^e.
$$

Moreover, from (4.10), we get

$$
(\overline{v}_2 - \overline{v}_1)_+ F(\nabla\overline{v}_2) = 0 \quad \text{a.e. in } \Omega^e.
$$

Finally,

$$
\begin{aligned}
\frac{1}{2}\left|\nabla((\overline{v}_2 - \overline{v}_1)_+)^2\right| &= |(\overline{v}_2 - \overline{v}_1)_+\nabla(\overline{v}_2 - \overline{v}_1)| \leq (\overline{v}_2 - \overline{v}_1)_+\left(|\nabla\overline{v}_1| + |\nabla\overline{v}_2|\right) \\
&\leq \frac{1}{c}(\overline{v}_2 - \overline{v}_1)_+\left(F(\nabla\overline{v}_1) + F(\nabla\overline{v}_2)\right) = 0 \quad \text{a.e. in } \Omega^e.
\end{aligned}
$$

This implies that $(\overline{v}_2 - \overline{v}_1)_+$ is equivalent to a constant in Ω^e. Since $(\overline{v}_2 - \overline{v}_1)_+ = 0$ for $|x|$ large enough, this constant has to be zero, which contradicts with (4.9). $\square$

Remark 4.7. We actually do not need to assume $\lim_{|x|\to\infty} v_2(x) = \infty$.

4.3. Uniqueness of the p-capacitary potentials

As a by-product of the techniques in the previous proof, we manage to achieve uniqueness of solutions of (1.5), which enhances our existence and regularity results in [12]. More precisely, we show that the transformation (1.2) allows us to obtain a comparison principle which yields uniqueness of solutions of this problem. In fact, setting $p^* = \frac{pN}{N-p}$,

Theorem 4.8. *Suppose that u_1 and u_2 are a weak subsolution and a weak supersolution of (1.8) with $D = \Omega^e$ that have boundary values $\varphi_1 \le \varphi_2 \in W^{1-\frac{1}{p},p}(\partial\Omega)$, respectively, on $\partial\Omega$, i.e., $u_i \in L^{p^*}(\Omega^e)$, $\nabla u_i \in L^p(\Omega^e)$, its trace on $\partial\Omega$ is equal to φ_i, $\lim\limits_{|x|\to\infty} u_i(x) = 0$ and there exist $\mathbf{z}_i \in \partial F(\nabla u_i)$, for $i = 1, 2$, such that*

$$\operatorname{div}\big(F^{p-1}(\nabla u_2)\mathbf{z}_2\big) \le 0 \le \operatorname{div}\big(F^{p-1}(\nabla u_1)\mathbf{z}_1\big)$$

in the weak sense. Then $u_1 \le u_2$ a.e. in Ω^e.

Proof. Let $T_{h_1}^{h_2} : \mathbb{R} \to [h, +\infty)$ be the truncation operator defined by

$$T_{h_1}^{h_2}(s) := \begin{cases} h_1 & \text{if } s \le h_1, \\ s & \text{if } h_1 \le s \le h_2, \\ h_2 & \text{if } s \ge h_2. \end{cases}$$

Suppose by contradiction that $u_1 > u_2$ holds on a set of positive measure. Then, there exist $0 < h < 1$ and $h < h'$ such that either $T_h^{h'}(u_1) > T_h^{h'}(u_2)$ or $T_h^{h'}(-u_1) < T_h^{h'}(-u_2)$ holds on a set of positive measure. We may assume without loss of generality that the former holds, since otherwise we can consider the solutions $\tilde{u}_i = -u_i$, $i = 1, 2$, of (1.5) and get to a contradiction in the same way. Moreover, we can assume that $h' = 1$, since otherwise we consider the solutions $\tilde{u}_i = \frac{1}{h'} u_i$, $i = 1, 2$.

Now, let $v_i = v_i^h := (1-p)\log(T_h^1(u_i)) \in W^{1,p}_{\mathrm{loc}}(\Omega^e)$, $i = 1, 2$. Note that $0 \le v_i \le (1-p)\log h$ and $v_1 \ge v_2$ on $\partial\Omega$. We claim that, similarly as in Proposition 4.4, one can show the inequalities

$$\operatorname{div}\big(F^{p-1}(\nabla v_1)(-\mathbf{z}_1)\big) \le F^p(\nabla v_1) \quad \text{in the weak sense,}$$

and

$$\operatorname{div}\big(F^{p-1}(\nabla v_2)(-\mathbf{z}_2)\big) \ge F^p(\nabla v_2) \quad \text{in the weak sense.}$$

From this point on, the proof follows exactly the one of Theorem 4.6. Therefore, we are led to a contradiction (we omit the details). □

Recall that the solutions of (1.8) with $D = \Omega^e$ and $\varphi \equiv 1$ are known to be minimizers of the anisotropic p-capacity (see [12] for more details). Accordingly,

Corollary 4.9. *There exists a unique minimizer in* $\{u \in L^{p^*}(\mathbb{R}^N) : \nabla u \in L^p(\mathbb{R}^N),\, u \geq 1$ *in* $\Omega\}$ *of* $\mathrm{Cap}_p^F(\overline{\Omega})$ *defined as in* (1.9)*, that is, p-capacitary functions are unique.*

With a similar proof to that of Theorem 4.8 we obtain a comparison principle, thus uniqueness, of weak solutions of (1.8) on bounded domains.

Theorem 4.10. *Suppose that u_1 and u_2 are two weak solutions of* (1.8) *with $D = \Omega$ and boundary values $\varphi_1 \leq \varphi_2 \in W^{1-\frac{1}{p},p}(\partial\Omega)$, respectively. Then $u_1 \leq u_2$ a.e. in Ω.*

Summing up, the results in this subsection give altogether a proof of Corollary 1.2.

5. Existence of solutions to A-IMCF

Theorem 5.1. *If $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz-continuous boundary, there exists a weak solution v of the problem* (1.7)*. Moreover, v satisfies the estimates*

$$(N-1)\log\left(F^\circ\left(\frac{x}{r_2}\right)\right) \leq v(x) \leq (N-1)\log\left(F^\circ\left(\frac{x}{r_1}\right)\right) \quad \text{a.e. in } \Omega^e, \tag{5.1}$$

for any constants $0 < r_1 < r_2$ such that $\mathcal{W}_{r_1} \subseteq \Omega \subseteq \mathcal{W}_{r_2}$.

Proof. By Corollary 4.5, we know that for each $p > 1$ there exists a weak solution v_p of (1.6) in the sense of Definition 4.3. Moreover, by uniqueness, $v_p = (1-p)\log u_p$, where u_p is the unique weak solution of (1.5).

In particular, (4.3) holds for some $\mathbf{z}_p \in \partial F(\nabla v_p)$. We aim to construct a solution of problem (3.1) as a subsequential limit of $\{v_p\}_{p>1}$ as $p \searrow 1$. We divide the proof into several steps.

Step 1: BV-estimate. Let $B \subset \Omega^e$ be an open and bounded set and $0 \leq \varphi \in C_c^\infty(\Omega^e)$ such that $\varphi \equiv 1$ in B. Then we take φ^p as a test function in (4.3) and, by means of Young's inequality with $p' := \frac{p}{p-1}$, we get

$$\begin{aligned}\int_{\Omega^e} F^p(\nabla v_p)\varphi^p &= -p\int_{\Omega^e} F^{p-1}(\nabla v_p)\varphi^{p-1}\mathbf{z}_p \cdot \nabla\varphi \\ &\leq p\int_{\Omega^e} \big(F(\nabla v_p)\varphi\big)^{p-1} F^\circ(\mathbf{z}_p)F(\nabla\varphi) \\ &\leq \frac{p}{p'}\int_{\Omega^e} F^p(\nabla v_p)\varphi^p + \int_{\Omega^e} F^p(\nabla\varphi).\end{aligned}$$

Here we have used (2.3) and $F^\circ(\mathbf{z}_p) \le 1$, where the latter follows from (2.5). Consequently, for all $1 < p < 3/2$, we have

$$\int_B |\nabla v_p|^p \le \left(\frac{C}{c}\right)^p \frac{1}{2-p} \int_{\operatorname{supp}\varphi} |\nabla \varphi|^p \le 2\left(\frac{C}{c}\right)^{\frac{3}{2}} \int_{\operatorname{supp}\varphi} (1 + |\nabla\varphi|^{\frac{3}{2}}) \le \mathcal{C}, \tag{5.2}$$

where c, C are the constants from (2.1). Hereafter $\mathcal{C}$ denotes any constant which depends on B, c and C, but is independent of p, whose concrete meaning may change from line to line. Now Hölder's inequality leads to

$$\int_B |\nabla v_p| \le \mathcal{C} \qquad \text{for all} \quad 1 < p < 3/2.$$

Since, by (4.5), $\{v_p\}$ is pointwise bounded independently of p for $p \in (1, 3/2)$, we conclude that $\{v_p\}$ is bounded in $W^{1,1}(B)$ for every $B \subset \Omega^e$ open and bounded, and hence by the embedding theorem [3, Corollary 3.49], we get a function $v \in BV_{\mathrm{loc}}(\Omega^e)$ such that, up to subsequences and as $p \searrow 1$,

$$\begin{aligned} &v_p \to v \quad \text{in } L^q_{\mathrm{loc}}(\Omega^e) \quad \text{for every } 1 \le q < \frac{N}{N-1} \quad \text{and a.e. in} \ \ \Omega^e, \\ &\nabla v_p \llcorner B \rightharpoonup Dv \llcorner B \text{ weakly-}* \text{ as measures for each open bounded } B \subset \Omega^e. \end{aligned} \tag{5.3}$$

Notice that the behaviour of v as $|x| \to \infty$, as well as the estimates in (5.1), follow directly by taking limits in (4.5).

Step 2: Existence of z. The aim is to find $\mathbf{z} \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega^e)$ with $F^\circ(\mathbf{z}) \le 1$ and satisfying

$$F^{p-1}(\nabla v_p)\mathbf{z}_p \rightharpoonup \mathbf{z} \quad \text{weakly in } L^1_{\mathrm{loc}}(\Omega^e; \mathbb{R}^N) \quad \text{as } p \searrow 1. \tag{5.4}$$

First, for $B \Subset \Omega^e$ fixed, we show that the sequence $\{F^{p-1}(\nabla v_p)\mathbf{z}_p\}_{1<p<3/2}$ is equibounded in $L^1(B; \mathbb{R}^N)$. Indeed, by (2.1), (2.2), Hölder's inequality and (5.2), we have for $1 \le q < \frac{p}{p-1}$

$$\begin{aligned} \int_B \left|F^{p-1}(\nabla v_p)\mathbf{z}_p\right|^q &\le C^q \int_B F^{q(p-1)}(\nabla v_p) F^\circ(\mathbf{z}_p)^q \le C^{pq} \int_B |\nabla v_p|^{q(p-1)} \\ &\le C^{pq} \|\nabla v_p\|^{q(p-1)}_{L^p(B)} |B|^{\alpha(p)} \le \mathcal{C} \end{aligned}$$

with $\alpha(p) \to 1$ as $p \to 1$. Therefore, for any $1 \le q < \infty$, we conclude that there exists a subsequence so that

$$F^{p-1}(\nabla v_p)\mathbf{z}_p \rightharpoonup \mathbf{z}_B \quad \text{weakly in } L^q(B; \mathbb{R}^N).$$

Claim 1. $\mathbf{z}_B \in L^\infty(B; \mathbb{R}^N)$ *with* $F^\circ(\mathbf{z}_B) \le 1$.

Proof of Claim 1. We adapt to our setting an argument from the proof of [4, Lemma 1]. For any $k > 0$, define

$$B_{p,k} = \{x \in B \ : \ F(\nabla v_p(x)) > k\}.$$

By means of (5.2) and (2.1), we can estimate the measure of this set by

$$|B_{p,k}| \leq \int_{B_{p,k}} \frac{F^p(\nabla v_p(x))}{k^p} \leq \frac{\mathcal{C}}{k^p} \quad \text{for every } 1 < p < 3/2 \text{ and } k > 0. \tag{5.5}$$

Now,

$$F^{p-1}(\nabla v_p)\mathbf{z}_p \chi_{B_{p,k}} \rightharpoonup \mathbf{g}_k := \mathbf{z}_B\, \chi_{B_{p,k}} \qquad \text{weakly in } L^1(B;\mathbb{R}^n) \text{ as } p \searrow 1.$$

Choose $\Phi \in L^\infty(B;\mathbb{R}^N)$ with $F(\Phi) \leq 1$. Then, since $F^\circ(\mathbf{z}_p) \leq 1$ because $\mathbf{z}_p \in \partial F(\nabla v_p)$, Cauchy-Schwarz (2.3) and Hölder's inequality lead to

$$\left| \int_{B_{p,k}} F^{p-1}(\nabla v_p)\mathbf{z}_p \cdot \Phi \right| \leq C^{p-1} \big\|\nabla v_p\big\|^{p-1}_{L^p(B)} \big|B_{p,k}\big|^{1/p} \leq \frac{\mathcal{C}}{k},$$

where we used (5.2) and (5.5). Taking limits as $p \searrow 1$ and choosing $\Phi \propto \operatorname{sgn}(\mathbf{g}_k)$, we get

$$\int_B |\mathbf{g}_k| \leq \frac{\mathcal{C}}{k}, \quad \text{for every } k > 0.$$

On the other hand, we have that

$$F^\circ\big(F^{p-1}(\nabla v_p)\mathbf{z}_p\, \chi_{B\setminus B_{p,k}}\big) = F^{p-1}(\nabla v_p)\chi_{B\setminus B_{p,k}} F^\circ(\mathbf{z}_p) \leq k^{p-1}$$

for any $p > 1$. Accordingly,

$$F^{p-1}(\nabla v_p)\mathbf{z}_p\, \chi_{B\setminus B_{p,k}} \rightharpoonup \boldsymbol{f}_k := \mathbf{z}_B\, \chi_{B\setminus B_{p,k}} \qquad \text{weakly in } L^1(B;\mathbb{R}^N) \text{ as } p \searrow 1,$$

with $F^\circ(\boldsymbol{f}_k) \leq 1$. In short, for any $k > 0$, we can write $\mathbf{z}_{_B} = \mathbf{g}_k + \boldsymbol{f}_k$ so that

$$F^\circ(\mathbf{z}_{_B}) \leq F^\circ(\mathbf{g}_k) + F^\circ(\boldsymbol{f}_k) \leq 1 + o\big(\tfrac{1}{k}\big),$$

from which we conclude that $F^\circ(\mathbf{z}_{_B}) \leq 1$, as desired. □

Next, we choose an increasing sequence $\{B_n\}_{n\in\mathbb{N}}$ with $B_n \Subset \Omega^e$ such that $\cup_{n\in\mathbb{N}} B_n = \Omega^e$. One can perform a diagonal argument to establish that (up to a subsequence) (5.4) holds, where $\mathbf{z} : \Omega^e \to \mathbb{R}^N$ is a vector field such that $\mathbf{z}|_B = \mathbf{z}_B$ for each $B \Subset \Omega^e$ and, consequently, $\mathbf{z} \in L^\infty(\Omega^e;\mathbb{R}^N)$ with $F^\circ(\mathbf{z}) \leq 1$ by virtue of Claim 1.

It remains to check that $\mathbf{z} \in \mathcal{DM}^{\infty}_{\text{loc}}(\Omega^e)$. Now, (4.3) and (5.2) ensure that the sequence $\{\operatorname{div}(F^{p-1}(\nabla v_p)\mathbf{z}_p)\}$ is bounded in $L^1_{\text{loc}}(\Omega^e;\mathbb{R}^N)$. Therefore, we can find a Radon measure μ such that, as $p \searrow 1$, we have the subsequential convergence

$$\operatorname{div}(F^{p-1}(\nabla v_p)\mathbf{z}_p) \rightharpoonup \mu \quad \text{weakly-}* \text{ as measures.}$$

In particular, recalling also (5.4), we have that, for any open and bounded set $B \subset \Omega^e$ and $\varphi \in C^1_c(B)$,

$$\int_B F^{p-1}(\nabla v_p)\mathbf{z}_p \cdot \nabla\varphi \to -\int_B \varphi\, d\mu = \int_B \mathbf{z} \cdot \nabla\varphi,$$

thus $\mu = \operatorname{div}\mathbf{z}$. The latter implies that $\operatorname{div}\mathbf{z}$ has locally bounded total variation, as desired.

Step 3. $\operatorname{div}\mathbf{z} = (\mathbf{z}, Dv) = F(Dv)$ as measures. Take $0 \le \varphi \in C^\infty_c(\Omega^e)$ as a test function in (4.3) to have

$$\int_{\Omega^e} F^{p-1}(\nabla v_p)\mathbf{z}_p \cdot \nabla\varphi + \int_{\Omega^e} F^p(\nabla v_p)\varphi = 0.$$

On the other hand, Young's inequality yields

$$\begin{aligned}\int_{\Omega^e} F(\nabla v_p)\,\varphi &\le \frac{1}{p}\int_{\Omega^e} F^p(\nabla v_p)\varphi + \frac{1}{p'}\int_{\Omega^e}\varphi \\ &= -\frac{1}{p}\int_{\Omega^e} F^{p-1}(\nabla v_p)\mathbf{z}_p \cdot \nabla\varphi + \frac{p-1}{p}\int_{\Omega^e}\varphi.\end{aligned}$$

This inequality joint with lower semicontinuity (see Lemma 2.10 (b)) give

$$\int_{\Omega^e} \varphi\, dF(Dv) \le \liminf_{p\searrow 1}\int_{\Omega^e} F(\nabla v_p)\varphi \le -\int_{\Omega^e} \mathbf{z}\cdot\nabla\varphi,$$

which follows from (5.4). Hence,

$$-\operatorname{div}\mathbf{z} + F(Dv) \le 0. \tag{5.6}$$

To prove the reverse inequality, let us take $\varphi \in C^\infty_c(\Omega^e)$ and use $e^{-v_p}\varphi$ as a test function in (4.3). By means of (2.5), we reach

$$0 = \int_{\Omega^e} e^{-v_p} F^{p-1}(\nabla v_p)\mathbf{z}_p \cdot \nabla\varphi.$$

Since $e^{-v_p} \le 1$, by (5.3) and the dominated convergence theorem we have that e^{-v_p} converges to e^{-v} in $L^q(\operatorname{supp}\varphi)$ for all $1 \le q < \infty$. Then we may take limits as $p \searrow 1$, and get

$$0 = \int_{\Omega^e} e^{-v} \mathbf{z} \cdot \nabla \varphi,$$

where we applied (5.4). In particular, we have

$$\operatorname{div}(e^{-v}\mathbf{z}) = 0 \quad \text{in } \mathcal{D}'(\Omega^e). \tag{5.7}$$

Here we can use the product rule from Lemma 2.9 (a) and write

$$0 = e^{-v} \operatorname{div} \mathbf{z} + \big(\mathbf{z}, D(e^{-v})\big) \geq e^{-v} F(Dv) - F(De^{-v}),$$

where the inequality follows from (5.6), the Cauchy-Schwarz type inequality from Lemma 2.11 (a) and $F^\circ(\mathbf{z}) \leq 1$. By definition of the precise representative (cf. Lemma 2.3) and of $F(Dv)$ in (2.11), we can write

$$0 \geq e^{-v} F(Dv)^d + \frac{e^{-v^+} + e^{-v^-}}{2} F(\nu_v)|v^+ - v^-| \mathcal{H}^{N-1} \llcorner J_v - F(De^{-v}). \tag{5.8}$$

Notice that, as we are integrating over a compact subset by the choice of φ, we can apply the chain rule for BV functions (see [3, Theorem 3.99]) and the 1-homogeneity of F to reach

$$F(De^{-v}) = e^{-v} F(Dv)^d + \left| e^{-v^+} - e^{-v^-} \right| F(\nu_v) \mathcal{H}^{N-1} \llcorner J_v.$$

Substitution of the latter into (5.8) yields

$$0 \geq \left(\frac{e^{-v^+} + e^{-v^-}}{2} |v^+ - v^-| - \left| e^{-v^+} - e^{-v^-} \right| \right) F(\nu_v) \mathcal{H}^{N-1} \llcorner J_v. \tag{5.9}$$

Given $a \geq 0$, we let

$$f(h) = e^{-a} \left(\frac{h}{2}(e^{-h} + 1) + e^{-h} - 1 \right).$$

It holds that $f(h) \geq 0$ in $\mathbb{R}^+$ and $f(h) = 0$ if, and only if, $h = 0$. Therefore, taking $a = \min\{v^-(x), v^+(x)\}$ and $h = |v^+ - v^-|$, having in mind (5.9), we obtain that $v^- = v^+$ $\mathcal{H}^{N-1}$-a.e in J_v; i.e. $v \in DBV_{\mathrm{loc}}(\Omega^e)$. Consequently, from (5.7) and Lemma 3.4 (b1), we get

$$0 = \operatorname{div}(-e^{-v}\mathbf{z}) = -e^{-v} \operatorname{div} \mathbf{z} + e^{-v}(\mathbf{z}, Dv) \quad \text{in } \mathcal{D}'(\Omega^e),$$

which implies, by means of (5.6) and Lemma 2.11 (a), that

$$F(Dv) \leq \operatorname{div} \mathbf{z} = (\mathbf{z}, Dv) \leq F(Dv) \quad \text{in } \mathcal{D}'(\Omega^e).$$

Finally, an approximation argument ensures

$$\operatorname{div} \mathbf{z} = (\mathbf{z}, Dv) = F(Dv), \quad \text{as measures.}$$

Step 4. Boundary condition. The aim is to show that $v = 0$ on $\partial\Omega$ in the sense of traces. Take $R > 0$ big enough so that $\Omega \subset B_R(0)$ and take ζv_p as a test function in (4.3), where $\zeta \in C_c^\infty(\mathbb{R}^N)$ given by

$$0 \le \zeta \le 1 \qquad \zeta(x) = \begin{cases} 1, & \text{if } |x| \le R \\ 0, & \text{if } |x| \ge 2R. \end{cases}$$

Doing so, we obtain

$$\int_{\Omega^e} F^p(\nabla v_p) v_p \zeta = -\int_{\Omega^e} F^p(\nabla v_p)\zeta - \int_{\Omega^e} F^{p-1}(\nabla v_p) v_p \, \mathbf{z}_p \cdot \nabla\zeta.$$

From here, by means of Young's inequality, we reach

$$\begin{aligned} &\int_{\Omega^e} \zeta F(\nabla v_p) + \int_{\Omega^e} F(\nabla v_p) v_p \zeta \\ &\qquad \le \frac{1}{p}\int_{\Omega^e} \zeta F^p(\nabla v_p) + \frac{1}{p'}\int_{\Omega^e} \zeta + \frac{1}{p}\int_{\Omega^e} v_p \zeta F^p(\nabla v_p) + \frac{1}{p'}\int_{\Omega^e} v_p\zeta \\ &\qquad = \frac{p-1}{p}\int_{\Omega^e} (1+v_p)\zeta - \frac{1}{p}\int_{\Omega^e} v_p F^{p-1}(\nabla v_p)\mathbf{z}_p \cdot \nabla\zeta. \end{aligned} \tag{5.10}$$

Set $\Omega_R := B_{2R} \setminus \overline{\Omega}$ and recall that v_p vanishes on $\partial\Omega$. Then, by lower semicontinuity (see [34, Theorem 5 and 7]), we can write

$$\int_{\Omega_R} \zeta dF(Dv) + \int_{\partial\Omega} vF(\nu_v) \le \liminf_{p \searrow 1} \int_{\Omega_R} \zeta F(\nabla v_p),$$

as well as

$$\int_{\Omega_R} \frac{\zeta}{2}\, dF\left(Dv^2\right) + \int_{\partial\Omega} \frac{v^2}{2} F(\nu_v) \le \liminf_{p \searrow 1} \int_{\Omega_R} \zeta v_p F(\nabla v_p).$$

Adding these inequalities up and taking limits in (5.10), we have

$$\int_{\Omega_R} \zeta dF(Dv) + \int_{\partial\Omega} F(\nu_v)v + \int_{\Omega_R} \zeta v\, dF(Dv) + \int_{\partial\Omega} F(\nu_v)\frac{v^2}{2} \le -\int_{\Omega_R} v\,\mathbf{z}\cdot\nabla\zeta, \tag{5.11}$$

where we also applied (5.4).

On the other hand, if we combine the product rule from Lemma 2.9 and the generalized Green's formula (2.9), we reach

$$-\int_{\Omega_R} v\,\mathbf{z}\cdot\nabla\zeta = \int_{\Omega_R} \zeta v\, d(\operatorname{div}\mathbf{z}) + \int_{\Omega_R} \zeta\, d(\mathbf{z}, Dv) - \int_{\partial\Omega} v[\mathbf{z},\nu_v]$$
$$= \int_{\Omega_R} \zeta v\, dF(Dv) + \int_{\Omega_R} \zeta dF(Dv) - \int_{\partial\Omega} v[\mathbf{z},\nu_v],$$

which follows from the equalities in Step 3.

Accordingly, by substitution of the latter into (5.11), all the integrals on Ω_R cancel out, and we conclude

$$\int_{\partial\Omega} \big(F(\nu_v)v + v[\mathbf{z},\nu_v]\big) + \frac{1}{2}\int_{\partial\Omega} F(\nu_v)v^2 \leq 0.$$

Notice that $v[\mathbf{z},\nu_v] \geq -vF(\nu_v)\|F^\circ(\mathbf{z})\|_\infty \geq -vF(\nu_v)$ by Lemma 2.11(b), and hence on the left hand side we have the sum of two non-negative quantities. Thus we get that v vanishes on $\partial\Omega$ in the sense of traces, as desired. □

6. Extra regularity for domains satisfying a uniform interior ball condition

We recall that, if Ω satisfies the $\mathcal{W}_r$-condition stated in Definition 1.3, then solutions to the approximating problems (1.5) are Lipschitz continuous ([12, Theorem 1.5]). Accordingly, it is natural to wonder if for the limiting problem (1.7) we also get additional regularity in case that one can slide a Wulff shape of uniform size along the boundary of Ω.

In this section we show that, under these circumstances, the solution to (1.7) is actually continuous and satisfies a Harnack inequality. The strategy to prove this closely follows that of the corresponding statements for the Euclidean IMCF with obstacles ([36, Theorem 2.5 and Proposition 2.7]).

Theorem 6.1. *Let $r > 0$ and suppose that Ω satisfies the $\mathcal{W}_r$-condition. Then the unique solution of (1.7) is continuous.*

Proof. Let $x_0 \in \partial\Omega$. Then, there exists $y_0 \in \mathbb{R}^N$ such that

$$\mathcal{W}_r + y_0 \subseteq \overline{\Omega} \quad \text{and} \quad x_0 \in \partial\left(\mathcal{W}_r + y_0\right).$$

Define

$$\eta_p(x) := \left(\frac{F^\circ(x-y_0)}{r}\right)^{\frac{p-N}{p-1}}.$$

Then η_p is a solution of (1.8) with $\eta_p \leq 1$ on $\partial\Omega$. Therefore, by Theorem 4.8, $\eta_p \leq u_p$ in Ω^e. In particular,

$$v_p(x) = (1-p)\log u_p(x) \leq (N-p)\log\left(\frac{F^\circ(x-y_0)}{r}\right) \quad \text{in } \Omega^e.$$

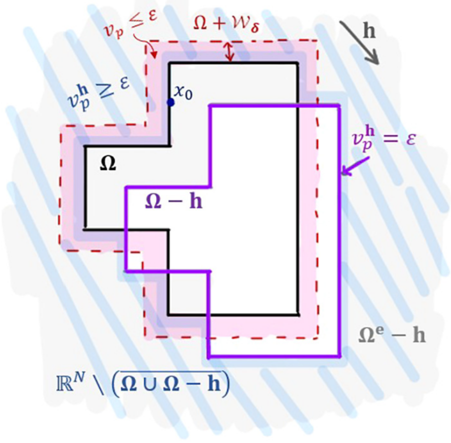

Fig. 1. Different regions for the comparison arguments.

Then, for any $\delta > 0$ and $x \in (\mathcal{W}_\delta + x_0) \cap \Omega^e$, we have

$$\begin{aligned} v_p(x) &\leq (N-p)\log\left(\frac{F^\circ(x-x_0)}{r} + \frac{F^\circ(x_0-y_0)}{r}\right) \\ &\leq (N-p)\log\left(\frac{\delta}{r} + 1\right). \end{aligned}$$

Consequently, for any $\varepsilon > 0$ we may choose $\delta > 0$ small enough, depending on r, ε and N (but independent of x_0), such that

$$v_p(x) \leq \varepsilon \quad \text{for } x \in (\mathcal{W}_\delta + x_0) \cap \Omega^e.$$

For $|\mathbf{h}| < \delta$, define

$$v_p^{\mathbf{h}}(x) := v_p(x + \mathbf{h}) + \varepsilon \quad \text{for} \quad x \in \Omega - \mathbf{h}.$$

Then (see Fig. 1),

$$v_p^{\mathbf{h}} = \varepsilon \;\text{ on }\; \partial(\Omega - \mathbf{h}) \qquad \text{and} \qquad v_p^{\mathbf{h}} \geq \varepsilon \;\text{ on }\; \partial\Omega \cap (\Omega^e - \mathbf{h}).$$

Now, v_p and $v_p^{\mathbf{h}}$ are two solutions of (4.6) on the exterior domain $\mathbb{R}^N \setminus \overline{E}$, with $E = \Omega \cup (\Omega - \mathbf{h})$, so that $v_p \leq v_p^{\mathbf{h}}$ on ∂E. Therefore, by Theorem 4.6 we get that

$$v_p \leq v_p^{\mathbf{h}} \quad \text{on} \quad \mathbb{R}^N \setminus \overline{E} \supset \mathbb{R}^N \setminus \overline{(\Omega + \tfrac{1}{c}\mathcal{W}_\delta)}.$$

Since the same is true for $-\mathbf{h}$ instead of $\mathbf{h}$, we conclude that $|v_p(x+\mathbf{h}) - v_p(x)| \leq \varepsilon$ for every $|\mathbf{h}| < \delta$ and $x \in \mathbb{R}^N \setminus \overline{(\Omega + \frac{1}{c}\mathcal{W}_\delta)}$. That is, the v_p are equicontinuous outside of any neighbourhood of Ω. In particular, they are equicontinuous on any compact subset of Ω^e, thus the Arzelà-Ascoli theorem ensures that v is continuous on Ω^e. $\square$

Theorem 6.2. *Let v the unique solution to (1.7) for a given domain Ω which satisfies a $\mathcal{W}_s$ condition for some $s > 0$. There exists $\mathcal{C} > 0$ such that, if $x_0 \in \Omega^e$ and $\rho > 0$ satisfy $\mathcal{W}_{2\rho}(x_0) \subset \Omega^e$, then*

$$\operatorname{osc}_{\mathcal{W}_\rho(x_0)} v \leq \mathcal{C}.$$

Proof. Assume that $p \in \left(1, \frac{3}{2}\right]$. Let $u_p \in W^{1,\infty}(\Omega^e)$ be a weak solution of (1.5). Given $x_0 \in \Omega^e$, let $\rho > 0$ be such that $\mathcal{W}_{2\rho}(x_0) \subset \Omega^e$. Since the equation in (1.5) is invariant under translations and dilations, we may suppose that $x_0 = 0$ and $\rho = 1$.

For $q \in \mathbb{R} \setminus \{0\}$ and $r \in (0, 2]$ we denote

$$I(q, r) := \left(\fint_{\mathcal{W}_r} u_p^q \right)^{\frac{1}{q}}.$$

Now, by Hölder's inequality, if $q_1 < q_2$,

$$I(q_1, r) \leq \frac{1}{|\mathcal{W}_r|^{\frac{1}{q_1}}} \left(\int_{\mathcal{W}_r} u_p^{q_2} \right)^{\frac{1}{q_2}} |\mathcal{W}_r|^{\frac{q_2 - q_1}{q_1 q_2}} = I(q_2, r),$$

thus $I(q, r)$ is non-decreasing in q. Moreover, since u_p is continuous,

$$I(\infty, r) = \lim_{q \to \infty} I(q, r) = \sup_{\mathcal{W}_r} u_p$$

and

$$I(-\infty, r) = \lim_{q \to -\infty} I(q, r) = \inf_{\mathcal{W}_r} u_p.$$

Let $v_p := (1 - p) \log u_p$. Following the proof of Proposition 4.4, we get that v_p satisfies

$$\operatorname{div}\left(F^{p-1}(\nabla v_p) \partial F(\nabla v_p) \right) = F^p(\nabla v_p) \quad \text{in } \mathcal{W}_2$$

in the weak sense; that is, there exists $\mathbf{z} \in \partial F(\nabla v_p)$ such that

$$- \int_{\mathcal{W}_2} F^{p-1}(\nabla v_p) \mathbf{z} \cdot \nabla \varphi = \int_{\mathcal{W}_2} F^p(\nabla v_p) \varphi,$$

for every $\varphi \in L^\infty(\mathcal{W}_2)$ with compact support in $\mathcal{W}_2$ whose distributional gradient $\nabla \varphi$ belongs to $L^p(\mathcal{W}_2; \mathbb{R}^N)$. Now, for any such φ, by (2.3) and Hölder's inequality we have

$$\begin{aligned}
\int_{\mathcal{W}_2} F^p(\nabla v_p) \varphi^p &= - \int_{\mathcal{W}_2} p \varphi^{p-1} F^{p-1}(\nabla v_p) \mathbf{z} \cdot \nabla \varphi \\
&\leq \left(\int_{\mathcal{W}_2} F^p(\nabla v_p) \varphi^p \right)^{\frac{p-1}{p}} \left(\int_{\mathcal{W}_2} p^p F^p(\nabla \varphi) \right)^{\frac{1}{p}}.
\end{aligned}$$

Thus

$$\int_{\mathcal{W}_2} F^p(\nabla v_p)\varphi^p \leq p^p \int_{\mathcal{W}_2} F^p(\nabla\varphi).$$

In particular, for any $y_0 \in \mathcal{W}_2$ and r such that $\mathcal{W}_{2r}(y_0) \subseteq \mathcal{W}_2$, we may appropriately choose φ to obtain

$$\fint_{\mathcal{W}_r(y_0)} F^p(\nabla v_p) \leq \frac{2^N p^p}{r^p}.$$

Then, the Poincaré inequality for anisotropic balls (proved as in [20, Section 5.8 Theorem 2]) yields

$$\fint_{\mathcal{W}_r(y_0)} |v_p(x) - \overline{v_p}^r|\, dx \leq \mathcal{C}, \quad \text{with} \quad \overline{v_p}^r = \fint_{\mathcal{W}_r(y_0)} v_p(y)dy.$$

Hereafter $\mathcal{C}$ denotes any constant depending only on N. Then, by [25], there exists $\alpha > 0$ independent of N such that

$$\left(\fint_{\mathcal{W}_{\frac{3}{2}}} e^{\alpha v_p}\right)\left(\fint_{\mathcal{W}_{\frac{3}{2}}} e^{-\alpha v_p}\right) \leq \left(\fint_{\mathcal{W}_{\frac{3}{2}}} e^{\alpha|v_p - \overline{v_p}^{3/2}|}\right)^2 \leq \mathcal{C}.$$

That is,

$$I\left((p-1)\alpha, 3/2\right) \leq \mathcal{C}^{\frac{1}{p-1}} I\left((1-p)\alpha, 3/2\right). \tag{6.1}$$

Now, by the Sobolev inequality, for any $s \in \mathbb{R}$ and non-negative function $\varphi \in C_0^1(\mathcal{W}_2)$, we have

$$\begin{aligned}\left(\int_{\mathcal{W}_2} \varphi^{p^*} u_p^{sp^*}\right)^{\frac{1}{p^*}} &\leq \mathcal{C}\left(\int_{\mathcal{W}_2} F^p(\nabla(\varphi u_p^s))\right)^{\frac{1}{p}} \\ &\leq \mathcal{C}\left(\int_{\mathcal{W}_2} F^p(\nabla\varphi) u_p^{sp}\right)^{\frac{1}{p}} + \mathcal{C}|s|\left(\int_{\mathcal{W}_2} \varphi^p u_p^{sp-p} F^p(\nabla u_p)\right)^{\frac{1}{p}},\end{aligned} \tag{6.2}$$

where $p^* = \frac{Np}{N-p}$. Moreover, if $s < 0$, and as by (4.4) $-\mathbf{z} \in \partial F(\nabla u_p)$, then

$$\begin{aligned}\int_{\mathcal{W}_2} \varphi^p u_p^{sp-p} F^p(\nabla u_p) &= \int_{\mathcal{W}_2} \varphi^p u_p^{sp-p} F^{p-1}(\nabla u_p)(-\mathbf{z}) \cdot \nabla u_p \\ &= \frac{1}{sp-p+1} \int_{\mathcal{W}_2} \varphi^p F^{p-1}(\nabla u_p)(-\mathbf{z}) \cdot \nabla u_p^{sp-p+1} \\ &= \frac{p}{|sp-p+1|} \int_{\mathcal{W}_2} \varphi^{p-1} u_p^{sp-p+1} F^{p-1}(\nabla u_p)(-\mathbf{z}) \cdot \nabla\varphi,\end{aligned}$$

which follows by taking $u_p^{sp-p+1}\varphi^p$ as a test function in (4.1). Now using (2.3) and $F^\circ(-\mathbf{z}) \le 1$ it holds

$$\int_{\mathcal{W}_2} \varphi^p u_p^{sp-p} F^p(\nabla u_p) \le \frac{p}{|sp-p+1|}\left(\int_{\mathcal{W}_2} \varphi^p u_p^{sp-p} F^p(\nabla u_p)\right)^{\frac{1}{p'}} \left(\int_{\mathcal{W}_2} F^p(\nabla\varphi) u_p^{sp}\right)^{\frac{1}{p}}.$$

Therefore,

$$\int_{\mathcal{W}_2} \varphi^p u_p^{sp-p} F^p(\nabla u_p) \le \frac{p^p}{|sp-p+1|^p}\int_{\mathcal{W}_2} F^p(\nabla\varphi) u_p^{sp},$$

which, together with (6.2), gives

$$\left(\int_{\mathcal{W}_2} \varphi^{p^*} u_p^{sp^*}\right)^{\frac{1}{p^*}} \le \mathcal{C}\left(1+\frac{|s|p}{|sp-p+1|}\right)\left(\int_{\mathcal{W}_2} F^p(\nabla\varphi) u_p^{sp}\right)^{\frac{1}{p}}.$$

Now, for $1 \le r < R \le 2$, choosing φ appropriately we get

$$\left(\int_{\mathcal{W}_r} u_p^{sp^*}\right)^{\frac{1}{p^*}} \le \frac{\mathcal{C}}{R-r}\left(1+\frac{|s|p}{|sp-p+1|}\right)\left(\int_{\mathcal{W}_R} u_p^{sp}\right)^{\frac{1}{p}}.$$

Hence, letting $\theta := N/(N-p)$ and noticing that $\frac{|s|p}{|sp-p+1|} \le 1$ for $s < 0$,

$$I(\theta sp, r)^s = I(sp^*, r)^s \le \frac{2\,\mathcal{C}}{R-r} I(sp, R)^s. \tag{6.3}$$

Set $s_0 := (\alpha(1-p))/p$ and $r_0 := 3/2$. For $k \in \mathbb{N}$, let $s_k := \theta^k s_0$ and $r_k = 1 + 2^{-(k+1)}$. Letting $s = s_k$, $r = r_{k+1}$ and $R = r_k$ in (6.3), since $s_{k+1} = \theta s_k$, $r_k - r_{k+1} = 2^{-(k+2)}$ and $s_k < 0$, we obtain

$$I(s_{k+1}p, r_{k+1}) \ge \mathcal{C}^{\frac{1}{s_k}} 2^{\frac{k+3}{s_k}} I(s_k p, r_k)$$

for every k. Consequently,

$$\begin{aligned} I((1-p)\alpha, 3/2) = I(s_0 p, r_0) &\le 2^{-\frac{3}{s_0}} \mathcal{C}^{-\frac{1}{s_0}} I(s_0 p, r_1) \\ &\le 2^{-\frac{3}{s_0}-\frac{4}{s_1}} \mathcal{C}^{-\frac{1}{s_0}-\frac{1}{s_1}} I(s_2 p, r_2) \le \cdots \le 2^\gamma \mathcal{C}^\beta I(-\infty, 1), \end{aligned}$$

where

$$\beta = -\sum_{k=0}^{\infty} \frac{1}{s_k} = \frac{N}{\alpha(p-1)}$$

and, as $p < N$,

$$\gamma = -\sum_{k=0}^{\infty} \frac{k+3}{s_k} = \frac{N^2+2Np}{\alpha p(p-1)} \le \frac{3N}{\alpha(p-1)}.$$

That is,

$$I((1-p)\alpha, 3/2) \le \mathcal{C}^{\frac{1}{p-1}} I(-\infty, 1),$$

which, together with (6.1), gives

$$I((p-1)\alpha, 3/2) \le e^{\frac{\mathcal{C}}{p-1}} I(-\infty, 1).$$

This is equivalent to

$$\sup_{\mathcal{W}_1} v_p \le \mathcal{C} - \frac{1}{\alpha} \log\left(\fint_{\mathcal{W}_{\frac{3}{2}}} e^{-\alpha v_p} \right).$$

We want to consider positive values of s too. However, the factor

$$g(s) := \frac{|s|p}{|sp-p+1|}$$

appearing in the previous computations is unbounded near $\frac{1}{p'} = \frac{p-1}{p}$. Therefore, set

$$\sigma_0 = \frac{2N-p}{2Np'}$$

so that

$$\sigma_0 < \frac{1}{p'} < \sigma_1 = \theta\sigma_0 = \frac{2N-p}{2(N-p)p'}$$

and $g(\sigma_0) = g(\sigma_1) = 2N/p - 1$. Hence, for any $l \in \mathbb{Z}$ and $s = \theta^l \sigma_0$ we have

$$g(s) \le \frac{2N}{p} - 1.$$

Let $k_0 \in \mathbb{Z}$ such that

$$\theta^{k_0}\sigma_0 \le \frac{\alpha}{p'} < \theta^{k_0+1}\sigma_0.$$

For $s_k := \theta^{k_0+k}\sigma_0$ and $r_k := 1 + 2^{-(k+1)}$, we may repeat the arguments above to get

$$\sup_{\mathcal{W}_1} u_p = I(\infty, 1) \le e^{\frac{\mathcal{C}}{p-1}} I(-\infty, 1) = e^{\frac{\mathcal{C}}{p-1}} \inf_{\mathcal{W}_1} u_p.$$

This is equivalent to

$$\mathrm{osc}_{\mathcal{W}_1} v_p \le \mathcal{C}.$$

Since, by the proof of Theorem 6.1, v_p converges (up to a subsequence, not relabelled) locally uniformly to v, we obtain that

$$\mathrm{osc}_{\mathcal{W}_1} v \le \mathcal{C}. \quad \square$$

7. About the anisotropic perimeter of sublevel sets

Concerning notation, all the sets appearing in this and the subsequent sections are considered to be Borel measurable subsets of $\mathbb{R}^N$.

7.1. Anisotropic perimeter, densities and coarea formula

A set E within an open subset $U \subset \mathbb{R}^N$ is said to be of finite perimeter in U if $\chi_E \in BV(U)$. In this case, $P_F(E;U)$ denotes the anisotropic perimeter (or F-perimeter, for brevity) of E in U, and is defined as

$$P_F(E;U) := \big|D\chi_E\big|_F(U). \tag{7.1}$$

If $U = \mathbb{R}^N$ we simply write $P_F(E) = P_F(E;\mathbb{R}^N)$. We say that E has finite F-perimeter in U if $P_F(E;U) < \infty$. In turn, E has locally finite F-perimeter in U if, for every open set $B \Subset U$, it holds that $P_F(E;B) < \infty$. By (2.1), the finiteness of Euclidean and anisotropic perimeter are equivalent.

If E has locally finite perimeter, it makes sense to consider the reduced boundary

$$\partial^* E = \Big\{ x \in \mathbb{R}^N : \lim_{r \searrow 0} \frac{D\chi_E(B_r(x))}{\big|D\chi_E\big|(B_r(x))} =: \nu_E(x) \text{ exists and belongs to } \mathbb{S}^{N-1} \Big\},$$

where $\nu^E(x)$ is known as the measure-theoretic normal to E at x, and clearly $\nu_E = \nu_{\chi_E}$. Then by de Giorgi structure theorem (see e.g. [31, Theorem 5.9]) and (2.11), it is known that

$$P_F(E;U) = \int_{U \cap \partial^* E} F(\nu^E)\, d\mathcal{H}^{N-1}. \tag{7.2}$$

Recall that, for $\alpha \in [0,1]$, the set of points of density α in E is given by

$$E^\alpha := \Big\{ x \in \mathbb{R}^N : \lim_{r \searrow 0} \frac{\big|B_r(x) \cap E\big|}{|B_r(x)|} = \alpha \Big\}.$$

Typically, E^1 is known as measure theoretic interior of E. Hereafter $E\Delta A := (E \setminus A) \cup (A \setminus E)$ denotes the symmetric difference of the sets E and A.

Lemma 7.1. *If E is a set of finite perimeter, the following properties hold:*

(a) ***(Federer)*** *[3, Theorem 3.61] $\partial^* E \subset E^{1/2}$. Moreover, E has density either 0 or 1/2 or 1 $\mathcal{H}^{N-1}$-a.e. in $\mathbb{R}^N$.*

(b) *[31, Example 5.17 and Remark 15.2] $\left|E\Delta E^1\right| = 0$. Consequently, $P_F(E) = P_F(E^1)$ and $\partial^* E = \partial^* E^1$.*

The goal of this section is to deal with the sublevel sets $E_t, G_t \subset \Omega^e$ of solutions of A-IMCF, defined as in (1.10), and show that their F-perimeters are finite and coincide.

Notice that we can extend the solution v of (3.1) to be identically zero within $\overline{\Omega}$ (in fact, the boundary condition in (3.2) guarantees that this is well-defined and hence the extended function belongs to $DBV_{\mathrm{loc}}(\mathbb{R}^N) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$). We will use the extension when needed without further mention, but the corresponding vector field $\mathbf{z}$ will not be extended.

Accordingly, G_t and E_t are redefined as

$$G_t = \{x : \bar{v}(x) \le t\} \quad \text{for} \quad \bar{v}(x) = \begin{cases} v(x), & \text{if } x \in \Omega^e, \\ 0, & \text{if } x \in \Omega, \end{cases}$$

so that $\overline{\Omega} \subset G_t$ for all $t \ge 0$, and similarly $E_t = \{x : \bar{v}(x) < t\}$.

Remark 7.2. Observe that, as the measure $|Dv|_F \llcorner \{v = t\}$ vanishes and v has no jump, by the previous extension, it also holds

$$|Dv|_F(\{v = 0\}) = |Dv|_F(G_0) = 0. \tag{7.3}$$

As the solution v from Theorem 5.1 belongs to $BV_{\mathrm{loc}}(\mathbb{R}^N)$, the anisotropic version of the coarea formula [1, Remark 4.4] yields

$$|Dv|_F(U) = \int_{\mathbb{R}} P_F(\{v > t\}; U)\, dt \quad \text{for every open set } U \subset \mathbb{R}^N. \tag{7.4}$$

In particular, this ensures that the measures $\int_{\mathbb{R}} \left|D\chi_{\{v>t\}}\right|_F(A)\, dt$ and $|Dv|_F(A)$ agree for any open subset A of U, and thus coincide. Hence, for any $B \subset U$,

$$|Dv|_F(B) = \int_{\mathbb{R}} \left|D\chi_{\{v>t\}}\right|_F(B)\, dt. \tag{7.5}$$

Notice that (7.4) guarantees that the superlevel sets $\{v > t\}$, and thus G_t have locally finite F-perimeter for a.e. $t \ge 0$. On the other hand, since v tends to ∞ as $|x| \to \infty$, we can find $r(t) > 0$ such that

$$G_t \subset B_{r(t)}(0) \tag{7.6}$$

and hence the sublevel sets G_t have finite F-perimeter for a.e. $t > 0$. The goal is to show that they actually have finite F-perimeter for all $t > 0$.

7.2. Reduced boundary of the sublevel sets

We need an auxiliary result about the reduced boundary of the sublevel sets. Let us first prove it for a.e. $t \geq 0$.

Lemma 7.3. *Let $t \geq 0$ be such that G_t has finite F-perimeter. Then*

$$v = t \quad \mathcal{H}^{N-1}\text{-a.e. in } \partial^* G_t.$$

Therefore, given $0 \leq \tau < t$ such that G_τ and G_t have finite F-perimeter, $\mathcal{H}^{N-1}(\partial^* G_t \cap \partial^* G_\tau) = 0$; in particular, $\mathcal{H}^{N-1}(\partial^* G_t \cap \partial^* \Omega) = 0$, leading to

$$P_F(G_t; \Omega^e) = P_F(G_t). \tag{7.7}$$

Proof. Since $\mathcal{H}^{N-1}(S_v) = 0$ as there is no jump part (cf. Lemma 3.3), and by Lemma 7.1 (a), it is enough to prove the claims for Lebesgue points of density $1/2$, that is, for x so that

$$\lim_{r \searrow 0} \frac{\big|B_r(x) \cap G_t\big|}{|B_r(x)|} = \frac{1}{2} \quad \text{and} \quad \lim_{r \searrow 0} \frac{\big|B_r(x) \cap \{v > t\}\big|}{|B_r(x)|} = \frac{1}{2}. \tag{7.8}$$

The first equality leads to

$$\lim_{r \searrow 0} \fint_{B_r(x) \cap G_t} |v - v(x)| \leq 2 \lim_{r \searrow 0} \fint_{B_r(x)} |v - v(x)| = 0,$$

where the latter follows because x is a Lebesgue point of v (recall Definition 2.2). In particular,

$$v(x) = \lim_{r \searrow 0} \fint_{B_r(x) \cap G_t} v \leq t.$$

By an analogous argument but with the second equality in (7.8), we reach

$$v(x) = \lim_{r \searrow 0} \fint_{B_r(x) \cap \{v > t\}} v \geq t,$$

which finishes the proof of the first claim. Now (7.7) follows directly from the fact that $\partial G_t \cap \overline{\Omega} = \emptyset$. $\square$

7.3. Anisotropic perimeter of the sublevel sets

Lemma 7.4. *Let v be a weak solution in the sense of Definition 3.1. Then the sublevel sets G_t have finite anisotropic perimeter for every $t \geq 0$.*

Proof. By the general coarea formula in [17, Theorem 4.2] and equation (3.1), we can write, for any set $B \subset \Omega^e$,

$$\int_{\mathbb{R}} (\mathbf{z}, D\chi_{\{v>t\}})(B)\, dt = (\mathbf{z}, Dv)(B) = |Dv|_F(B) = \int_{\mathbb{R}} \left|D\chi_{\{v>t\}}\right|_F(B)\, dt,$$

where the latter follows by (7.5). From here, since $-(\mathbf{z}, D\chi_{G_t}) \leq \left|D\chi_{G_t}\right|_F$, we have the equality

$$-(\mathbf{z}, D\chi_{G_t}) = \left|D\chi_{G_t}\right|_F \quad \text{as measures in } \Omega^e \quad \text{for a.e. } \; t > 0. \tag{7.9}$$

Claim 1. $P_F(G_t) - |Dv|_F(G_t) = \Lambda$, where Λ is constant, for a.e. $t > 0$.

Take two times $T > \tau > 0$ satisfying (7.9) and so that G_T and G_τ have finite F-perimeter, and hence (7.7) holds. Now by substituting (7.9) in (7.1) and applying Green's formula in $B_{r(T)+1}(0) \cap \Omega^e$, where $r(T)$ is the radius from (7.6), we get

$$\begin{aligned} P_F(G_T) - P_F(G_\tau) &= -\int_{\Omega^e} (\mathbf{z}, D\chi_{G_T}) + \int_{\Omega^e} (\mathbf{z}, D\chi_{G_\tau}) \\ &= -\int_{\Omega^e \cap B_{r(T)+1}(0)} (\mathbf{z}, D\chi_{\{\tau < v \leq T\}}) \\ &= \int_{\Omega^e \cap B_{r(T)+1}(0)} \chi^*_{\{\tau < v \leq T\}}\, d(\operatorname{div} \mathbf{z}) = \int_{\{\tau < v \leq T\}} d(\operatorname{div} \mathbf{z}), \end{aligned}$$

where for the last equality we use that $\chi^*_{\{\tau<v<T\}} = 1$ on $\{\tau < v < T\}$ and Corollary 3.5. Hence equation (3.1) yields

$$\begin{aligned} |Dv|_F(G_T) - |Dv|_F(G_\tau) &= \int_{\{\tau<v\leq T\}} d|Dv|_F = \int_{\{\tau<v\leq T\}} d(\operatorname{div} \mathbf{z}) \\ &= P_F(G_T) - P_F(G_\tau). \end{aligned}$$

Thus the claim follows for a.e. $t > 0$ because we argued for any pair of times satisfying (7.9).

Claim 2. $\dfrac{d^+}{dt}|Dv|_F(G_t) = \Lambda + |Dv|_F(G_t)$ for every $t \geq 0$, where $\frac{d^+}{dt}$ denotes the right derivative.

First, notice that by (7.5) we have

$$|Dv|_F(G_t) = \int_0^t \int_{\mathbb{R}^N} \left|D\chi_{\{v>s\}}\right|_F ds = \int_0^t \int_{\mathbb{R}^N} \left|D\chi_{\{v\leq s\}}\right|_F ds = \int_0^t P_F(G_s)\, ds.$$

In particular,

$$t \mapsto |Dv|_{F}(G_t) \text{ is absolutely continuous on each bounded interval.} \tag{7.10}$$

Moreover, for any $h > 0$ and any $t \geq 0$, Claim 1 leads to

$$\frac{|Dv|_{F}(G_{t+h}) - |Dv|_{F}(G_t)}{h} = \frac{1}{h}\int_t^{t+h} P_{F}(G_s)\,ds = \Lambda + \frac{1}{h}\int_t^{t+h} |Dv|_{F}(G_s)\,ds.$$

Taking limits as $h \to 0^+$, this implies that there exists the right derivative and the claimed equality follows.

Claim 3. $P_{F}(G_t) \leq \dfrac{d^+}{dt}|Dv|_{F}(G_t)$ for every $t \geq 0$.

Fix $t \geq 0$ and $h > 0$, for which we consider the Lipschitz continuous function $\eta : \mathbb{R} \to [0,1]$ given by

$$\eta(s) = \begin{cases} 1, & \text{if } s \leq t, \\ 1 + \dfrac{t-s}{h}, & \text{if } s \in [t, t+h], \\ 0, & \text{if } s \geq t+h. \end{cases} \tag{7.11}$$

Notice that, for $s \neq t, t+h$, it holds $\eta'(s) = -\frac{1}{h}\chi_{(t,t+h)}(s)$. Using this, together with Corollary 3.5, we reach

$$\begin{aligned} \frac{|Dv|_{F}(G_{t+h}) - |Dv|_{F}(G_t)}{h} &= \frac{1}{h}\int_{\{t<v<t+h\}} dF(Dv) = \int_{\mathbb{R}^N} -\eta'(v)dF(Dv) \\ &= \int_{\mathbb{R}^N} dF(D(\eta \circ v)) \geq \int_{\mathbb{R}^N} \eta(v)\operatorname{div}\mathbf{w}, \end{aligned}$$

for any test function $\mathbf{w} \in C_c^1(\mathbb{R}^N; \mathbb{R}^N)$ with $F^\circ(\mathbf{w}) \leq 1$. Notice that the first equality in the second line follows from the chain rule in Lemma 3.4 (a), while the last inequality comes from (2.12).

Accordingly, taking limits as $h \to 0^+$, we have that

$$\frac{d^+}{dt}|Dv|_{F}(G_t) \geq \int_{\mathbb{R}^N} \chi_{G_t}\operatorname{div}\mathbf{w}$$

and, taking the supremum over all such vector fields $\mathbf{w}$,

$$\frac{d^+}{dt}|Dv|_{F}(G_t) \geq \left|D\chi_{G_t}\right|_{F}(\mathbb{R}^N) = P_{F}(G_t),$$

thus the claimed inequality follows.

Summing up, if we join Claims 2 and 3, by means of (7.6) one can conclude

$$P_{_F}(G_t) \leq \Lambda + |Dv|_{_F}(G_t) < \infty \quad \text{for every } t \geq 0. \quad \square$$

Remark 7.5. Now the conclusions of Lemma 7.3 are valid for all times. Moreover, as we are interested in properties involving the F-perimeter, which are invariant under modifications of 0-Lebesgue measure by Lemma 7.1, hereafter we will identify any finite perimeter set (in particular, G_t for any $t \geq 0$) with its measure theoretic interior.

Lemma 7.6. *Let v be the weak solution from Definition 3.1. Then the sublevel sets E_t have finite anisotropic perimeter $P_{_F}$ for every $t > 0$.*

Proof. Arguing exactly as before, but for $h < 0$, one gets that, for the left derivative and the same constant from Claim 1 above, it holds

$$\frac{d^-}{dt}|Dv|_{_F}(G_t) = \Lambda + |Dv|_{_F}(G_t).$$

One can also mimic the proof of Claim 3, but using the following function instead of the one introduced in (7.11):

$$\xi(s) = \begin{cases} 1, & \text{if } s \leq t+h, \\ \dfrac{s-t}{h}, & \text{if } s \in [t+h, t], \\ 0, & \text{if } s \geq t. \end{cases}$$

Doing so, we conclude

$$P_{_F}(E_t) \leq \frac{d^-}{dt}|Dv|_{_F}(G_t) \quad \text{for all } t > 0,$$

and arguing as before the statement follows. $\square$

7.4. The generalized unit outward normal and continuity of F-perimeter

Here we show that, in case of extra regularity, the vector field $\mathbf{z}$ corresponding to the weak solution v of (3.1) represents the generalized outward unit normal to $\partial^* G_t$. Recall that

$$\nu_{G_t} = \nu_{\partial^* G_t} = \nu_v \quad \mathcal{H}^{N-1}\text{-a.e. in } \partial^* G_t.$$

Proposition 7.7. *With the same notation of the previous subsections, we have the equality*

$$-(\mathbf{z}, D\chi_{G_t}) = \left|D\chi_{G_t}\right|_{_F} \quad \textit{as measures on } \Omega^e \quad \textit{for all } t \geq 0. \tag{7.12}$$

In particular, if $\mathbf{z} \cdot \nu_v$ *is continuous, this leads to*

$$\mathbf{z} \cdot \nu_v = -F(\nu_v) \quad \mathcal{H}^{N-1}\text{-a.e. in } \partial^* G_t.$$

Remark 7.8. Notice that the last equality means that $-\mathbf{z} \in \partial F(\nu_v)$. In case that F is smooth, this implies that $\mathbf{z}$ points in the direction of the anisotropic outward normal to $\partial^* G_t$ $\mathcal{H}^{N-1}$-a.e., and then the left hand side of (3.1) is actually the anisotropic mean curvature $H_F = \operatorname{div} \mathbf{z}$. Hence our definition of A-IMCF recovers that of [19].

Proof. Consider the function η defined in (7.11) for any $t \geq 0$. By means of Green's formula applied on $\Omega_r := \Omega^e \cap B_{r(t)+1}$ so that (7.6) holds, we get

$$\int_{\Omega_r} (\eta \circ v)\, d(\operatorname{div} \mathbf{z}) = -\int_{\Omega_r} \big(\mathbf{z}, D(\eta \circ v)\big) + \int_{\partial\Omega} [z, \nu],$$

where we have applied that $\eta(v) = 1$ on $\partial\Omega$ since v vanishes on the boundary.

Now, as $-\eta$ is a Lipschitz non-decreasing function, by Lemma 3.4 (b1),

$$\big(\mathbf{z}, D(-\eta \circ v)\big) = -\eta'(v)(\mathbf{z}, Dv) \quad \text{as measures.}$$

Hence, by means of equation (3.1), we obtain

$$\begin{aligned}\int_{\Omega_r} (\eta \circ v)\, d(\operatorname{div} \mathbf{z}) - \int_{\partial\Omega} [\mathbf{z}, \nu] = \int_{\Omega_r} -\eta'(v) dF(Dv) &= \frac{1}{h} \int_{\{t<v<t+h\}} dF(Dv) \\ &= \frac{|Dv|_F(G_{t+h}) - |Dv|_F(G_t)}{h},\end{aligned}$$

where we applied Corollary 3.5. Letting $h \to 0^+$ yields by Green's formula

$$\begin{aligned}\frac{d^+}{dt}|Dv|_F(G_t) &= \int_{\Omega_r} \chi^*_{G_t}\, d(\operatorname{div} \mathbf{z}) - \int_{\partial\Omega} [\mathbf{z}, \nu] \\ &= -\int_{\Omega^e} d(\mathbf{z}, D\chi_{G_t}) \leq \int_{\Omega^e} dF(D\chi_{G_t}) \leq P_F(G_t),\end{aligned} \tag{7.13}$$

which follows from the Cauchy-Schwarz type inequality in Lemma 2.11 (a) because $F^\circ(\mathbf{z}) \leq 1$. This means that we have equality in Claim 3 within the proof of Lemma 7.4, that is,

$$P_F(G_t) = \frac{d^+}{dt}|Dv|_F(G_t) = \Lambda + |Dv|_F(G_t) \quad \text{for all } t \geq 0. \tag{7.14}$$

Hence we get equality also in (7.13), that is, $0 = \int_{\Omega^e} d(-\mathbf{z}, D\chi_{G_t}) - dF(D\chi_{G_t})$ which, again by Cauchy-Schwarz from Lemma 2.11 (a), proves the first assertion of the statement.

For the second claim about the vector field $\mathbf{z}$, when $\mathbf{z}\cdot\nu_v$ is continuous, by using (2.8) and (2.11), we have

$$\mathbf{z}\cdot\nu_v = [\mathbf{z},\nu_v] = \frac{[\mathbf{z},\nu_v]^+ + [\mathbf{z},\nu_v]^-}{2} = -F(\nu_v) \quad \mathcal{H}^{N-1}\text{-a.e. in } \partial^* G_t,$$

where the first equality comes from [16, Corollary 4.6]. □

7.5. Continuity and exponential growth of the F-perimeter

Next, we show that, despite possible *fattening* of the weak solution of the A-IMCF, the F-perimeter is preserved.

Corollary 7.9 *(Continuity of the anisotropic perimeter). Let v be a weak solution of the A-IMCF in the sense of Definition 3.1, then the function $t \mapsto P_F(G_t)$ is right-continuous for every $t \geq 0$ and $t \mapsto P_F(E_t)$ is left-continuous for every $t > 0$. Moreover, it holds*

$$P_F(G_t) = P_F(E_t) \qquad \textit{for all } t > 0. \tag{7.15}$$

Proof. First, notice that by (7.14), we have that for any $s \geq 0$

$$P_F(G_s) = \Lambda + |Dv|_F(G_s),$$

and taking limits as $s \searrow t$, thanks to (7.10), we get

$$\lim_{s\searrow t} P_F(G_s) = \Lambda + |Dv|_F(G_t) = P_F(G_t),$$

which gives the claimed right-continuity.

Repeating the arguments from the proof of Proposition 7.7 for the sublevel sets E_t, one can show that

$$P_F(E_t) = \frac{d^-}{dt}|Dv|_F(G_t) = \Lambda + |Dv|_F(G_t) \quad \text{for all } t > 0,$$

which leads to the claimed equality thanks to (7.14). The left continuity follows arguing as before. □

Summing up, if we introduce the functional

$$\mathcal{J}_v(E) := P_F(E) - |Dv|_F(E) \quad \text{for a bounded set } E \subset \mathbb{R}^N,$$

we conclude from (7.14) that

$$P_F(G_0) = \mathcal{J}_v(G_0) = \Lambda = \mathcal{J}_v(G_t) = \mathcal{J}_v(E_t) \qquad \text{for every} \;\; t > 0. \tag{7.16}$$

Here the first equality comes from (7.3), and the last one using Corollary 3.5, as well as (7.15).

As in the classical (smooth and isotropic) case, it follows that the rescaled F-perimeter $e^{-t}P_{F}(G_t)$ stays constant.

Corollary 7.10. *Let v be a weak solution of the A-IMCF in the sense of Definition 3.1, then it holds*

$$P_{F}(G_t) = e^t P_{F}(G_0) = e^t P_{F}(\{v = 0\}) \quad \textit{for all} \quad t > 0. \tag{7.17}$$

Proof. By (5.7), we have that $\operatorname{div}(e^{-v}\mathbf{z}) = 0$ as measures in Ω^e, and hence by application of Green's formula in $B_{r(t)+1}(0) \cap \Omega^e$ satisfying (7.6), we reach

$$0 = -\int_{\{s<v\le t\}} \operatorname{div}(e^{-v}\mathbf{z}) = \int_{\Omega^e} \big(e^{-v}\mathbf{z}, D\chi_{\{s<v\le t\}}\big) = \int_{\Omega^e} e^{-v}\big(\mathbf{z}, D\chi_{\{s<v\le t\}}\big),$$

where the latter follows from (2.10). As $\chi_{\{s<v\le t\}} = \chi_{G_t} - \chi_{G_s}$, (7.12) yields

$$\begin{aligned} 0 &= \int_{\Omega^e} e^{-v}\big(\mathbf{z}, D\chi_{G_t}\big) - \int_{\Omega^e} e^{-v}\big(\mathbf{z}, D\chi_{G_s}\big) \\ &= -\int_{\Omega^e} e^{-v}\big|D\chi_{G_t}\big|_{F} + \int_{\Omega^e} e^{-v}\big|D\chi_{G_s}\big|_{F}. \end{aligned} \tag{7.18}$$

On the other hand, for $0 < s < t$, we can apply Lemma 7.3 recalling that by (7.2) the Radon measure $\big|D\chi_{G_s}\big|_{F}$ has support in $\partial^* G_s$, and hence

$$\begin{aligned} e^{-s}P_{F}(G_s) - e^{-t}P_{F}(G_t) &= e^{-s}\int_{\mathbb{R}^N} \big|D\chi_{G_s}\big|_{F} - e^{-t}\int_{\mathbb{R}^N} \big|D\chi_{G_t}\big|_{F} \\ &= \int_{\mathbb{R}^N} e^{-v}\big|D\chi_{G_s}\big|_{F} - \int_{\mathbb{R}^N} e^{-v}\big|D\chi_{G_t}\big|_{F} = 0, \end{aligned}$$

which comes from (7.18). The statement follows by letting $s \to 0^+$. $\square$

8. Outward minimizing properties of solutions

8.1. Variational definition of weak solutions

As in [23,37] we consider a variational version of the definition of weak solutions, and we will show that the solution from Definition 3.1 is also a solution in the following sense:

Definition 8.1. A function $v \in BV_{\rm loc}(\Omega^e) \cap L^\infty_{\rm loc}(\Omega^e)$ is a weak variational solution of A-IMCF if for every compact set $K \subset \Omega^e$ and for every $w \in BV_{\rm loc}(\Omega^e) \cap L^\infty_{\rm loc}(\Omega^e)$ with $w = v$ in $\Omega^e \setminus K$ we have

$$\mathcal{I}_v^K(v) \le \mathcal{I}_v^K(w),$$

where $\mathcal{I}_v^K$ is the functional given by

$$\mathcal{I}_v^K(w) := \int_K dF(Dw) + \int_K w\, dF(Dv).$$

Proposition 8.2. *The weak solution v in the sense of Definition 3.1 is a weak variational solution of A-IMCF.*

Proof. Fix any function $w \in BV_{\rm loc}(\Omega^e) \cap L^\infty_{\rm loc}(\Omega^e)$ such that $\{w \neq v\}$ is contained in a compact set $K \subset \Omega^e$. We also take an open and bounded set B with Lipschitz boundary and so that $K \subset B \subset \Omega^e$.

We know that there exists a vector field $\mathbf{z} \in \mathcal{DM}^\infty_{\rm loc}(\Omega^e)$ with $F^\circ(\mathbf{z}) \le 1$ and satisfying $\operatorname{div} \mathbf{z} = F(Dv)$. If we multiply this equation by w, integrate over B and apply Green's formula (2.9), we get

$$-\int_B d(\mathbf{z}, Dw) + \int_{\partial B} [\mathbf{z}, \nu] w\, d\mathcal{H}^{N-1} = \int_B w\, dF(Dv).$$

Now, Lemma 2.11 (a) ensures that $(\mathbf{z}, Dw) \le F(Dw)$, and hence

$$\int_{\partial B} [\mathbf{z}, \nu] w\, d\mathcal{H}^{N-1} \le \int_B dF(Dw) + \int_B w\, dF(Dv).$$

On the other hand, arguing as before with v instead of w, we have that

$$\int_{\partial B} [\mathbf{z}, \nu] v\, d\mathcal{H}^{N-1} = \int_B dF(Dv) + \int_B v\, dF(Dv),$$

where we have also applied the second equality in (3.1).

Next, notice that $v = w$ on ∂B, which leads to

$$\int_B dF(Dv) + \int_B v\, dF(Dv) \le \int_B dF(Dw) + \int_B w\, dF(Dv).$$

As this holds for all B, we have that $\mathcal{I}_v^K(v) \le \mathcal{I}_v^K(w)$, as desired. □

8.2. Geometric definition of weak solutions

To achieve a geometric notion of weak solutions as in [23], we consider the minimization problem

$$\inf \left\{ \mathcal{J}_v(E) \,:\, \Omega \subset E \Subset \mathbb{R}^N \right\}. \tag{8.1}$$

The goal is to prove the following result:

Proposition 8.3. *Let $v \in BV_{\mathrm{loc}}(\Omega^e) \cap L^\infty_{\mathrm{loc}}(\Omega^e)$ be a weak solution of (1.7). Then, for each $\tau \geq 0$ and $t > 0$, the minimum in (8.1) is attained for E_t and G_τ.*

Proof. We start by considering the following slight modification of (8.1):

$$\inf\left\{\mathcal{J}_v(E) : \overline{\Omega} \subset E \Subset \mathbb{R}^N\right\}, \tag{8.2}$$

and proving that the corresponding minimum is achieved at G_t with $t > 0$.

Let E be a bounded set containing $\overline{\Omega}$. Then,

$$\mathcal{J}_v(E) - \mathcal{J}_v(G_t) = P_F(E) - P_F(G_t) - |Dv|_F(E \setminus G_t) + |Dv|_F(G_t \setminus E). \tag{8.3}$$

In turn, Lemma 7.3 ensures that $P_F(G_t) = P_F(G_t; \Omega^e)$, which implies by means of Cauchy-Schwarz from Lemma 2.11 (b) that

$$\begin{aligned} P_F(E) - P_F(G_t) &\geq \left(\left|D\chi_E\right|_F - \left|D\chi_{G_t}\right|_F\right)(\Omega^e) \\ &\geq -\int_{\Omega^e} d(\mathbf{z}, D\chi_E) + \int_{\Omega^e} d(\mathbf{z}, D\chi_{G_t}) \\ &= -\int_{\Omega^e} d(\mathbf{z}, D\chi_{E\setminus G_t}) + \int_{\Omega^e} d(\mathbf{z}, D\chi_{G_t\setminus E}). \end{aligned}$$

Now, Green's formula on an open bounded set with Lipschitz-continuous boundary $B \supset G_t \cup E$ and (3.1) allow us to conclude

$$\begin{aligned} P_F(E) - P_F(G_t) &= \int_{E\setminus G_t} d(\operatorname{div}\mathbf{z}) - \int_{G_t\setminus E} d(\operatorname{div}\mathbf{z}) \\ &= |Dv|_F(E \setminus G_t) - |Dv|_F(G_t \setminus E), \end{aligned}$$

which, by substitution into (8.3), leads to

$$\mathcal{J}_v(E) - \mathcal{J}_v(G_t) \geq 0.$$

Let us now prove that, in fact, the minimum in (8.1) is attained for G_t. Given $E \supset \Omega$, consider $\tilde{E} = E \cup \partial\Omega \supset \overline{\Omega}$. Then, by the corresponding claim for G_t concerning (8.2),

$$\mathcal{J}_v(G_t) \leq \mathcal{J}_v(\tilde{E}) = P_F(E \cup \partial\Omega) - |Dv|_F(E \cup \partial\Omega) = P_F(E) - |Dv|_F(E) = \mathcal{J}_v(E),$$

because $|\partial\Omega| = 0$ and, by (7.3), $|Dv|_F(\partial\Omega) = 0$.

Finally, the statement for G_0 and E_t follows by (7.16). $\square$

8.3. Definition and proof of the outward minimizing properties

We first define the anisotropic analogue of the notion of minimizing hull or outward minimizing set.

Definition 8.4. A measurable subset $E \subset \mathbb{R}^N$ of finite F-perimeter is said to be outward F-minimizing if

$$P_{_F}(E) \leq P_{_F}(A) \quad \text{for all} \quad E \subset A \Subset \mathbb{R}^N.$$

We say that it is strictly outward F-minimizing if the above inequality is strict unless $|A \setminus E| = 0$.

Notice that for smooth F and ∂E of class C^2, the above definition implies that E is F-mean convex (i.e., $H_F \geq 0$).

On the other hand, as pointed out in [15, p.625], it holds

Lemma 8.5. *If E is outward F-minimizing, then E^1 is open.*

Recall that we consider E^1 as a representative of each finite perimeter set (cf. Remark 7.5), thus we may assume that outward F-minimizing sets are open.

Proposition 8.6. *Let $v \in BV_{\mathrm{loc}}(\Omega^e) \cap L^\infty_{\mathrm{loc}}(\Omega^e)$ be a weak solution of the A-IMCF. Then*

(a) *E_t is outward F-minimizing for any $t > 0$.*
(b) *G_t is strictly outward F-minimizing for any $t \geq 0$.*

Proof. (a) Fix $t \geq 0$ and take any bounded $A \supset E_t$. By Proposition 8.3

$$P_{_F}(E_t) - |Dv|_{_F}(E_t) \leq P_{_F}(A) - |Dv|_{_F}(A).$$

Hence

$$P_{_F}(E_t) \leq P_{_F}(E_t) + |Dv|_{_F}(A \setminus E_t) \leq P_{_F}(A)$$

as desired. Similarly, G_t are outward F-minimizing for any $t \geq 0$.

(b) Let us see that G_t is strictly outward F-minimizing. With this aim, take a bounded set A such that $G_t \subset A$ and

$$P_{_F}(G_t) = P_{_F}(A),$$

which implies that A is also outward F- minimizing. Recall then that we take open representatives for both G_t and A. Arguing as before,

$$P_{_F}(G_t) \leq P_{_F}(G_t) + |Dv|_{_F}(A \setminus G_t) \leq P_{_F}(A).$$

This implies that $|Dv|_F(A \setminus G_t) = 0$, which yields that v is equivalent to a constant on each connected component of the open set $A \setminus G_t$.

However, as G_t is outward F-minimizing, the reduced boundary of these components intersect $\partial^* G_t$ on a set of positive $\mathcal{H}^{N-1}$-measure. Indeed, suppose otherwise that C is a connected component of $A \setminus G_t$ with $\mathcal{H}^{N-1}(\partial^* C \cap \partial^* G_t) = 0$. Then, this yields $\mathcal{H}^{N-1}(\partial^* C \cap \partial^*(A \setminus C)) = 0$, which implies that

$$P_F(A \setminus C) = P_F(A) - P_F(C) < P_F(A) = P_F(G_t);$$

but $G_t \subset A \setminus C$, this is in contradiction with G_t being outward F-minimizing.

Thus $\mathcal{H}^{N-1}(\partial^* C \cap \partial^* G_t) > 0$ for each connected component C of $A \setminus \overline{G_t}$. Therefore, as $v \in DBV$ is constant on each of these connected components and, by Lemma 7.3, $v = t$ $\mathcal{H}^{N-1}$-a.e. on $\partial^* G_t$, we conclude that $v = t$ on $A \setminus G_t$. Consequently, $A \subset G_t$ and, as we are assuming the other inclusion, equality holds, as desired. □

Now we introduce the notion of F-minimal hull (strictly outward F-minimizing hull) of a bounded set $E \subset \mathbb{R}^N$.

$$E' = \bigcap \big\{ S \,:\, S \text{ bounded, strictly outward } F\text{-minimizing and } S \supset E \big\}.$$

Remark 8.7. As we are working up to measure zero modification, the above definition may be realized by a countable intersection. Then arguing as in [7, Proposition 1.3], as the proof basically relies on subadditivity of perimeters, one gets that E' is strictly outward F-minimizing.

Corollary 8.8. *With the same assumptions from Proposition 8.6, it holds*

$$\Omega' = G_0 \quad \textit{and} \quad E_t' = G_t \quad \textit{up to null sets and for all } t > 0.$$

In particular, $P_F(G_0) \le P_F(\Omega)$.

Proof. Since $E_t \subset G_t$ and, by Proposition 8.6 (b), G_t is strictly outward F-minimizing, $E_t' \subset G_t$. Similarly, $\Omega' \subset G_0$.

We now argue by contradiction and assume that $|G_t \setminus E_t'| > 0$ for $t > 0$. Then, since E_t' is strictly outward F-minimizing and $E_t' \subset G_t$, $P_F(E_t') < P_F(G_t)$. However, since $G_t \setminus E_t' \subset \{v = t\}$, Lemma 7.3 (see also Corollary 3.5) implies $|Dv|_F(G_t \setminus E_t') = 0$ thus, by Proposition 8.3,

$$P_F(G_t) \le P_F(E_t') + |Dv|_F(G_t \setminus E_t') = P_F(E_t'),$$

which contradicts $P_F(E_t') < P_F(G_t)$. Hence the statement follows.

Analogously, if we assume that $|G_0 \setminus \Omega'| > 0$, arguing as before (using also Remark 7.2) we reach a contradiction. □

9. Explicit examples of solutions

We begin by showing that Wulff shapes play the role of spheres in the case of the A-IMCF (3.1); i.e. the level sets of the solution are Wulff shapes with exponentially growing radius.

Example 9.1. Let $\Omega = \mathcal{W}_R$, for some $R > 0$. Then, the solution to (1.7) is given by

$$v(x) = (N-1)\log\left(\frac{F^\circ(x)}{R}\right).$$

Note that $v \in \mathrm{Lip}_{\mathrm{loc}}(\Omega^e)$, $\nabla v(x) = \frac{(N-1)\nabla F^\circ(x)}{F^\circ(x)}$ a.e. in Ω^e and $v(x) = 0$ if $x \in \partial\Omega$. Then, by Lemma 2.1 (i), we have $F(\nabla v(x)) = \frac{N-1}{F^\circ(x)}$.

On the other hand, considering $\mathbf{z}(x) := \frac{x}{F^\circ(x)}$, it is immediate to check that $\mathrm{div}(\mathbf{z}) = \frac{N-1}{F^\circ(x)}$. Finally, since by Lemma 2.1, $\mathbf{z} \in \partial F(\nabla F^\circ(x))$, we conclude that v is the solution to (1.7).

Example 9.2. Take a rectangle $\Omega =]-\frac{1}{2}, \frac{1}{2}[\times]-1,1[$ in $\mathbb{R}^2$ with norm $F = \ell_1$, and hence $\mathcal{W}_1 =]-1,1[^2$. The solution is given by

$$v(x,y) = \begin{cases} \dfrac{1}{2}\log\left(\dfrac{|y| + \sqrt{y^2 - 3/4}}{3(|y| - \sqrt{y^2 - 3/4})}\right), & \text{if } y^2 > x^2 + 3/4, \\[2ex] \dfrac{1}{2}\log\left(\dfrac{|x| + \sqrt{x^2 + 3/4}}{3(\sqrt{x^2 + 3/4} - |x|)}\right), & \text{if } y^2 < x^2 + 3/4. \end{cases}$$

It suffices to define

$$\mathbf{z}(x,y) := \begin{cases} \left(\dfrac{x}{\sqrt{y^2 - 3/4}}, \mathrm{sign}(y)\right), & \text{if } y^2 > x^2 + 3/4, \\[2ex] \left(\mathrm{sign}(x), \dfrac{y}{\sqrt{x^2 + 3/4}}\right), & \text{if } y^2 < x^2 + 3/4. \end{cases}$$

Notice that the vertices of the level sets do not evolve linearly (see Fig. 2), they instead move along the curve $y^2 = x^2 + 3/4$.

We finish this section with an example of fattening:

Example 9.3. Let $\Omega \subset \mathbb{R}^2$ be the union of 3 squares centred at $(0,2), (\pm 2, -2)$ with sides of length 1 and consider $F(x,y) = \|(x,y)\|_1$. In this case, $\Omega = \mathcal{W}_{\frac{1}{2}}(0,2) \cup \mathcal{W}_{\frac{1}{2}}(-2,-2) \cup \mathcal{W}_{\frac{1}{2}}(2,-2)$. We claim that the solution to (1.7) is the following function (see Fig. 3):

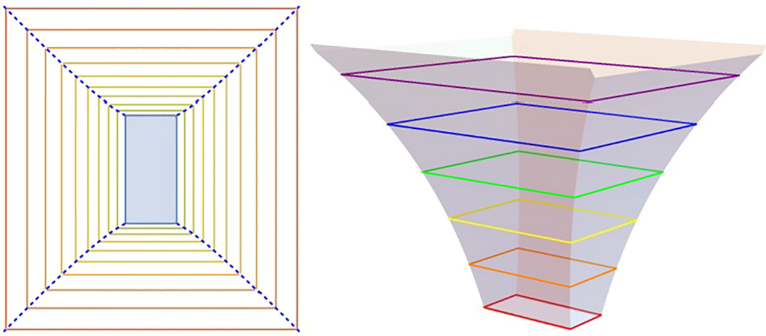

Fig. 2. Evolution of an initial rectangle with respect to the ℓ_1-anisotropy. Plot of the level sets.

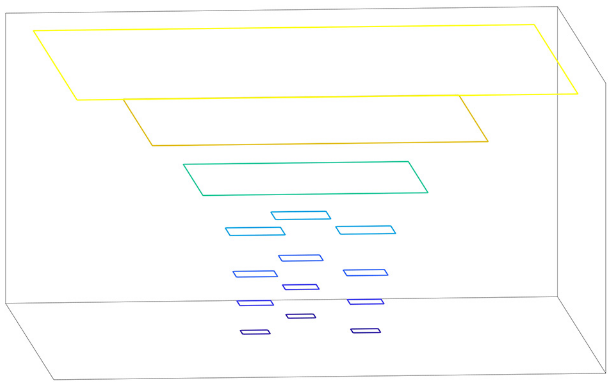

Fig. 3. Example of fattening.

$$v(x,y) = \begin{cases} \log(2), & \text{if } (x,y) \in \mathcal{W}_3 \setminus \overline{\Omega}, \\ \log(\|x, y-2\|_\infty), & \text{if } (x,y) \in \mathcal{W}_1(0,2) \setminus \overline{\mathcal{W}_{\frac{1}{2}}(0,2)}, \\ \log(2\|x+2, y+2\|_\infty), & \text{if } (x,y) \in \mathcal{W}_1(-2,-2) \setminus \overline{\mathcal{W}_{\frac{1}{2}}(-2,2)}, \\ \log(2\|x-2, y+2\|_\infty), & \text{if } (x,y) \in \mathcal{W}_1(2,-2) \setminus \overline{\mathcal{W}_{\frac{1}{2}}(2,2)}, \\ \log\left(\frac{2}{3}\|x,y\|_\infty\right), & \text{if } (x,y) \in \mathbb{R}^2 \setminus \overline{\mathcal{W}_3}. \end{cases}$$

Notice that there is fattening at time $t^* = \log 2$: $E_{t^*} = \mathcal{W}_1(0,2) \cup \mathcal{W}_1(-2,2) \cup \mathcal{W}_1(2,-2)$ while $G_{t^*} = \mathcal{W}_3$. Moreover, it holds $P_F(G_{t^*}) = P_F(E_{t^*}) = 24 = e^{t^*} P_F(\Omega)$, consistent with (7.17).

A direct computation shows that

$$\tilde{\mathbf{z}}(x,y) := \begin{cases} \dfrac{(x, y-2)}{\|x, y-2\|_\infty}, & (x,y) \in \mathcal{W}_1(0,2) \setminus \overline{\mathcal{W}_{\frac{1}{2}}(0,2)}, \\ \dfrac{(x+2, y+2)}{\|x+2, y+2\|_\infty}, & (x,y) \in \mathcal{W}_1(-2,2) \setminus \overline{\mathcal{W}_{\frac{1}{2}}(-2,2)}, \\ \dfrac{(x-2, y+2)}{\|x-2, y+2\|_\infty}, & (x,y) \in \mathcal{W}_1(2,-2) \setminus \overline{\mathcal{W}_{\frac{1}{2}}(2,-2)}, \\ \dfrac{(x,y)}{\|x,y\|_\infty}, & (x,y) \in \mathbb{R}^2 \setminus \overline{\mathcal{W}_3}, \end{cases}$$

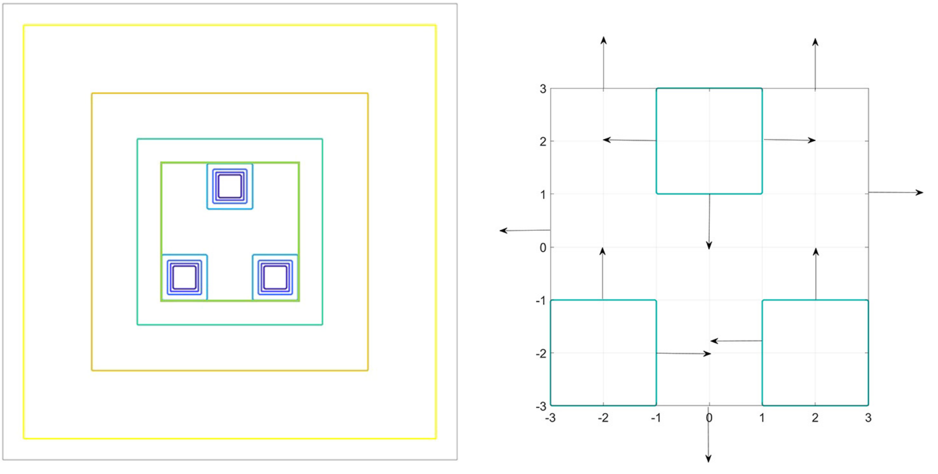

Fig. 4. Left: level sets of the solution. Right: Compatibility conditions of the vector field at the time of fattening.

satisfies all the requirements for v to be a solution in $(\mathbb{R}^2 \setminus \overline{\mathcal{W}_3}) \cup (\mathcal{W}_1(0,2) \cup \mathcal{W}_1(-2,2) \cup \mathcal{W}_1(2,-2)) \setminus \overline{\Omega}$. In order to complete the proof, we need to extend $\tilde{\mathbf{z}}$ to $A := \mathcal{W}_3 \setminus \overline{(\mathcal{W}_1(0,2) \cup \mathcal{W}_1(-2,2) \cup \mathcal{W}_1(2,-2))}$. To achieve this aim, we need a vector field $\mathbf{z}^A : A \to \mathcal{W}_1$ such that

$$\begin{cases} \operatorname{div} \mathbf{z}^A = 0 & \text{in } A, \\ \text{compatibility conditions} & \text{on } \partial A. \end{cases}$$

Next, we briefly sketch how to specify the latter on ∂A. Let

$$\mathbf{z} := \begin{cases} \tilde{\mathbf{z}} & \text{in } \mathbb{R}^2 \setminus (\overline{\Omega} \cup A), \\ \mathbf{z}^A = (z_1^A, z_2^A) & \text{in } A. \end{cases}$$

Then, since $\operatorname{div} \mathbf{z} = F(Dv) << \mathcal{L}^2(\mathbb{R}^2 \setminus \overline{\Omega})$, it follows that $\mathbf{z}^A \cdot \nu = \tilde{\mathbf{z}} \cdot \nu$, with ν being the outer unit normal to ∂A. Therefore, we need

$$\begin{cases} z_1^A(\pm 3, y) = \pm 1, & z_1^A(\pm 1, y) = \pm \operatorname{sign}(y) \\ z_2^A(x, \pm 3) = 1, & z_2^A(x, \pm 1) = \mp 1, \end{cases} \tag{9.1}$$

for $(x, y) \in \partial A$. In Fig. 4, we plot the compatibility conditions at ∂A. The arrows represent the orientation of the normal component of $\mathbf{z}^A$.

The construction of $\mathbf{z}^A$ follows as in [9, Theorem 5] (see [30, Theorem 5] for the ℓ_1 anisotropy). In particular, it is obtained as a minimizer of

$$\int_A (\operatorname{div} \boldsymbol{\xi})^2 \, d\mathcal{L}^2$$

among vector fields $\boldsymbol{\eta} \in L^\infty(A)$ satisfying (9.1). Moreover, $\text{div}\boldsymbol{\xi} = \text{div}\boldsymbol{\eta}$ is constant for any two minimizers $\boldsymbol{\xi}$, $\boldsymbol{\eta}$. Now, using integration by parts,

$$\int_A \operatorname{div} \mathbf{z}^A \, d\mathcal{L}^2 = \int_{\partial A} \mathbf{z}^A \cdot \nu \, ds = 0,$$

and, since $\operatorname{div} \mathbf{z}^A$ is constant, necessarily $\operatorname{div} \mathbf{z}^A = 0$. This finally shows that v is the solution to (1.7) with associated vector field $\mathbf{z}$.

Data availability

No data was used for the research described in the article.

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