Periodic solutions to second order nonlinear differential equations in Banach spaces via measures of noncompactness

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Abstract

In this article we study the existence of solutions for the following second order differential equation:

$$\begin{cases} u''(t) = f(t, u(t), u'(t)) + h(t) \\ u(a) = u(b), \ u'(a) = u'(b) \end{cases}$$

where $\mathbb{B}$ is a reflexive real Banach space, $f : [a, b] \times \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ is a continuous mapping and $h \in L^1([a, b], \mathbb{B})$. The proofs are obtained by using a recent generalization of the well known Bolzano-Poincaré-Miranda theorem to infinite dimensional Banach spaces as well as the fixed point theory involving measures of noncompactness.

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1. Introduction

This paper is mainly concerned with the second-order boundary value dynamical equation

$$\begin{cases} u''(t) = f(t, u(t), u'(t)) + h(t) \\ u(a) = u(b), \\ u'(a) = u'(b) \end{cases} \tag{1}$$

The state $u(\cdot)$ takes values in a reflexive Banach space $\mathbb{B}$, $f : [a, b] \times \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ is a continuous mapping and $h \in L^1([a, b], \mathbb{B})$. During the previous decades, some classical tools have been used in the study of periodic solutions of Equation (1) in finite dimensional Banach spaces, including the method of upper and lower solutions [8, 11, 14], degree theory and fixed point theory [9, 13]. In this sense, in [18] the author gave a generalization of Miranda-Poincaré theorem and using this generalization proved theorems about the existence for systems of k equations $x'' = f(t, x, x')$, where $f : [0, 1] \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ is a vector function, subject to various boundary conditions.

In [19] the existence and uniqueness of solutions of a second order Dirichlet boundary value problem posed in a general Banach space is studied by using Sadovskii's fixed point theorem and precise computation of measure of noncompactness. Recently, the existence of periodic solutions second order differential equations on a general Banach space has been treated in several paper, for instance see [15, 16, 3]. In [15] operator-valued Fourier multiplier theorems are used in order

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to study this problem. On the other hand, in [16] the authors developed an infinite-dimensional version of Poincaré-Miranda theorem and they showed their relation with viability theory for differential inclusions, apply them, in particular, in the context of constrained PDEs. In [3] by using a generalization of Miranda-Poincare theorem, see [2], and Schauder fixed point theorem, Problem (1) under the condition $f(t,x,y) = f(t,x)$ is considered.

The aim of this paper is to give sufficient conditions in order to obtain existence of periodic solutions for Problem 1 which, among other things, allow us to recapture the main result in [3].

2. Notations and Preliminaries

In this section we shall recall some definitions and results that are needed later on.

Throughout this article $\mathbb{B}$ will denote a real reflexive Banach space with norm $\|\cdot\|$ and we shall denote its dual by $\mathbb{B}^*$. The closed ball of center x_0 and radius r is denoted by $B_r[x_0]$ and the sphere of center x_0 and radius r by $\partial B_r[x_0]$. The symbol " $\rightharpoonup$ " is used to denote the convergence of a sequence (or net) in $\mathbb{B}$ with respect to the weak topology $\sigma(\mathbb{B},\mathbb{B}^*)$, as well as the symbol " $\to$ " is used to denote the convergence of a sequence (or net) in $\mathbb{B}$ with respect to the norm topology.

Recall that the *normalized duality mapping* $J : \mathbb{B} \to 2^{\mathbb{B}^*}$ is defined by

$$J(x) := \{j \in \mathbb{B}^* : \langle x, j\rangle = \|x\|^2, \|j\| = \|x\|\}, \;\; x \in \mathbb{B}.$$

Let $\langle y, x\rangle_+ := \max\{\langle y, j\rangle : j \in J(x)\}$ and let $\langle y, x\rangle_- := \min\{\langle y, j\rangle : j \in J(x)\}$. For more details, see [7].

Denote by $\mathcal{C}([a,b],\mathbb{B})$ the space of $\mathbb{B}$-valued continuous functions on $[a,b]$ with the norm $\|u\|_\infty = \sup\{\|u(t)\| \;:\; t \in [a,b]\}$, and denote by $L^1(a,b,\mathbb{B})$ the space of $\mathbb{B}$-valued Bochner integrable function on $[a,b]$ with the norm $\|u\|_1 = \int_a^b \|u(t)\|dt$. It is well known (for instance, see [17]) that if $u \in \mathcal{C}([a,b],\mathbb{B})$ then, $u \in L^1(a,b,\mathbb{B})$ and for each $t \in [a,b]$ we have that $\frac{d}{dt}(\int_a^t u(s)ds) = u(t)$, furthermore, if $u \in L^1(a,b,\mathbb{B})$ the function $g : [a,b] \to \mathbb{B}$ given by $g(t) := \int_a^t u(s)ds$ is a continuous, i.e, $g \in \mathcal{C}([a,b],\mathbb{B})$.

Let K be a nonempty subset of $\mathbb{B}$. A mapping $F : K \to \mathbb{B}$ is said to be *compact* if $F(C)$ is relatively compact whenever C is a bounded subset of K. If, moreover, F is continuous in K, then F is called *completely continuous*. Clearly, every continuous mapping with closed domain in a finite dimensional Banach space is completely continuous. On the other hand, F is said to be weak-strong sequentially continuous mapping if given any sequence (x_n) in K such that $x_n \rightharpoonup x \in K$ the strong convergence $F(x_n) \to F(x)$ holds. It is clear that if $\mathbb{B}$ is a reflexive Banach space, K is a closed convex subset of $\mathbb{B}$ and $F : K \to \mathbb{B}$ is weak-strong continuous mapping, then F is completely continuous.

Let $\mathcal{B}(\mathbb{B})$ denote the family of all nonempty bounded subsets of $\mathbb{B}$, $\mathcal{BC}(\mathbb{B})$ the family of all nonempty closed and bounded subsets of $\mathbb{B}$, and $\mathcal{K}(\mathbb{B})$ the family of all nonempty relatively compact subsets of $\mathbb{B}$. For each nonempty subset M of $\mathbb{B}$, we denote by $\overline{\text{conv}}(M)$ the closure of the convex hull of M.

Definition 2.1. By a measure of non-compactness on $\mathbb{B}$, we mean a function $\mu : \mathcal{B}(\mathbb{B}) \to \mathbb{R}^+$ satisfying that, for every $M, N \in \mathcal{B}(\mathbb{B})$:

(*i*) (Consistence). The family $ker(\mu) = \{M \in \mathcal{B}(\mathbb{B}) : \mu(M) = 0\}$ is nonempty and $ker(\mu) \subseteq \mathcal{K}(\mathbb{B})$,

(*ii*) (Invariance). $\mu(\overline{\text{conv}}(M)) = \mu(M)$,

(*iii*) (Monotonicity). If $M \subseteq N$, then $\mu(M) \leq \mu(N)$, Sometimes it is necessary to assume that a measure of non-compactness μ enjoys some additional properties:

(iv) (Cantor's intersection). If $(M_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{BC}(\mathbb{B})$ such that $M_{n+1} \subseteq M_n$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} \mu(M_n) = 0$, then $\bigcap\limits_{n\in\mathbb{N}} M_n$ is nonempty,

(v) (Maximum). For every $x \in E$, $\mu(M \cup \{x\}) = \mu(M)$.

(vi) (Permanence under translations). For every $x \in \mathbb{B}$, $\mu(M + x) = \mu(M)$.

(vii) (Homogeneity) $\mu(\lambda M) = |\lambda|\mu(M)$, where $\lambda M = \{\lambda x : \ x \in M\}$,

$(viii)$ (Algebraic semi-additivity) $\mu(M+N) \leq \mu(M)+\mu(N)$, where $M+N = \{x+y : \ x \in M, \ y \in N\}$

(ix) $\mu(M \cup N) = \max\{\mu(M), \mu(N)\}$.

When $ker(\mu) = \mathcal{K}(\mathbb{B})$, the measure of non-compactness μ is called *full*. Let us notice that if a measure of non-compactness μ satisfies (v), then μ also satisfies (iv), see for instance [4].

The most important examples of measures of noncompactness are the *Kuratowski measure of noncompactness* (or set measure of noncompactness)

$$\alpha(\Omega) = \inf\{r > 0 \, : \, \Omega \ \text{ may be covered by finitely many sets of diameter} \leq r\}.$$

and the *Hausdorff measure of noncompactness* (or ball measure of noncompactness)

$$\chi(\Omega) = \inf\{r > 0 : \ \text{there exist a finite } r\text{-net for } \ \Omega \ \text{ in } E\}.$$

These two measures of non-compactenss are full and satisfy all the above conditions.

Let $\mathbb{B}$ be a Banach space and let $\alpha : \mathcal{B}(\mathbb{B}) \to \mathbb{R}^+$ be the Kuratowski measure of noncompactness on $\mathbb{B}$. We can define the following function: $\beta : \mathcal{B}(\mathcal{C}([a, b], \mathbb{B})) \to [0, \infty)$ given by

$$\beta(B) := W_0(B) + \sup\{\alpha(B(t)) : \ t \in [a, b]\}, \tag{2}$$

where $B(t) := \{u(t) : \ u \in B\}$ and $W_0(B) = \lim_{\epsilon\to 0^+} W(B, \epsilon)$ being $W(u, \epsilon) = \sup\{\|u(t) - u(s)\| : |t - s| \leq \epsilon\}$ and $W(B, \epsilon) = \sup\{W(u, \epsilon) : \ u \in B\}$. It is not difficult to see that β is a measure of noncompactness on $\mathcal{C}([a, b], \mathbb{B})$ which satisfies maximum property and thus it also has Cantor's intersection property. To discuss the existence we shall need the following results:

Lemma 2.1. *([19, lemma 2.12]) Assume that Ω is a nonempty bounded subset of $\mathcal{C}([a, b], \mathbb{B})$. If $W_0(\Omega) = 0$, then $W_0(\overline{conv}(\Omega)) = 0$.*

Lemma 2.2. *([10]) If B is a bounded subset of $\mathcal{C}([a, b], \mathbb{B})$ and $W_0(B) = 0$, then $\alpha(B(t))$ is a continuous function on $[a, b]$ and*

$$\alpha(\{\int_a^b u(t)dt : \ u \in B\}) \leq \int_a^b \alpha(B(t))dt.$$

Lemma 2.3. *([5, Corollary 2.10]) Let Ω be a nonempty bounded subset of $\mathcal{C}([a, b], \mathbb{B})$. Then we have*

$$\alpha(\int_a^b \Omega(t)dt) \leq 4 \int_a^b \alpha(\Omega(t))dt.$$

If μ is a measure of noncompactness on $\mathbb{B}$ (Definition 2.1) satisfying Cantor's intersection in [1, Proposition 3.2] the following fixed point result was obtained:

Theorem 2.1 ([1])**.** *Let K be a nonempty, bounded, closed and convex subset of a Banach space $\mathbb{B}$ and let $F : K \to K$ be a continuous function. If there exists a decreasing sequence $\{C_n)\}$ of convex closed and F-invariant subsets of K such that $\mu(C_n) \to 0$ as $n \to \infty$, then F has a fixed point.*

Recently, in [2, Theorem 3.1] an extension of the well-known Bolzano-Poincaré-Miranda theorem to infinite dimensional Banach spaces is obtained. A particular version of this result, which we will used, is the following one (see Remark 3.1 and Corollary 3.1 in [2]).

Proposition 2.1. *Given a reflexive Banach space $\mathbb{B}$, a point $z \in X$ and a number $R > 0$, let $F : B_R[z] \to \mathbb{B}$ be a weak-strong continuous mapping. Assume that there exists a functional $[\cdot,\cdot]_c : \mathbb{B} \times \mathbb{B} \to \mathbb{R}$ satisfying*

(C_1) $[x,x]_c > 0$ for all $x \in \mathbb{B}$ with $x \neq 0$;

(C_2) $[\lambda\, x,x]_c = \lambda\, [x,x]_c$ for all $x \in \mathbb{B}$ and $\lambda \in \mathbb{R}$.

If $[F(x), x - z]_c$ has a constant sign for all $x \in \partial B_R[z]$, then F has at least a zero in $B_R[z]$.

3. Preparatory results

Let $f : [a,b] \times \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ be a continuous function such that for every $z \in \mathbb{B}$, the function $f_z : [a,b] \times \mathbb{B} \to \mathbb{B}$ given by $f_z(t,x) = f(t,x,z)$ is weak-strong sequentially continuous.

For our study we introduce, for each element of Banach space $\mathcal{C}([a,b],\mathbb{B})$, an auxiliary mapping and we will prove the existence and uniqueness of zeros of this mapping. Let $v \in \mathcal{C}([a,b],\mathbb{B})$. We define $F_v : \mathbb{B} \to \mathbb{B}$ given by

$$F_v(x) = \int_a^b f(\tau, \textstyle\int_a^\tau v(s)\, ds + x, v(\tau))\, d\tau. \tag{3}$$

Lemma 3.1. *Let $v \in \mathcal{C}([a,b],\mathbb{B})$ suppose that:*

$$\begin{array}{l} \textit{for each } M > 0 \textit{ there exists } \delta_M > 0 \textit{ such that } \langle f_z(t, y + x), x\rangle_- \geq 0 \\ \textit{for all } t \in [a,b], y \in B_M[0], z \in B_{\frac{M}{b-a}}[0] \textit{ and } \|x\| \geq \delta_M. \end{array} \tag{4}$$

Then, the mapping $F_v : \mathbb{B} \to \mathbb{B}$ given by (3) has at least a zero in $\mathbb{B}$.

Proof. Let $v \in \mathcal{C}([a,b],\mathbb{B})$. First, we note that F_v is weak-strong sequentially continuous because if $\{x_n\}$ is a bounded sequence in $\mathbb{B}$, then there exists $r > 0$ such that $x_n \in B_r$ for all $n \in \mathbb{N}$. Since $\mathbb{B}$ is reflexive, there exists a weakly convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$, say $x_{n_k} \rightharpoonup x \in B_r$. We define a sequence of functions given by $g_{n_k}(t) := f(t, \int_a^t v(s)\, ds + x_{n_k}, v(t))$ for $t \in [a,b]$. By hypothesis, $f_{v(t)}$ is weak-strong sequentially continuous, then for each $t \in [a,b]$ $g_{n_k}(t)$ converges strongly to $g(t) := f(t, \int_a^t v(s)\, ds + x, v(t))$. Furthermore, since f is a continuous function and $\mathbb{B}$ is a reflexive Banach space, for each n_k we get to $\|g_{n_k}(t)\| \leq K$ for all $t \in [a,b]$, where $K \geq 0$ is the constant of boundedness of $f([a,b] \times B_{r+(b-a)\|v\|_\infty}[0] \times B_{\|v\|_\infty}[0])$. Thus, for each $k \in \mathbb{N}$,

$$\begin{aligned} \|F_v(x_{n_k}) - F_v(x)\| &\leq \int_a^b \left\| f(t, \textstyle\int_a^t v(s)\, ds + x_{n_k}, v(t)) - f(t, \int_a^t v(s)\, ds + x, v(t)) \right\| dt \\ &= \int_a^b \|g_{n_k}(t) - g(t)\|\, dt \end{aligned} \tag{5}$$

which implies, by dominated convergence theorem, the strong convergence of $\{F_v(x_{n_k})\}$ to $F_v(x)$.

On the other hand, if we consider the functional $[\cdot,\cdot]_c : \mathbb{B}\times\mathbb{B}\to\mathbb{R}$ given by

$$[x,y]_c = \langle x,y\rangle_+ := \sup_{j\in J(y)} \langle x,j\rangle = \max_{j\in J(y)} \langle x,j\rangle,$$

then $[\cdot,\cdot]_c$ satisfies (C_1)-(C_2). Moreover, taking $M := (b-a)\|v\|_\infty$ we know that for all $\tau\in[a,b]$, $\int_a^\tau v(s)ds \in B_M[0]$ and $v(\tau)\in B_{\frac{M}{b-a}}[0]$. By (4) there exists $\delta_M>0$ such that $\langle f(t,\int_a^t v(s)ds + x, v(t)), x\rangle_- \geq 0$ for all $t\in[a,b]$ and $\|x\|\geq\delta_M$. Therefore for all $x\in\partial B_{\delta_M}[0]$ we have that

$$[F_v(x),x]_c = \langle F_v(x),x\rangle_+ \geq \int_a^b \langle f(\tau,\textstyle\int_a^\tau v(s)ds + x, v(\tau)), x\rangle_- \, d\tau \geq 0$$

Therefore, by Proposition 2.1 we deduce that F_v has at least a zero in $B_{\delta_M}[0]$. ☐

Remark 3.1. *Bearing in mind Proposition 2.1 and the proof of Lemma 3.1, In Lemma 3.1 we can, clearly, replace Condition (4) by*

$$\begin{aligned} &\text{for each } M>0 \text{ there exists } \delta_M>0 \text{ such that } \langle f_z(t,y+x),x\rangle_+ \leq 0 \\ &\text{for all } t\in[a,b], y\in B_M[0], z\in B_{\frac{M}{b-a}}[0] \text{ and } \|x\|\geq\delta_M. \end{aligned} \tag{6}$$

Lemma 3.2. *Let $v\in\mathcal{C}([a,b],\mathbb{B})$ and suppose that for every $z\in\mathbb{B}$ the function f_z is a strictly accretive operator with respect to the second variable, that is,*

$$\langle f_z(t,x)-f_z(t,y), x-y\rangle_- > 0 \text{ for all } t\in[a,b],\ x,y\in\mathbb{B},\ \text{with } x\neq y. \tag{7}$$

Then, if there is a zero of the operator $F_v:\mathbb{B}\to\mathbb{B}$, given by (3), it is unique.

Proof. Note that, by (7), F_v is strictly accretive, that is, $\langle F_v(x)-F_v(y), j\rangle > 0$ for all $x,y\in\mathbb{B}$, with $x\neq y$ and for all $j\in J(x-y)$. Indeed, for each $x,y\in\mathbb{B}$ and $j\in J(x-y)$, we have that

$$\begin{aligned}\langle F_v(x)-F_v(y), j\rangle &= \langle \int_a^b \big(f(\tau,\textstyle\int_a^\tau v(s)\,ds + x, v(\tau)) - f(\tau,\int_a^\tau v(s)\,ds + y, v(\tau))\big)d\tau, j\rangle \\ &= \int_a^b \langle f(\tau,\textstyle\int_a^\tau v(s)\,ds + x, v(\tau)) - f(\tau,\int_a^\tau v(s)\,ds + y, v(\tau)), j\rangle\, d\tau\end{aligned}$$

Since $j\in J(x-y) = J\big(x-\int_a^\tau v(s)\,ds - (y-\int_a^\tau v(s)\,ds)\big)$, from (7) we obtain that $\langle F_v(x)-F_v(y), j\rangle>0$ for all $x,y\in\mathbb{B}$, with $x\neq y$ and for all $j\in J(x-y)$.

Now suppose that $z_1,z_2\in\mathbb{B}$ are two zeros of F_v with $x_1\neq z_2$. Since F_v is strictly accretive, $\langle F_v(z_1)-F_v(z_2), j\rangle>0$ for all $j\in J(z_1-z_2)$ which contradicts to $F_v(z_i)=0$, $i=1,2$. Therefore, $z_1=z_2$, that is, if F_v has a zero, it is unique. ☐

Remark 3.2. *Checking the proof of Lemma 3.2 we can replace Condition (7) by let f_z be a strictly dissipative operator with respect to the second variable for every $z\in\mathbb{B}$, that is,*

$$\langle f_z(t,x)-f_z(t,y), x-y\rangle_+ < 0 \text{ for all } t\in[a,b],\ x,y\in\mathbb{B},\ \text{with } x\neq y. \tag{8}$$

4. Main result

Now we can prove the main result of this work related to the existence of a periodic solution of Problem (1). To this end, we introduce the following hypotheses:

(H_1) $f : [a,b] \times \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ be a uniformly continuous function on each bounded subset of $\mathbb{B}$,

(H_2) for every $z \in \mathbb{B}$ the function $f_z : [a,b] \times \mathbb{B} \to \mathbb{B}$ is weak-strong sequentially continuous,

(H_3) let α be the Kuratowski measure of on compactness on $\mathbb{B}$. For any $A, C \in \mathcal{B}(\mathbb{B})$, there exists $l \in L^1_+([a,b],\mathbb{R})$ such that $\alpha(f(t,A,C)) \leq l(t)\alpha(C)$,

(H_4) there exist $k \in L^1_+([a,b],\mathbb{R})$, $\alpha \in [0,1]$ and $\Gamma : [0,\infty) \to [0,\infty)$ nondecreasing such that $\|f(t,x,y)\| \leq \Gamma(\|x\|) + k(t)\|y\|^\alpha$ for all $t \in [a,b]$, $x, y \in \mathbb{B}$.

Theorem 4.1. *Let $f : [a,b] \times \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ be a function satisfying conditions $(H_1) - (H_4)$ and (4) and (7) (or (6) and (8), respectively). Let $h \in L^1([a,b],\mathbb{B})$ be a function with $\int_a^b h(t)\,dt = 0$. Then, Problem (1) has at least one solution $u \in \mathcal{C}^2([a,b],\mathbb{B})$ whenever the following assumptions holds.*

(A_1) *[A growth condition]*

$$\lim_{s\to\infty} \frac{\Gamma(s+\delta_s)}{s} < \begin{cases} \frac{1-2\int_a^b k(t)dt}{2(b-a)^2}, & \text{if, } \alpha = 1 \\ \frac{1}{2(b-a)^2}, & \text{if, } 0 \leq \alpha < 1 \end{cases}$$

where δ_s is the constant given in (4).

Proof. For the sake of clarity, the proof will be divided into seven steps.

Step 1. Notice that if consider $v(t) = u'(t)$ then Problem (1) is equivalent to solve the following problem

$$\begin{cases} v'(t) = f(t, \int_a^t v(s)\,ds + D, v(t)) + h(t) \\ v(a) = v(b), \\ \int_a^b v(t)\,dt = 0, \end{cases} \tag{9}$$

where D is any element of $\mathbb{B}$ depending of v, because v_0 is a solution for (9) if and only if, $u_0(t) = \int_a^t v_0(s)ds + D$ is a solution for (1).

Step 2. By Lemmas 3.1 and 3.2, for each $v \in \mathcal{C}([a,b],\mathbb{B})$, there exists a unique zero D_v for F_v in $B_R[0]$ where $R := \delta_M$ and $M := (b-a)\|v\|_\infty$.

Integrating Equation 9, we deduce that the previous problem is equivalent to the following integral equation

$$v(t) = \int_a^t f(\tau, \int_a^\tau v(s)\,ds + D_v, v(\tau))\,d\tau + \int_a^t h(\tau)\,d\tau + v(a),$$

where we take

$$v(a) := -\frac{1}{b-a}\int_a^b \left(\int_a^t (f(\tau, \int_a^\tau v(s)\,ds + D_v, v(\tau)) + h(\tau))\,d\tau \right)\,dt,$$

because in this case it is clear that $\int_a^b v(t)\ dt = 0$. Moreover, notice that since

$$\int_a^b f(\tau, \textstyle\int_a^\tau v(s)\, ds + D_v, v(\tau))d\tau) = 0 \text{ and } \displaystyle\int_a^b h(\tau)d\tau = 0,$$

then $v(a) = v(b)$. Thus, we consider the operator $T : \mathcal{C}([a,b],\mathbb{B}) \to \mathcal{C}([a,b],\mathbb{B})$, given by

$$T(v)(t) := \int_a^t f(\tau, \textstyle\int_a^\tau v(s)\, ds + D_v, v(\tau))\, d\tau + \displaystyle\int_a^t h(\tau)\, d\tau + v(a),$$

and therefore to find a fixed point for the operator T is equivalent to find a solution of Equation (9).

Step 3. Let us see that T is continuous. Let $\{v_n\}$ be a sequence in $\mathcal{C}([a,b],\mathbb{B})$ that converges strongly to $v \in \mathcal{C}([a,b],\mathbb{B})$. Note that (D_{v_n}) is a bounded sequence and since $\mathbb{B}$ is a reflexive Banach space we may assume that $D_{v_n} \rightharpoonup D$. Now we can define $g_n(t) := f(t, \int_a^t v_n(s)\, ds + D_{v_n}, v_n(t))$ and $g(t) = f(t, \int_a^t v(s)\, ds + D, v(t))$. For each $\tau \in [a,b]$ we obtain

$$\|g_n(\tau) - g(\tau)\| \quad \le \left\|g_n(\tau) - f(\tau, \textstyle\int_a^\tau v_n(s)ds + D_{v_n}, v(\tau))\right\| + \left\|f(\tau, \textstyle\int_a^\tau v_n(s)ds + D_{v_n}, v(\tau)) - g(\tau)\right\|$$

Since there exists $r > 0$ such that for all $t \in [a,b]$, $v_n(t), v(t), D_{v_n}, D \in B_r$ and f satisfies (H_1), then f is uniformly continuous on $[a,b] \times B_{r(b-a)+r} \times B_r$. Therefore,

$$\left\|g_n(\tau) - f(\tau, \int_a^\tau v_n(s)ds + D_{v_n}, v(\tau))\right\| \to 0 \text{ as } n \to \infty, \tag{10}$$

moreover, since f satisfies (H_2) and $\int_a^\tau v_n(s)ds + D_{v_n} \rightharpoonup \int_a^\tau v(s)ds + D$, we obtain

$$\left\|f(\tau, \int_a^\tau v_n(s)ds + D_{v_n}, v(\tau)) - g(\tau)\right\| \to 0 \text{ as } n \to \infty. \tag{11}$$

Hence, by (10) and (11) we conclude that $\|g_n(\tau) - g(\tau)\| \to 0$ as $n \to \infty$ for all $\tau \in [a,b]$ and, furthermore, $\|g_n(t)\| \le K$ for all $t \in [a,b]$ where $K \ge 0$ is the constant of boundedness of $f([a,b] \times B_{(b-a)r+r} \times B_r)$. Then, by dominated convergence theorem, we obtain $F_{v_n}(D_{v_n}) \to F_v(D)$ as $n \to \infty$, and by Lemma 3.2 we derive that $D = D_v$.

Consequently,

$$\begin{aligned}
\|T(v_n)(t) - T(v)(t)\| &\le \int_a^t \|g_n(\tau) - g(\tau)\|\ d\tau + \|v_n(a) - v(a)\| \\
&\le \int_a^b \|g_n(\tau) - g(\tau)\|\ d\tau + \|v_n(a) - v(a)\| \\
&\le 2\int_a^b \|g_n(\tau) - g(\tau)\|\ d\tau,
\end{aligned}$$

for all $n \in \mathbb{N}$. Then, $\|T(v_n) - T(v)\|_\infty \le 2\int_a^b \|g_n(\tau) - g(\tau)\|\ d\tau$, for all $n \in \mathbb{N}$, which means that T is continuous (applying again dominated convergence theorem).

Step 4 We next show that there exists a ball in $\mathcal{C}([a,b],\mathbb{B})$ T-invariant. Indeed, for a function $v \in C([a,b],\mathbb{B})$, it is clear that for all $t \in [a,b]$

$$\|T(v)(t)\| \leq 2\int_a^b \left\| f(\tau, \textstyle\int_a^\tau v(s)\,ds + D_v, v(\tau)) \right\| d\tau + 2\,\|h\|_1\,,$$

Since f satisfies (H_4), taking $M := (b-a)\|v\|_\infty$, from the above inequality, we derive

$$\begin{aligned} \|T(v)(t)\| &\leq 2(\int_a^b (\Gamma(M+\delta_M) + k(t)\|v\|_\infty^\alpha)dt + \|h\|_1) \\ &\leq 2(b-a)\Gamma(M+\delta_M) + 2\|h\|_1 + 2\|v\|_\infty^\alpha \int_a^b k(t)dt \end{aligned} \tag{12}$$

Now, from (12) we have that

$$\frac{\|T(v)\|_\infty}{\|v\|_\infty} \leq 2(b-a)^2\,\frac{\Gamma\big((M+\delta_M\big)}{M} + 2\frac{\|h\|_1}{\|v\|_\infty} + 2\|v\|_\infty^{\alpha-1}\int_a^b k(t)dt. \tag{13}$$

Bearing in mind growth condition (A_1) and (13), we get

$$\lim_{\|v\|_\infty \to\infty} \frac{\|T(v)\|_\infty}{\|v\|_\infty} < \begin{cases} 2(b-a)^2 \frac{1-2\int_a^b k(t)dt}{2(b-a)^2} + 2\int_a^b k(t)dt = 1 \text{ if } \alpha = 1 \\ 2(b-a)^2 \frac{1}{2(b-a)^2} = 1 \text{ if } 0 \leq \alpha < 1 \end{cases}$$

This case yields the existence of $K > 0$ such that $B_K[0]$ is T-invariant.

Step 5. We claim that if $B \in \mathcal{B}(C([a,b],\mathbb{B}))$ then $W_0(\overline{conv}(T(B))) = 0$. Since B is bounded, there exists $s > 0$ such that $\|u\|_\infty \leq s$ for all $u \in B$, moreover we call $M = (b-a)s$. If we suppose that $t, s \in [a,b]$ and $s < t$, we have

---

$$\|Tu(t) - Tu(s)\| \leq \int_s^t \left\| f(\tau, \int_a^\tau u(r)dr + D_u, u(\tau)) \right\| d\tau + \int_s^t \|h(\tau)\|d\tau \leq |t-s|S + \int_s^t \|h(\tau)\|\,d\tau,$$

where S is the boundeness of $f([a,b]\times B_{M+\delta_M} \times B_s)$, which means that $W_0(T(B)) = 0$ because the function $t \to \int_a^t \|h(\tau)\|d\tau$ is uniformly continuous.

Finally, applying Lemma 2.1, we obtain that the claim holds.

Step 6. We next prove that there is a decreasing sequence of bounded, closed, convex and T-invariant subsets (C_n) such that $\beta(C_n) \to 0$ as $n \to \infty$.

By step 4, we can find $K > 0$ such that $B_K[0]$ is T-invariant. Let

$$C_0 := B_K[0], \quad C_1 = \overline{conv}(T(C_0)), \;\cdots\; C_n = \overline{conv}(T(C_{n-1})),\; n \in \mathbb{N}. \tag{14}$$

we shall show

(*a*) $C_0 \supseteq C_1 \supseteq \cdots \supseteq C_n \supseteq \cdots$,

(*b*) $\beta(C_n) \to 0$, as $n \to \infty$.

By (14), we know that $C_1 = \overline{conv}(T(C_0)) \subseteq C_0$, hence $T(C_1) \subseteq T(C_0)$, therefore $C_2 \subseteq C_1$. Now the method of induction allows us to prove (a).

By step 5, we have that $W_0(C_n) = 0$ for $n = 1, 2, \cdots$. This fact says that

$$\beta(C_n) = \sup\{\alpha(C_n(t)) : \quad t \in [a, b]\}, \quad n = 1, 2, \cdots .$$

The properties of the Kuratowski measure of noncompactness yield:

$$\begin{aligned}
\alpha(C_1(t)) = \alpha(T(C_0(t))) &\leq \alpha(\{\int_a^t f(\tau, \int_a^\tau u(s)ds + D_u, u(\tau))d\tau + \int_a^t h(\tau)d\tau : \ u \in C_0\}) \\
&+ \frac{1}{b-a}\alpha(\int_a^b (\int_a^t f(\tau, \int_a^\tau u(s)ds + D_u, u(\tau))d\tau + \int_a^t h(\tau)d\tau)dt : \ u \in C_0\}) \\
&= \alpha(\{\int_a^t f(\tau, \int_a^\tau u(s)ds + D_u, u(\tau))d\tau : \ u \in C_0\}) \\
&+ \frac{1}{b-a}\alpha(\int_a^b (\int_a^t f(\tau, \int_a^\tau u(s)ds + D_u, u(\tau))d\tau)dt : \ u \in C_0\}) = I_1 + I_2,
\end{aligned} \tag{15}$$

where $I_1 = \alpha(\{\int_a^t f(\tau, \int_a^\tau u(s)ds + D_u, u(\tau))d\tau : \ u \in C_0\})$ and $I_2 = \frac{1}{b-a}\alpha(\int_a^b(\int_a^t f(\tau, \int_a^\tau u(s)ds + D_u, u(\tau))d\tau)dt : \ u \in C_0\})$.

By Lemma 2.3 and (H_3) we obtain that

$$I_1 \leq 4\int_a^t \alpha(\{f(\tau, \int_a^\tau u(s)ds + D_u, u(\tau)) : \ u \in C_0\})\}d\tau \leq 4\int_a^t l(\tau)\alpha(C_0(\tau))d\tau$$

which means that

$$I_1 \leq 4\int_a^b l(\tau)d\tau \sup\{\alpha(C_0(t)) : \ t \in [a, b])\} \leq 4\int_a^b l(\tau)d\tau\beta(C_0).$$

On the other hand, By Lemmas 2.2, 2.3 and hypothesis (H_3) we have

$$I_2 \leq \frac{1}{b-a}\int_a^b I_1 dt \leq 4\int_a^b l(\tau)d\tau\beta(C_0).$$

Consequently

$$\beta(C_1) \leq 2(4\int_a^b l(\tau)d\tau)\beta(C_0).$$

Let us obtain $\beta(C_2)$. By the above argument we know that $\beta(C_2) = \sup\{\alpha(T(C_1(t))) : \ t \in [a, b]\}$.

$$\begin{aligned}
\alpha(T(C_1(t))) &\leq \alpha(\{\int_a^t f(\tau, \int_a^\tau u(s)ds + D_u, u(\tau))d\tau + \int_a^t h(\tau)d\tau : \ u \in C_1\}) \\
&+ \frac{1}{b-a}\alpha(\int_a^b (\int_a^t f(\tau, \int_a^\tau u(s)ds + D_u, u(\tau))d\tau + \int_a^t h(\tau)d\tau)dt : \ u \in C_1\}) \\
&= \alpha(\{\int_a^t f(\tau, \int_a^\tau u(s)ds + D_u, u(\tau))d\tau : \ u \in C_1\}) \\
&+ \frac{1}{b-a}\alpha(\int_a^b (\int_a^t f(\tau, \int_a^\tau u(s)ds + D_u, u(\tau))d\tau)dt : \ u \in C_1\}) = J_1 + J_2,
\end{aligned} \tag{16}$$

where $J_1 = \alpha(\{\int_a^t f(\tau, \int_a^\tau u(s)ds + D_u, u(\tau))d\tau : \ u \in C_1\})$ and $J_2 = \frac{1}{b-a}\alpha(\int_a^b(\int_a^t f(\tau, \int_a^\tau u(s)ds + D_u, u(\tau))d\tau)dt : \ u \in C_1\})$.

By Lemma 2.3 and (H_3) we have that

$$\begin{aligned} J_1 \leq & 4\int_a^t \alpha(\{f(\tau, \int_a^\tau u(s)ds + D_u, u(\tau)) : \ u \in C_1\})\}d\tau \leq 4\int_a^t l(\tau)\alpha(C_1(\tau))d\tau \\ & \leq 4^2 \int_a^t l(\tau)(\int_a^\tau l(s)ds)d\tau \sup\{\alpha(C_0(t)) : \ t \in [a,b]\} \\ & = \frac{4^2(\int_a^t l(\tau)d\tau)^2}{2} \sup\{\alpha(C_0(t)) : \ t \in [a,b]\} \end{aligned} \tag{17}$$

which means that

$$J_1 \leq \frac{4^2(\int_a^b l(\tau)d\tau)^2}{2}\beta(C_0).$$

On the other hand, By Lemmas 2.2, 2.3 and (H_3) we obtain

$$J_2 \leq \frac{1}{b-a}\int_a^b J_1 dt \leq \frac{4^2(\int_a^b l(\tau)d\tau)^2}{2}\beta(C_0).$$

Therefore,

$$\beta(C_2) \leq 2\frac{4^2(\int_a^b l(\tau)d\tau)^2}{2}\beta(C_0).$$

Finally, by using a recurrence argument we get that

$$\beta(C_n) \leq 2\frac{4^n(\int_a^b l(\tau)d\tau)^n}{n!}\beta(C_0). \tag{18}$$

Inequality (18) implies that $\beta(C_n) \to 0$ as $n \to +\infty$ and therefore (b) holds.

Step 7. Notice that the above steps allow us to apply Theorem 2.1 and thus we can guarantee that T has a fixed point.

□

4.1. Consequences

Corollary 4.1. *Let $\mathbb{B}$ be a finite dimensional Banach space and let $f : [a,b] \times \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ be a continuous function satisfying condition (H_4) and (4) and (7) (or (6) and (8), respectively). Let $h \in L^1([a,b],\mathbb{B})$ be a function with $\int_a^b h(t)\,dt = 0$. Then, Problem (1) has at least one solution $u \in \mathcal{C}^2([a,b],\mathbb{B})$ whenever the following assumptions holds.*

(A_1) *[A growth condition]*

$$\lim_{s\to\infty} \frac{\Gamma(s+\delta_s)}{s} < \begin{cases} \frac{1-2\int_a^b k(t)dt}{2(b-a)^2}, & \text{if, } \alpha = 1 \\ \frac{1}{2(b-a)^2}, & \text{if, } 0 \leq \alpha < 1 \end{cases}$$

where δ_s is the constant given in (4).

The main result in [3] can be obtained as a consequence of Theorem 4.1. Indeed, let $g:[a,b]\times\mathbb{B}\to\mathbb{B}$ be a sequentially weak-strong continuous function satisfying that

$$\begin{aligned} &\text{for each } M>0, \text{ there exists } \delta_M>0 \text{ such that } \langle g(t,y+x),x\rangle_-\geq 0,\\ &\text{for all } t\in[a,b], y\in B_M[0] \text{ and } \|x\|\geq\delta_M. \end{aligned} \tag{19}$$

and it is a strictly accretive operator with respect to the second variable, that is,

$$\langle g(t,x)-g(t,y),x-y\rangle_- > 0 \text{ for all } t\in[a,b],\ x,y\in\mathbb{B},\ \text{with } x\neq y. \tag{20}$$

If g satisfies conditions (19) and (20), then the mapping $f:[a,b]\times\mathbb{B}\times\mathbb{B}\to\mathbb{B}$ defined by $f(t,x,z)=g(t,x)$ fulfills conditions (4) and (7). Let us see that f also fulfills conditions $(H_1)-(H_3)$.

By hypothesis g is weak-strong sequentially compact, then f satisfies (H_2) and moreover since $\mathbb{B}$ is reflexive $\alpha(f(t,A,C))=\alpha(g(t,A))=0\leq 0\cdot\alpha(C)$, hence f also satisfies (H_3).

Next we will show that f verifies condition (H_1). Otherwise, there exist $A\in\mathcal{B}(\mathbb{B})$ and $\epsilon>0$ such that for every $n\in\mathbb{N}$ we can find $(t_n,x_n),(s_n,y_n)\in[a,b]\times A$ with $\|(t_n,x_n)-(s_n,y_n)\|\leq\frac{1}{n}$ and with $\|g(t_n,x_n)-g(s_n,y_n)\|>\epsilon$. Since $\mathbb{B}$ is reflexive, we can assume that $(t_n,x_n)\rightharpoonup(t_0,x_0)$ and $(s_n,y_n)\rightharpoonup(t_0,x_0)$. Because g is weak-stong sequentially continuous, we have

$$g(t_n,x_n)\to g(t_0,x_0),\quad g(s_n,y_n)\to g(t_0,x_0),$$

therefore

$$\|g(t_n,x_n)-g(s_n,y_n)\|\to 0,$$

which is a contradiction.

Finally, if there exists a non-decreasing function $\Gamma:[0,\infty)\to[0,\infty)$ such that

$$\|g(t,x)\|\leq\Gamma(\|x\|),\qquad \text{for all } (t,x)\in[a,b]\times\mathbb{B}.$$

It is clear that

$$\|f(t,x,y)\|\leq\Gamma(\|x\|)+0\cdot\|y\|\,,\qquad \text{for all } (t,x,y)\in[a,b]\times\mathbb{B}\times\mathbb{B}$$

and hence f satisfies (H_4).

Summarizing, as a consequence of Theorem 4.1, we may affirm:

Corollary 4.2. *([3]) Problem*

$$\begin{cases} u''(t)=g(t,u(t))+h(t)\\ u(a)-u(b)=u'(a)-u'(b)=0, \end{cases} \tag{21}$$

has at least one solution $u\in\mathcal{C}^2([a,b],\mathbb{B})$ whenever the following assumptions hold.

1. *$g:[a,b]\times\mathbb{B}\to\mathbb{B}$ be a sequentially weak-strong continuous function satisfying (19) and (20),*
2. *Let $h\in L^1([a,b],\mathbb{B})$ with $\int_a^b h(t)\,dt=0$,*
3. *there exists a non-decreasing function $\Gamma:[0,\infty)\to[0,\infty)$ satisfying*

$$\lim_{s\to\infty}\frac{\Gamma(s+\delta_s)}{s}<\frac{1}{2(b-a)^2},$$

where δ_s is the constant given in (4), such that

$$\|f(t,x)\|\leq\Gamma(\|x\|),\qquad \textit{for all } (t,x)\in[a,b]\times\mathbb{B}.$$

Next, we will show that in Theorem 4.1, neither the hypothesis $\int_a^b h(t)\,dt = 0$ nor condition (4) cannot be omitted:

Example 4.1.
$$\begin{cases} u''(t) = \dfrac{u(t)}{1+|u(t)|} + 2 \\ u(0) = u(1), \\ u'(0) = u'(1), \end{cases} \tag{22}$$

Does not have solution.

Suppose that u_0 is a solution of (22), then if we call $v(t) = u_0'(t)$, we have

$$v'(t) = \frac{\int_0^t v(s)ds + u_0(0)}{1 + \left|\int_0^t v(s)ds + u_0(0)\right|} + 2,$$

Then,

$$0 = u_0'(1) - u_0'(0) = v(1) - v(0) = \int_0^1 \frac{\int_0^t v(s)ds + u_0(0)}{1 + \left|\int_0^t v(s)ds + u_0(0)\right|} dt + 2,$$

This means that

$$\int_0^1 \frac{\int_0^t v(s)ds + u_0(0)}{1 + \left|\int_0^t v(s)ds + u_0(0)\right|} dt = -2,$$

which is a contradiction.

On the other hand, it is not difficult to see that the function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ defined by $f(t,x) = \frac{x}{1+|x|}$ is under the conditions of the main result (see [3, Exampe 5.1]), therefore hypothesis $\int_a^b h(t)dt = 0$ cannot be omitted.

If we call $f(t,x) = \frac{x}{1+|x|} + 2$, we may say that in the main result condition (4) cannot be removed.

5. Examples

In this section we present two example of application of Theorem 4.1 for a systems of $n \in \mathbb{N}$ second order differential equations.

Example 5.1. Consider the Banach space $(\mathbb{R}^n, \|\cdot\|_1)$. If $h \in L^1([a,b], \mathbb{R}^n)$ with $\int_a^b h(t)\,dt = 0_{\mathbb{R}^n}$ (for instance, $h(t) = (\frac{1}{\sqrt{|\frac{b+a}{2} - t|}}, \frac{1}{\sqrt[3]{\frac{b+a}{2} - t}}, 0, \cdots, 0))$ and $k \in [0,1)$, then the problem

$$\begin{cases} u''(t) = \dfrac{ku'(t)}{1 + \|u'(t)\|_1} + \dfrac{u(t)}{1 + \|u(t)\|_1} + h(t) \\ u(a) = u(b), \\ u'(a) = u'(b), \end{cases} \tag{23}$$

has at least a solution in $\mathcal{C}^2([a,b],\mathbb{R}^n)$. Indeed, just to note that (23) can be rewritten as Problem (1) where $f:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ is given by $f(x,z)=\frac{x}{1+\|x\|_1}+\frac{kz}{1+\|z\|_1}$. Here, $\|\cdot\|_1$ denotes to the 1-norm of $\mathbb{R}^n$.

It is clear that f is a continuous function. On the other hand, for each $M>0$, taking $\delta_M=\frac{(M+1)k+M}{1-k}$ we have that $\langle f(y+x,z),x\rangle_-\geq 0$ for all $y\in B_M[0]$, $z\in B_{\frac{M}{b-a}}[0]$ and $\|x\|_1\geq\delta_M$. Indeed, fix $y\in B_M[0]$, $z\in B_{\frac{M}{b-a}}[0]$ and $\|x\|_1\geq\delta_M$, for any $j\in J(x)$,

$$\begin{aligned}\langle f(y+x,z),j\rangle &= \frac{1}{1+\|y+x\|_1}\langle y+x,j\rangle+k\frac{\langle z,j\rangle}{1+\|z\|_1}\\ &\geq \frac{1}{1+\|y+x\|_1}\big(\|x\|_1^2-\|x\|_1\|y\|_1\big)-k\frac{\|x\|_1\|z\|}{1+\|z\|_1}\\ &\geq \|x\|_1\big(\frac{\|x\|_1-M}{1+\|x\|_1+M}-k\big)\geq 0,\end{aligned}$$

since $\|x\|_1\geq\delta_M=\frac{(M+1)k+M}{1-k}$.

Furthermore, $\|f(x,y)\|\leq\Gamma(\|x\|)+k\|y\|^0$, where we can take $\Gamma(s)=1$. $f(\cdot,z)$ is strictly accretive (see [3]). By Corollary 4.1, we deduce that Problem (23) has at least one solution $u\in\mathcal{C}^2([a,b],\mathbb{R}^n)$.

Remark 5.1. *If one checks carefully the above example, other kinds of systems of equations can be solved. Just consider another norm $\|\cdot\|$ in $\mathbb{R}^n$. For example, we can ensure the existence of at least a periodic solution of a system of nonlinear second order differential equations of the following type:*

$$u''(t)=k\frac{u'(t)}{1+\|u'(t)\|}+\frac{u(t)}{1+\|u(t)\|}+h(t).$$

Example 5.2. Let Ω be an open convex and bounded subset of $\mathbb{R}^n$ and consider $\rho:[a,b]\times\Omega\to\mathbb{R}$ a function. This example deals to study the existence of solutions for the following partial differential equation:

$$\begin{cases}\displaystyle\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}\left(\frac{\partial^2\phi}{\partial t^2}(t,x)-\rho(t,x)\right)=k\frac{\partial\phi}{\partial t}(t,x)+\phi(t,x),\,for\ all\ (t,x)\in(a,b)\times\Omega\\ \phi(a,x)-\phi(b,x)=\dfrac{\partial\phi}{\partial t}(a,x)-\dfrac{\partial\phi}{\partial t}(b,x)=0,\,for\ all\ x\in\Omega\\ \dfrac{\partial^2\phi}{\partial t^2}(t,x)-\rho(t,x)=0\ for\ all\ (t,x)\in(a,b)\times\partial\Omega.\end{cases}\tag{24}$$

In order to do this study by using the above results, consider the operator $\Delta: L^2(\Omega)\to L^2(\Omega)$ with domain $D(\Delta):=W_0^{1,2}(\Omega)\cap W^{2,2}(\Omega)$ and, given $y\in D(\Delta)$, define $\Delta y=\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}y(x)$, where the partial derivatives are taken in the sense of distributions over Ω. Therefore, if we call $u(t):=\phi(t,\cdot)\in L^2(\Omega)$ and $h(t):=\rho(t,\cdot)\in L^2(\Omega)$. We interpret and rewrite Problem (24) as follows:

$$\begin{cases}\Delta\left(u''(t)-h(t)\right)=ku'(t)+u(t)\\ u(a)-u(b)=u'(a)-u'(b)=0.\end{cases}\tag{25}$$

Our idea is to show that there exists at least $u\in\mathcal{C}^2([a,b];L^2(\Omega))$ which is a solution of (25). To get it, we are going to assume that $h\in\mathcal{C}([a,b],L^2(\Omega))$ and moreover $\int_a^b h(t)dt=0$.

It is well known (for instance, see [**?**]) that Δ is an m-dissipative operator in $L^2(\Omega)$. Moreover, by using Green's formula, we obtain that for $u \in D(\Delta)$

$$\langle \Delta u, u\rangle = -\int_\Omega \langle \nabla u, \nabla u\rangle = -\|\nabla u\|_2^2,$$

this implies, by Poincaré inequality (see Proposition **??**), that

$$\|\Delta u\|_2 \|u\|_2 \geq \|\nabla u\|_2^2 \geq \frac{1}{\boldsymbol{c}}\,\|\nabla u\|_2\,\|u\|_2$$

and, bearing in mind (**??**), we obtain that

$$\frac{1}{\boldsymbol{c}\sqrt{\boldsymbol{c}^2+1}}\,\|u\|_{1,2} \leq \|\Delta u\|_2 \qquad \text{for all } u \in D(\Delta). \tag{26}$$

The inequality given in (26) says that Δ is an m-ψ-expansive dissipative operator (it is enough to take $\psi(t) = t/(\boldsymbol{c}\sqrt{\boldsymbol{c}^2+1})$). Then, Remark 4.12 in [**?**] guarantees that there exists $\Delta^{-1} : L^2(\Omega) \to D(\Delta)$ and it is a strictly m-dissipative operator in $L^2(\Omega)$. This fact allows us to rewrite Problem (25) as follows:

$$\begin{cases} u''(t) = \Delta^{-1}(ku'(t) + u(t)) + h(t) \\ u(a) - u(b) = u'(a) - u'(b) = 0. \end{cases} \tag{27}$$

Finally, in order to show that Problem (27) admits at least a solution, we only have to check that $f : L^2(\Omega) \times L^2(\Omega) \to D(\Delta)$ given by $f(u,v) = \Delta^{-1}(u+v)$satisfies the conditions of Theorem 4.1.

First, let us see that $\Delta^{-1} : L^2(\Omega) \to D(\Delta)$ is sequentially weak-strong continuous. Indeed, from inequality (26) it is clear that

$$\|\Delta^{-1}u\|_2 \leq \|\Delta^{-1}u\|_{1,2} \leq \boldsymbol{c}\sqrt{\boldsymbol{c}^2+1}\,\|u\|_2, \tag{28}$$

which means that $\Delta^{-1} : L^2(\Omega) \to D(\Delta)$ is continuous because it is linear, then this operator is also weak-weak-continuous. On the other hand, if M is a bounded subset of $L^2(\Omega)$, from (28), we have that $\Delta^{-1}(M)$ is a bounded subset of $W^{1,2}(\Omega)$ and since the embedding $W^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$ is compact (Rellich-Kondrachov's theorem), we have that $\Delta^{-1}(M)$ relatively compact in $L^2(\Omega)$, since Δ^{-1} is continuous we may conclude that Δ is a weak-strong continuous map.

On the other hand, let us see that $\Delta^{-1} : L^2(\Omega) \to D(\Delta)$ satisfies Condition (6). Indeed, let $M > 0$ and consider $g \in B_M[0] \subseteq L^2(\Omega)$, we have to check that $\langle \Delta^{-1}(u+g), u\rangle = \langle \Delta^{-1}u, u\rangle + \langle \Delta^{-1}(g), u\rangle$.

If we call $z := \Delta^{-1}(u) \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$, we may consider that $\Delta z = u$, Therefore by using Green's formula, Cauchy–Bunyakovsky–Schwarz inequality and (28), we derive that

$$\begin{aligned}\langle \Delta^{-1}(u+g), u\rangle &\leq -\|\nabla z\|_2^2 + \|\Delta^{-1}g\|_2 \|u\|_2 \\ &\leq -\|\nabla z\|_2^2 + \boldsymbol{c}\sqrt{\boldsymbol{c}^2+1}\,\|g\|_2\,\|u\|_2.\end{aligned}$$

This means that, for every $g \in B_M[0]$,

$$\langle \Delta^{-1}(u+g), u\rangle \leq -\|\nabla z\|_2^2 + \boldsymbol{c}\sqrt{\boldsymbol{c}^2+1}\,M\,\|u\|_2.$$

Now, by using (**??**), we get

$$\langle \Delta^{-1}(u+g), u\rangle \leq -\frac{1}{\boldsymbol{c}^2+1}\,\|z\|_{1,2}^2 + \boldsymbol{c}\sqrt{\boldsymbol{c}^2+1}\,M\,\|u\|_2.$$

Finally, since it is clear that $\|\Delta z\|_2 \leq \|\Delta z\|_{1,2} \leq \|z\|_{1,2}$ we conclude that

$$\begin{aligned}\langle \Delta^{-1}(u+g), u\rangle &\leq -\frac{1}{\boldsymbol{c}^2+1}\|u\|_2^2 + \boldsymbol{c}\sqrt{\boldsymbol{c}^2+1}\,M\,\|u\|_2 \\ &= \|u\|_2(\boldsymbol{c}\sqrt{\boldsymbol{c}^2+1}\,M - \frac{1}{\boldsymbol{c}^2+1}\|u\|_2) \leq 0,\end{aligned}$$

whenever $M\,\boldsymbol{c}\,(\boldsymbol{c}^2+1)^{3/2} \leq \|u\|_2$, hence we may take $\delta_M := M\,\boldsymbol{c}\,(\boldsymbol{c}^2+1)^{3/2}$.

Second, we have already seen that $\Delta^{-1} : L^2(\Omega) \to D(\Delta)$ is a strictly dissipative operator and thus it satisfies Condition (8).

Last step, if we define $\Gamma : [0,\infty) \to [0,\infty)$ by $\Gamma(s) := \boldsymbol{c}\sqrt{\boldsymbol{c}^2+1}\,s$. From (28) we have that $\|\Delta^{-1}u\|_2 \leq \Gamma(\|u\|_2)$ for all $u \in L^2(\Omega)$. Thus,

$$\lim_{s\to\infty}\frac{\Gamma(s+\delta_s)}{s} = \boldsymbol{c}\sqrt{\boldsymbol{c}^2+1} + \boldsymbol{c}^2(\boldsymbol{c}^2+1)^2.$$

Therefore, as a consequence of Theorem 4.1 we have the following result which guarantees the existence of periodic solutions of the differential equation (27).

Proposition 5.1. *Let Ω be an open convex and bounded subset of $\mathbb{R}^n$ and $\phi, \rho : [a,b]\times\Omega \to \mathbb{R}$ be two functions. Assume that $\phi(t,\cdot) \in L^2(\Omega)$ for $t \in [a,b]$. If $h(t) := \rho(t,\cdot)$ satisfies that $h \in \mathcal{C}([a,b], L^2(\Omega))$ and $\int_a^b h(t)dt = 0$, then Problem (27) has at least one solution in $\mathcal{C}^2([a,b]; L^2(\Omega))$, whenever*

$$\boldsymbol{c}\sqrt{\boldsymbol{c}^2+1} + \boldsymbol{c}^2(\boldsymbol{c}^2+1)^2 < \frac{1}{2(b-a)^2},$$

where $\boldsymbol{c} = \boldsymbol{c}(\Omega)$ is the constant in Poincaré's inequality.

Remark 5.2. *The optimal constant c in the Poincaré inequality (Proposition* **??***) is sometimes known as the Poincaré constant for the domain Ω. Determining the Poincaré constant is, in general, a very hard task that depends upon the value of p and the geometry of the domain Ω. Nevertheless, it is well known, see [15, Theorem 3], that if $\Omega := \{(x,y) \in \mathbb{R}^2 : \; x \in (0,1), \; 0 < y < 1-x\}$ then the Poincaré constant $c = c(\Omega, 2) = \frac{1}{\pi}$. Therefore, from Proposition 5.1,we may derive that Problem (27) on this domain Ω has at least one solution whenever $0 < b-a \leq 1$.*

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