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Fixed points of continuous nonlinear mappings in Banach spaces

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Abstract. The purpose in this article is to discuss under what conditions a nonexpansive or a condensing mapping has a fixed point when its domain is a closed, convex subset of a Hilbert space and its range is not necessarily contained within its domain. When the mapping is completely continuous and its domain is a closed, convex subset of a uniformly convex Banach space a similar result is also investigated. We also deal with the existence of fixed points of condensing mappings whose domain is the closure of an open subset of an arbitrary Banach space.

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1. Introduction

From a mathematical point of view, many problems arising from diverse areas of natural science involve the existence of solutions of nonlinear equations with the form

$$f(x) = x, \text{ or } f(x) = 0, \quad x \in A, \tag{1.1}$$

where A is a closed and convex subset of a Banach space X, and $f : A \to X$ is a nonlinear mapping. Fixed point theory plays an important role to solve Eq. (1.1). Traditionally, fixed point theory has sought classes of spaces in which a given type of mapping (nonexpansive, continuous, condensing with respect to a measure of noncompactness etc.) from a nonempty closed, bounded, convex set into itself always has a fixed point. In this sense, in 1965 Kirk [24] established that in a reflexive Banach space with normal structure every nonexpansive mapping $T : K \to K$, where K is a nonempty, bounded, closed, convex subset, has a fixed point, this result generalizes previous results obtained by Browder [6], Göhde [18]. Since then, a Banach space X is said to enjoy the *fixed point property (FPP in short)* if every nonexpansive mapping defined on a closed, convex and bounded subset K of X into itself has a fixed point. Up to now many efforts are being made to know the Banach spaces with this property (for instance, see [15,25,30,31]). With respect to the fixed

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points of condensing mappings, in 1955, Darbo [9] showed that every α-set-contractive mapping $T : K \to K$, where α is the Kuratowskii measure of noncompactness and K is a nonempty, bounded, closed, convex subset of an arbitrary Banach space X, has a fixed point. In 1967, Sadovskii in [32] gave a generalization of Darbo's theorem showing that every α-condensing mapping from a nonempty, closed, convex and bounded subset of a Banach space into itself has a fixed point.

Nevertheless, in some situations the class of space as well as the required mapping are determined by the application and while the space can be chosen the required mapping has an unbounded domain and moreover its range is not contain within its domain.

With this in mind we seek to relax the conditions on the domain and range of the mapping by considering more restrictive types of spaces (Hilbert space, uniformly convex Banach space). In this sense, it is interesting to see [33, Theorem 3.3] where, among other things, is proved that if every nonexpansive mapping $T : C \to C$ with C a closed, convex subset of a Hilbert space has a fixed point, then C must be bounded. On the other hand, in [10] a characterization of the existence of fixed points for nonexpansive mappings from a nonempty, closed, convex subset of a Banach space with the (FPP) into itself is given. In [26] is showed that if C is a closed, convex subset of a Banach space X such that $0 \in C$ and $T : C \to C$ is an α-condensing mapping such that $Tx \neq \lambda x$ for any $\lambda > 1$ whenever $x \in C$ with $\|x\| = r$, then T has a fixed point in C (also see [13,20,28,29]). In the previous results, the considered mapping acts from its domain into itself. Here we do not assume such situation.

2. Preliminaries

Throughout this paper we suppose that $(X, \|\cdot\|)$ is a real Banach space and that X^* is its topological dual. We use $B_r(x_0)$ and $S_r(x_0)$ to denote the closed ball and the sphere centered at $x_0 \in X$ with radius $r > 0$ respectively. Given a nonempty subset A of X we denote by ∂A the boundary of A.

If $x \in X$, we will denote by $J(x)$ the normalized duality mapping at x defined by $J(x) := \{j \in X^* : j(x) = \|x\|^2, \|j\| = \|x\|\}$. We will often use the mappings $\langle \cdot, \cdot \rangle_+ : X \times X \to \mathbb{R}$ defined by $\langle y, x\rangle_+ := \max\{j(y) : j \in J(x)\}$ and $\langle \cdot, \cdot \rangle_- : X \times X \to \mathbb{R}$ defined by $\langle y, x\rangle_- := \min\{j(y) : j \in J(x)\}$

Proposition 2.1. (for instance see [11,17]) *Let X be a Banach space and J its normalized duality mapping, then for every $x, y \in X$ the following statements are equivalent:*

(a) $\|x\| \leq \|x + \lambda y\|$ *for all* $\lambda > 0$,
(b) $\|x\| \leq \|x + \lambda y\|$ *for all* $\lambda \in [0, 1]$,
(c) *there exists* $j \in J(x)$ *such that* $\langle y, j\rangle \geq 0$ *(i.e.* $\langle y, x\rangle_+ \geq 0$*).*

A Banach space $(X, \|\cdot\|)$ is uniformly convex if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x, y \in B_1(0_X)$

$$\|x - y\| \geq \varepsilon \Longrightarrow \|\frac{x+y}{2}\| \leq 1 - \delta.$$

Let A be a closed and convex subset of a Banach space X and let $x \in X$. A point $y \in A$ is a nearest point to x in A if $\operatorname{dist}(x, A) = \|x - y\|$. The set of all nearest points denoted by $P_A(x)$ (metric projection) is convex, and coincide with $A \cap B_{\operatorname{dist}(x,A)}(x)$. It is also clear that if $x \notin A$ and there exists $P_A(x)$, then $P_A(x) \in \partial A$.

If A is a closed, convex subset of a uniformly convex Banach space X, it is well known that P_A is a continuous mapping (see for instance, [16, Theorem 3.14]). When $(X, \langle \cdot, \cdot \rangle)$ is a Hilbert space, $P_A(x)$ is the unique point of A such that $\langle x - P_A(x), y - P_A(x) \rangle \leq 0$ for all $y \in A$. This fact allows us to see that, in this case, the metric projection, P_A, is a nonexpansive mapping (see also [16, Theorem 6.2]).

If $(X, \langle \cdot, \cdot \rangle)$ is a Hilbert space and A is a nonempty, closed and convex subset of X, the normal cone to A at $x \in A$, denoted by $N_A(x)$ is defined as

$$N_A(x) := \{u \in X : \ \langle u, v - x \rangle \leq 0 \text{ for all } v \in A\}.$$

Definition 2.2. Let $(X, \|\cdot\|)$ be a Banach space and $\mathcal{B}(X)$ the family of bounded subsets of X. By a measure of non-compactness on X, we mean a function $\Phi : \mathcal{B}(X) \to \mathbb{R}^+$ satisfying:

(1) $\Phi(\Omega) = 0$ if and only if Ω is relatively compact in X,
(2) $\Phi(\overline{\Omega}) = \Phi(\Omega)$,
(3) $\Phi(\operatorname{conv}(\Omega)) = \Phi(\Omega)$, for all bounded subsets $\Omega \in \mathcal{B}(X)$, where conv denotes the convex hull of Ω,
(4) for any subsets $\Omega_1, \Omega_2 \in \mathcal{B}(E)$ we have

$$\Omega_1 \subseteq \Omega_2 \Longrightarrow \Phi(\Omega_1) \leq \Phi(\Omega_2),$$

(5) $\Phi(\Omega_1 \cup \Omega_2) = \max\{\Phi(\Omega_1), \Phi(\Omega_2)\}$, $\Omega_1, \Omega_2 \in \mathcal{B}(X)$,
(6) $\Phi(\lambda\Omega) = |\lambda|\Phi(\Omega)$ for all $\lambda \in \mathbb{R}$ and $\Omega \in \mathcal{B}(X)$,
(7) $\Phi(\Omega_1 + \Omega_2) \leq \Phi(\Omega_1) + \Phi(\Omega_2)$.

The most important examples of measures of noncompactness are the *Kuratowski measure of noncompactness* (or set measure of noncompactness)

$$\alpha(\Omega) = \inf\{r > 0 : \ \Omega \text{ may be covered by finitely many sets of diameter} \leq r\},$$

and the *Hausdorff measure of noncompactness* (or ball measure of noncompactness)

$$\beta(\Omega) = \inf\{r > 0 : \ \text{there exists a finite } r\text{-net for } \ \Omega \ \text{ in } X\}.$$

A detailed account of theory and applications of measures of noncompactness may be found in the monographs [1,2,4,11].

Definition 2.3. Let Φ be a measure of non-compactness on X and let D be a nonempty subset of X. A mapping $T : D \to X$ is said to be a $\Phi - k$-set contraction, $k \in (0, 1]$, if T is continuous and if, for all bounded subsets C of D, $\Phi(T(C)) \leq k\Phi(C)$. T is said to be Φ-condensing if T is continuous and $\Phi(T(A)) < \Phi(A)$ for every bounded subset A of D with $\Phi(A) > 0$.

A mapping $T : D \subseteq X \to X$ is said to be nonexpansive if the inequality $\|T(x) - T(y)\| \leq \|x - y\|$ holds for every $x, y \in D$. Recall that if T is nonexpansive, then T is α-1-set contractive.

A mapping $f : K \to X$ is said to be *compact* if $f(C)$ is relatively compact whenever C is a bounded subset of K. If, moreover, f is continuous on K, then f is called *completely continuous.* It is clear that every continuous mapping with a closed domain in a finite dimensional Banach space is, in fact, completely continuous. Moreover, every completely continuous mapping is k-set contractive with respect to the Kuratowski measure of noncompactness.

Definition 2.4. Let C be a nonempty subset of X. We say that $T : C \subseteq X \to X$ satisfies the Leray–Schauder condition if there exist $R > 0$ and $x_0 \in C$ such that $Tx - x_0 \neq \lambda(x - x_0)$ for any $x \in C \cap S_R(x_0)$ and for any $\lambda > 1$.

We say that the mapping $T : C \to X$ is weakly inward on C if

$$\lim_{\lambda \to 0^+} \frac{1}{\lambda} d((1 - \lambda)x + \lambda T(x), C) = 0$$

for all $x \in C$ (see [25, Chapter 10]). Such condition is always weaker than the assumption of T mapping the boundary of C into C.

The following theorems will be the key in the proof of some of our results. The first one was proved in [10]. The second one was obtained in [13].

Theorem 2.5. [10, Theorem 4.5] *Let X be a Banach space with the (FPP), and C a nonempty, closed and convex subset of X. If $T : C \to X$ is a weakly inward nonexpansive mapping, the following conditions are equivalent:*

(i) *T has a fixed point in C,*
(ii) *Either there exist $x_1 \in C$ and $R > 0$ such that $S_R(x_1) \cap C = \emptyset$ or T fulfills the Leray–Schauder condition.*

Theorem 2.6. [13, Proposition 3.2, Lemma 3.3] *Suppose M is a nonempty, closed and convex subset of a Banach space X and suppose $T : M \to M$ is Φ-condensing. Then T has a fixed point whenever T satisfies the Leray–Schauder condition.*

Remark 2.7. In the sequel, we will assume that A is a nonempty subset of a Banach space with $\partial A \neq \emptyset$ since in other case $\overline{A}$ must be the whole space and then any mapping whose domain is $\overline{A}$ goes from $\overline{A}$ into itself.

3. Fixed points in Hilbert spaces

3.1. Nonexpansive mappings

It is well known that if X is a Hilbert space then it enjoys the (FPP) and therefore using Theorem 2.5 we may assure that if C is an unbounded, closed, convex subset of X and $T : C \to X$ is a weak inward nonexpansive mapping satisfying the Leray–Schauder condition then T has a fixed point in C. This conclusion can be maintained by replacing the hypotheses on the weak inwardness and on the Leray–Schauder condition in the following way.

Theorem 3.1. *Let X be a Hilbert space and A a nonempty, closed, convex subset of X. If $f : A \to X$ is a nonexpansive mapping satisfying:*

1. *there exist $R > 0$ and $x_0 \in A$ such that either $A \cap S_R(x_0) = \emptyset$ or $\langle f(x) - x_0, x - x_0\rangle \leq R^2$ for all $x \in A \cap S_R(x_0)$,*
2. *if $x \in \partial A$ and $f(x) \neq x$, there exists $\xi_x \in A$ such that $\langle f(x)-x, \xi_x - x\rangle > 0$.*

Then, f has a fixed point in A.

Proof. Let us define the mapping $h : A \to A$ by $h(x) := P_A(f(x))$. Since X is a Hilbert space and A is a nonempty, closed and convex subset of A, then P_A is nonexpansive and, therefore, h will be also a nonexpansive mapping on A.

We claim that h has a fixed point in A. Indeed, by Theorem 2.5 it will be enough to see that if $A \cap S_R(x_0) \neq \emptyset$, then $h(x) - x_0 \neq \lambda(x - x_0)$ for all $\lambda > 1$ and for all $x \in A \cap S_R(x_0)$.

Bearing in mind the properties of the metric projection, we know that

$$\langle f(x)-h(x), y-h(x)\rangle = \langle f(x)-P_A(f(x)), y-P_A(f(x))\rangle \leq 0, \quad \forall y \in A, \tag{3.1}$$

consequently if there exist $x_1 \in A \cap S_R(x_0)$ and $\lambda > 1$ such that $h(x_1) - x_0 = \lambda(x_1 - x_0)$, using inequality (3.1) and hypothesis (1), we have

$$\begin{aligned} 0 &\geq \langle f(x_1) - (x_0 + \lambda(x_1 - x_0)), x_0 - (x_0 + \lambda(x_1 - x_0))\rangle \\ &= \langle f(x_1) - x_0 - \lambda(x_1 - x_0), -\lambda(x_1 - x_0)\rangle \\ &= -\lambda\langle f(x_1) - x_0, x_1 - x_0\rangle + \lambda^2\|x_1 - x_0\|^2. \end{aligned} \tag{3.2}$$

Therefore,

$$R^2 \geq \langle f(x_1) - x_0, x_1 - x_0\rangle \geq \lambda R^2 > R^2,$$

which is a contradiction. Thus, we can apply Theorem 2.5 and h has a fixed point in A as we claimed.

Now, let $z \in A$ such that $h(z) = z$. Let us see that, in this case, z is also a fixed point of f. Otherwise, $f(z) \notin A$ and this means that $z = h(z) \in \partial A$, hence by hypothesis (2), we have that there exists $\xi_z \in A$ with

$$\langle f(z) - z, \xi_z - z\rangle > 0$$

but the properties of the metric projection say that

$$\langle f(z) - z, y - z\rangle \leq 0 \quad \forall y \in A$$

and since this is a contradiction, we obtain that $f(z) \in A$, but then $z = h(z) = f(z)$. □

If f goes from A into itself, then condition (2) in Theorem 3.1 can be removed since it is obviously satisfied. Next result states the following characterization.

Proposition 3.2. *Let X be a Hilbert space and A a nonempty, closed, convex subset of X. If $f : A \to A$ is a nonexpansive mapping, the following conditions are equivalent:*

1. *there exist $R > 0$ and $x_0 \in A$ such that either $A \cap S_R(x_0) = \emptyset$ or $\langle f(x) - x_0, x - x_0 \rangle \leq R^2$ for all $x \in A \cap S_R(x_0)$,*
2. *f has a fixed point in A.*

Proof. (1) $\Rightarrow$ (2). Since X is a Hilbert space, then it enjoys the (FPP); hence, to obtain this implication, we only have to see that condition (ii) in Theorem 2.5 is satisfied. To get a contradiction, suppose that there exists $x_1 \in A \cap S_R(x_0)$ and $\mu > 1$ such that $f(x_1) - x_0 = \mu(x_1 - x_0)$. In this case, we have

$$\langle f(x_1) - x_0, x_1 - x_0 \rangle = \langle \mu(x_1 - x_0), x_1 - x_0 \rangle = \mu \|x_1 - x_0\|^2 > R^2,$$

which contradicts assumption (1).

(2) $\Rightarrow$ (1) Let $x_0 \in A$ be a fixed point of f. In this case, given $R > 0$ either $S_R(x_0) \cap A = \emptyset$ or if $x \in S_R(x_0) \cap A$, since f is a nonexpansive mapping, we derive

$$\begin{aligned} \langle f(x) - x_0, x - x_0 \rangle &= \langle f(x) - f(x_0), x - x_0 \rangle \\ &\leq \|f(x) - f(x_0)\| \|x - x_0\| \\ &\leq \|x - x_0\|^2 = R^2. \end{aligned}$$

□

Remark 3.3. Condition (2) in Theorem 3.1 is always weaker than the assumption of $f(x) \in I_A(x) := \{x + \lambda(z - x) : \lambda \geq 1,\ z \in A\}$ for all $x \in \partial A$. This implies that if f is inward (see [7] and [25, Chapter 10]), then f satisfies condition (2) in Theorem 3.1. In particular, it is weaker than f maps the boundary of A into A.

Let C be a closed, convex subset of a Banach space X and $T : C \to C$ a nonexpansive mapping. Halpern [19] proposed the following algorithm.

$$x_1 \in C,\ x_{n+1} = \alpha_n x_1 + (1 - \alpha_n) T(x_n) \tag{3.3}$$

where (α_n) is a sequence of parameters in $[0, 1]$. Halpern was the first to study the convergence of the iterative process (3.3) in the framework of Hilbert spaces and under a suitable choice of parameters (α_n). Wittmann [35] improved Halpern's result and proved the strong convergence of this scheme (still in a Hilbert space) whenever $(\alpha_n)_{n \in \mathbb{N}}$ be a sequence in $(0, 1)$ satisfying:

(i) $\sum_{n=1}^{+\infty} \alpha_n = +\infty$,
(ii) $\lim_{n \to +\infty} \alpha_n = 0$,
(iii) $\sum_{n=1}^{+\infty} |\alpha_n - \alpha_{n-1}| < +\infty$, (for example $\alpha_n = \frac{1}{n}$).

In this sense, in [10, Theorem 4.10, Remark 4.11] the following result is obtained.

Theorem 3.4. *Let X be a uniformly smooth Banach space (i.e., X^* is uniformly convex), C be a closed, convex subset of X, and $T : C \to C$ be a nonexpansive. The following assumptions are equivalent:*

1. *The sequence* (x_n) *given in* (3.3), *where* (α_n) *satisfies* $(i), (ii)$ and (iii), *converges strongly to a fixed point of* T.
2. *The sequence* (x_n) *given in* (3.3), *where* (α_n) *satisfies* $(i), (ii)$ and (iii), *is bounded.*

In our context, if we consider the following iterative process

$$x_1 \in A, \quad x_{n+1} = \alpha_n x_1 + (1 - \alpha_n) P_A(f(x_n)), \tag{3.4}$$

where (α_n) is a sequence of parameters fulfilling $(i), (ii)$ and (iii). Condition (1) of Theorem 3.1 can be replaced by the following one:

Proposition 3.5. *Let* X *be a Hilbert space and* A *a nonempty, closed, convex subset of* X. $f : A \to X$ *is a nonexpansive mapping satisfying condition (2). If the sequence* (x_n) *given in* (3.4) *is bounded, then* (x_n) *converges strongly to a fixed point of* f.

Proof. Since X is a Hilbert space, then it is uniformly smooth and $h := P_A \circ f : A \to A$ is nonexpansive. Thus, since (x_n) is bounded we can apply Theorem 3.4 to obtain that (x_n) converges strongly to a fixed point of h. Now, the last part of the proof of the previous theorem shows that such fixed point is also a fixed point of f. □

Proposition 3.6. *Let* X *be a Hilbert space and* A *a nonempty, closed, convex subset of* X. $f : A \to X$ *is a nonexpansive mapping satisfying condition (2). If the sequence* (x_n) *defined by* $x_1 \in A$ *and* $\lambda \in (0, 1)$,

$$x_{n+1} = \lambda x_n + (1 - \lambda) P_A(f(x_n)),$$

is bounded, then (x_n) *converges weakly to a fixed point of* f.

Proof. Since X is a Hilbert space, the mapping $h := P_A \circ f : A \to A$ is nonexpansive. By hypothesis (x_n) is a bounded sequence and then according to [21, Lemma 2] we know that $\|x_n - h(x_n)\| \to 0$ i.e.; (x_n) is a bounded approximate fixed point sequence of h, this implies by [10, Theorem 4.5] that h has a fixed point in A. Condition (2) and Theorem 3.1 say that such fixed point is also a fixed point of f. Finally, we may apply [14, Theorem 6.4] to obtain that (x_n) converges weakly to a fixed point of f. □

3.2. α-condensing mappings

Theorem 2.6 is related to a theorem proved by W. V. Petryshyn in [26]. Such a result works when the mapping f goes from a nonempty, closed, convex subset of a Banach space into itself and moreover f is Φ-condensing (also see [20, Theorem 6.1]). Next, we will give conditions to guarantee the existence of some fixed point for an α-condensing mapping although the mapping does not leave invariant its domain. In this sense, we will see that the same assumptions of Theorem 3.1 are valid in this context.

Theorem 3.7. *Let* X *be a Hilbert space and* A *a nonempty, closed, convex subset of* X. *If* $f : A \to X$ *is an* α*-condensing mapping satisfying:*

1. *there exist* $R > 0$ *and* $x_0 \in A$ *such that either* $A \cap S_R(x_0) = \emptyset$ *or* $\langle f(x) - x_0, x - x_0 \rangle \le R^2$ *for all* $x \in A \cap S_R(x_0)$,

2. *if* $x \in \partial A$ *and* $f(x) \neq x$*, there exists* $\xi_x \in A$ *such that* $\langle f(x)-x, \xi_x - x\rangle > 0$.

Then, f *has a fixed point in* A.

Proof. Let us define the mapping $h : A \to A$ by $h(x) := P_A(f(x))$. Since X is a Hilbert space and A is a nonempty, closed and convex subset of A, then P_A is nonexpansive (and therefore α-1-set contractive) and since f is α-condensing h will be also an α-condensing mapping on A.

Following the first step in the proof of Theorem 3.1 we obtain that if $A \cap S_R(x_0) \neq \emptyset$, then $h(x) - x_0 \neq \lambda(x - x_0)$ for all $\lambda > 1$ and for all $x \in A \cap S_R(x_0)$. It means that h fulfills the hypotheses of Theorem 2.6, hence h has a fixed point.

Finally, let $z \in A$ such that $h(z) = z$. Let us see that, in this case, z is also a fixed point of f. Otherwise, $f(z) \notin A$ and this means that $z = h(z) \in \partial A$, hence by hypothesis (2), we have that there exists $\xi_z \in A$ with

$$\langle f(z) - z, \xi_z - z\rangle > 0$$

but the properties of the metric projection say that

$$\langle f(z) - z, y - z\rangle \leq 0 \quad \forall y \in A$$

and since this is a contradiction, we obtain that $f(z) \in A$, but then $z = h(z) = f(z)$. □

Corollary 3.8. *If* $A \subseteq \mathbb{R}^n$ *is a nonempty, closed and convex set, then every continuous mapping* $g : A \to \mathbb{R}^n$ *such that*

(i) *there exist* $R > 0$ *and* $x_0 \in A$ *such that either* $A \cap S_R(x_0) = \emptyset$ *or* $\langle g(x), x - x_0\rangle \leq 0$ *for all* $x \in A \cap S_R(x_0)$,

(ii) $g(x) \notin N_A(x)$ *for all* $x \in \partial A$,

has a zero in A *(i.e.; there is* $x \in A$ *such that* $g(x) = 0$*).*

Proof. Let us define the mapping $f : A \to \mathbb{R}^n$ by $f(x) := g(x) + x$. Since g is continuous in A, then f is completely continuous in A. Now, let us see that f satisfies the conditions (1) and (2) from Theorem 3.7. Indeed,

$$\langle f(x)-x_0, x-x_0\rangle = \langle x+g(x)-x_0, x-x_0\rangle = \|x-x_0\|^2 + \langle g(x), x-x_0\rangle, \quad (3.5)$$

if $x \in A \cap S_R(x_0)$ by condition (i) we know that $\langle g(x), x - x_0\rangle \leq 0$ therefore inequality (3.5) yields

$$\langle f(x) - x_0, x - x_0\rangle \leq R^2.$$

On the other hand, if $x \in \partial A$, condition (ii) implies $f(x) - x = g(x) \notin N_A(x)$ i,e,; there exists $\xi_x \in A$ such that $\langle f(x) - x, \xi_x - x\rangle > 0$, and this means that condition (2) is satisfied.

Finally, since f fulfills the hypothesis of Theorem 3.7, we know that there exists $x_1 \in A$ such that $f(x_1) = x_1$ and then $g(x_1) = 0$. □

It is worth noticing that Corollary 3.8 allows us to recapture [8, Theorem 5.1].

Corollary 3.9. *If $A \subseteq \mathbb{R}^n$ is a nonempty, closed, bounded and convex set, then every continuous mapping $g : A \to \mathbb{R}^n$ such that $g(x) \notin N_A(x)$ for all $x \in \partial A$, has a zero in A.*

Corollary 3.10. *If $A \subseteq \mathbb{R}^n$ is a nonempty, closed and convex set, then every continuous mapping $g : A \to \mathbb{R}^n$ such that*

(i) *there exist $R > 0$ and $x_0 \in A$ such that either $A \cap S_R(x_0) = \emptyset$ or $\langle g(x), x - x_0\rangle \geq 0$ for all $x \in A \cap S_R(x_0)$,*
(ii) *for each $x \in \partial A$ there exists $\xi_x \in A$ such that $\langle g(x), \xi_x - x\rangle < 0$,*

has a zero in A.

Proof. Define the mapping $f : A \to \mathbb{R}^n$ by $f(x) = x - g(x)$. It is easy to see that f satisfies all the hypothesis of Theorem 3.7, consequently there exists $x_1 \in A$ with $f(x_1) = x_1$, which allows us to conclude that $g(x_1) = 0$. □

4. Fixed points in uniformly convex Banach spaces

In this section we present a fixed point result for completely continuous mappings whose domain is a nonempty, closed and convex subset of a uniformly convex Banach space.

Lemma 4.1. *Let X be a uniformly convex Banach space, A is a nonempty, closed and convex subset of X and $P_A : X \to A$ the metric projection. Then, for every $x \in X$ the following condition fulfilled*

$$\langle v - P_A(x), x - P_A(x)\rangle_- \leq 0, \quad \forall v \in A.$$

Proof. Let $x \in X$ and $v \in A$. Since $P_A(x) \in A$ and A is a convex subset, then there exist $w \in A$ and $t_0 \in [0, 1]$ such that

$$v = (1 - t_0)P_A(x) + t_0 w. \tag{4.1}$$

By definition of metric projection, for every $t \in [0, 1]$, we have

$$\|x - P_A(x)\| \leq \|x - ((1 - t)P_A(x) + tw)\| = \|x - P_A(x) + t(P_A(x) - w)\|.$$

Thus, we know, see Proposition 2.1, that there exists $j \in J(x - P_A(x))$ such that $\langle P_A(x) - w, j\rangle \geq 0$.

On the other hand, from (4.1) it is clear that $v = P_A(x) - t_0(P_A(x) - w)$, and hence

$$v - P_A(x) = -t_0(P_A(x) - w).$$

Consequently

$$\begin{aligned}\langle v - P_A(x), x - P_A(x)\rangle_- &\leq \langle v - P_A(x), j\rangle \\ &= \langle -t_0(P_A(x) - w), j\rangle \\ &= -t_0\langle P_A(x) - w, j\rangle \\ &\leq 0.\end{aligned}$$

□

Theorem 4.2. *Let X be a uniformly convex Banach space and A a nonempty, closed and convex subset of X. If $f : A \to X$ is a continuous and compact mapping satisfying:*

(a) *there exist $R > 0$ and $x_0 \in A$ such that either $A \cap S_R(x_0) = \emptyset$ or*
$$\langle x - x_0, x_0 - f(x)\rangle_+ > 0, \quad \forall x \in A \cap S_R(x_0),$$

(b) *if $x \in \partial A$ and $f(x) \neq x$, there exists $\xi_x \in A$ such that $\langle \xi_x - x, f(x) - x\rangle_- > 0$.*

Then, f has a fixed point in A.

Proof. Let us define the mapping $h : A \to A$ by $h(x) := P_A(f(x))$. Since X is a uniformly convex Banach space and A is a nonempty, closed and convex subset of A, then P_A is continuous and since f is continuous and compact mapping, h will be also a continuous and compact mapping on A.

We claim that h has a fixed point in A. Indeed, let us see that if $A \cap S_R(x_0) \neq \emptyset$, then $h(x) - x_0 \neq \lambda(x - x_0)$ for all $\lambda > 1$ and for all $x \in A \cap S_R(x_0)$. Otherwise, if there exist $x_1 \in A \cap S_R(x_0)$ and $\lambda > 1$ such that $h(x_1) - x_0 = \lambda(x_1 - x_0)$, we have
$$\|h(x_1) - f(x_1)\| < \|x_0 - f(x_1)\|,$$
that means
$$\|x_0 - f(x_1)\| > \|x_0 + \lambda(x_1 - x_0) - f(x_1)\| = \|x_0 - f(x_1) + \lambda(x_1 - x_0)\|.$$
Therefore, by Proposition 2.1, we derive that $\langle x_1 - x_0, x_0 - f(x_1)\rangle_+ \leq 0$. Which is a contradiction with condition (a). This fact says that h is under the hypotheses of Theorem 2.6 and then h has a fixed point, as we claimed.

Finally, let $z \in A$ such that $h(z) = z$. Let us see that, in this case, z is also a fixed point of f. Otherwise, $f(z) \notin A$ (then $z \in \partial A$) and by hypothesis (b), we have that there exists $\xi_z \in A$ with
$$\langle \xi_z - z, f(z) - z\rangle_- > 0,$$
but by Lemma 4.1 we know that
$$\langle y - z, f(z) - z\rangle_- \leq 0 \;\; \forall y \in A$$
and since this is a contradiction, we obtain that $f(z) \in A$, but then $z = h(z) = f(z)$. □

5. Fixed points in an arbitrary Banach space

In the previous sections we have dealt with fixed point results for nonexpansive or condensing mappings where their domain can have empty interior, in this section we are going to assume that their domain is the closure of an open set (not necessarily convex). One of the first positive results in this direction can be found in [23, 26].

Theorem 5.1. *Let X be a Banach space and A a nonempty, open subset of X. If $f : \overline{A} \to X$ is a Φ-condensing mapping satisfying:*

(a) *there exists* $z \in A$ *such that for all* $x \in \partial A$,

$$f(x) - z \neq \lambda(x - z), \text{ for all } \lambda > 1, \tag{5.1}$$

(b) *there exists* $R > 0$ *such that either* $\overline{A} \cap S_R(z) = \emptyset$ *or* (5.1) *is satisfied for any* $x \in S_R(z) \cap \overline{A}$.

Then, f *has a fixed point in* $\overline{A}$.

Proof. Let us define $D := \{x \in \overline{A} : x = \lambda f(x) + (1-\lambda)z \text{ for some } \lambda \in [0,1]\}$. Taking $\lambda = 0$, we see that $z \in D$ and so D is nonempty and since f is a continuous mapping, D is also closed.

We claim that either f has a fixed point in ∂A or $D \subseteq A$. Indeed, if $x_0 \in D \cap \partial A$ there exists $\lambda \in (0,1]$ such that $x_0 = \lambda f(x_0) + (1-\lambda)z$. This fact implies that $x_0 - z = \lambda(f(x_0) - z)$, in this case, if $\lambda = 1$, we obtain that $f(x_0) = x_0$, otherwise, $\lambda \in (0,1)$ and therefore $f(x_0) - z = \frac{1}{\lambda}(x_0 - z)$ which is a contradiction by condition (a).

Suppose that f does not have fixed points in ∂A, then $D \cap X \backslash A = \emptyset$.

Now by Urysohn's lemma (see, for instance, [22, page 146]), there exists a continuous function $\varphi : X \to [0,1]$ such that $\varphi(x) = 1$ whenever $x \in D$ and $\varphi(x) = 0$ for all $x \in X \backslash A$. This allows us to define $F : X \to X$ by

$$F(x) := \varphi(x) f(x) + (1 - \varphi(x))z.$$

Since F is continuous, f is Φ-condensing and $F(B) \subseteq \overline{co}(\{z\} \cup f(B))$ we have

$$\Phi(F(B)) \leq \Phi(\overline{co}(\{z\} \cup f(B))) = \Phi(f(B)) < \Phi(B),$$

for all $B \subset X$ bounded and not relatively compact. This proves that F is a Φ-condensing mapping.

Now, if we take $x \in S_R(z)$ we have that

$$F(x) - z = \varphi(x) f(x) + (1 - \varphi(x)z - z = \varphi(x)(f(x) - z),$$

On one hand, if $x \notin \overline{A}$, then $F(x) = z$, which implies that $F(x) - z \neq \lambda(x - z)$ for all $\lambda > 1$. On the other hand, , if $x \in \overline{A}$ and there exists $\lambda > 1$ such that $F(x) - z = \lambda(x - z)$, we derive that

$$f(x) - z = \frac{\lambda}{\varphi(x)}(x - z),$$

which is a contradiction by condition (b).

The above argument allows us to apply Theorem 2.6 to obtain that F has a fixed point $x_0 \in X$.

If $x_0 \notin A$, then $\varphi(x_0) = 0$ and then $x_0 = z$, which contradicts the hypothesis $z \in A$. Thus $x_0 \in A$ and $x_0 = \varphi(x_0) f(x_0) + (1 - \varphi(x_0))z$, which implies that $x_0 \in D$, and so $\varphi(x_0) = 1$. Therefore $x_0 = f(x_0)$. □

As a consequence of Theorem 5.1 we may recapture Petryshyn's result [26].

Corollary 5.2. *Let* X *be a Banach space and* A *a nonempty, open and bounded subset of* X*. If* $f : \overline{A} \to X$ *is a* Φ*-condensing mapping satisfying condition (a) of* Theorem 5.1*, then* f *has a fixed point in* $\overline{A}$.

In connection to [8, Theorem 5.1,Corollary 5.2], we obtain the following result.

Corollary 5.3. *Let A be an open subset of $\mathbb{R}^n$ and $g : \overline{A} \to \mathbb{R}^n$ is a continuous function such that*

(i) *there exists $z \in A$ such that for any $x \in \partial A$*

$$g(x) \neq \lambda(x - z), \text{ for all } \lambda > 0, \text{ and} \tag{5.2}$$

(ii) *there exists $R > 0$ such that either $\overline{A} \cap S_R(z) = \emptyset$ or (5.2) is satisfied for any $x \in S_R(z) \cap \overline{A}$.*

Then, g has a zero in $\overline{A}$.

Proof. To get the proof it is enough to consider the function $f(x) = g(x) + x$ and checking that f is under the assumptions of Theorem 5.1. □

As a consequence of the last corollary, we can obtain the following version of [8, Corollary 5.2].

Corollary 5.4. *Let A be an open bounded subset of $\mathbb{R}^n$ with $0 \in A$ and $g : \overline{A} \to \mathbb{R}^n$ is a continuous function such that*

(a) *for each $x \in \partial A$, there exists $a(x) \in N_A(x)$ with $\|a(x)\| = 1$ such that $\langle a(x), g(x)\rangle \leq 0$.*

Then, g has a zero in $\overline{A}$.

Proof. Since A is bounded, then $\overline{A}$ is also bounded and thus condition (ii) in Corollary 5.3 is satisfied. To get the result, we only have to show that g fulfills condition (i) in corollary 5.3. To this end, consider $z = 0 \in A$, and $x \in \partial A$, we have to see that $g(x) \neq \lambda x$ for any $\lambda > 0$.

First, let us see that

$$\langle a(x), x\rangle > 0, \text{ for any } x \in \partial A. \tag{5.3}$$

Indeed, since $\|a(x)\| = 1$ and $0 \in A$, there exists $r > 0$ such that $ra(x) \in A$. therefore since $a(x) \in N_A(x)$, we have $0 \geq \langle a(x), ra(x) - x\rangle = r - \langle a(x), x\rangle$, i.e.; $\langle a(x), x)\rangle \geq r$.

Suppose to get a contradiction that there exist $x_1 \in \partial A$ and $\lambda > 0$ with $g(x_1) = \lambda x_1$. In this case, assumption (a) yields

$$0 \geq \langle a(x_1), g(x_1)\rangle = \langle a(x_1), \lambda x_1\rangle = \lambda\langle a(x_1), x_1\rangle,$$

which means that

$$\langle a(x_1), x_1\rangle \leq 0.$$

This contradicts inequality (5.3). □

Remark 5.5. The proof of [8, Corollary 5.2] uses the properties of the Brouwer degree theory and the authors ask if their version can be deduced from the Brouwer fixed point theorem. In this sense, the proof of the above corollary shows that fact can be done.

It was proved in [3] that given an infinite dimensional Banach space $(X, \|\cdot\|)$ and for an arbitrary measure of noncompactness Φ on X there exists a fixed point free Φ-1-set contraction self-mapping of the unit ball of X. Nevertheless, if we add the condition $I - f$ is a closed mapping (see [27, Theorem 1] and [12, Theorem 3.2]), the following result holds.

Definition 5.6. Let C be a nonempty subset of a Banach space X. A mapping $f : C \to X$ is said to be of "demiclosed-type" at zero if $0 \in R(f)$ whenever $0 \in \overline{R(f)}$.

Theorem 5.7. *Let X be a Banach space and A a nonempty, open subset of X. If $f : \overline{A} \to X$ is a Φ-1-set contraction mapping satisfying conditions (a) and (b) in* Theorem 5.1 *and $I - f$ is of demiclosed-type at zero. Then, f has a fixed point in $\overline{A}$.*

Proof. Given $n \in \mathbb{N}$, define the mapping $f_n : \overline{A} \to X$ by

$$f_n(x) = \frac{1}{n+1} z + \frac{n}{n+1} f(x).$$

It is clear that f_n is Φ-condensing (in fact, it is $\frac{n}{n+1}$-Φ-set contraction). Let us see that f_n has a fixed point in $\overline{A} \cap B_R(z)$. To this end, it will be enough to show that f_n satisfies the condition (a) and (b) in Theorem 5.1.

Suppose that $x_1 \in \overline{A}$ and there exists $\lambda > 1$ such that $f_n(x_1) - z = \lambda(x_1 - z)$. In this case, we obtain that

$$f(x_1) - z = \mu(x_1 - z),$$

where $\mu = \lambda \frac{n+1}{n} > 1$, which implies that conditions (a) and (b) in Theorem 5.1 for f_n are satisfied. Thus, there exists $x_n \in B_R(z) \cap \overline{A}$ with $f_n(x_n) = x_n$, this means that

$$f(x_n) - x_n = \frac{1}{n}(x_n - z),$$

and since $x_n \in B_R(z)$, we derive that $x_n - f(x_n) \to 0$ as $n \to +\infty$. Finally, since $I - f$ is of demiclosed-type at zero we may affirm that there exists $y \in \overline{A}$ such that $f(y) = y$. $\square$

Remark 5.8. If assumption, $I - f$ is of demiclosed at zero, in Theorem 5.7 is replaced by $I - f : \overline{A} \to X$ is ϕ-expansive (see [12, Definition 2.3]), then we can guarantee that f has a unique fixed point in $\overline{A}$ (see [12, Theorem 3.2]).

In [23, Theorem 4.5] additional conditions for a nonexpansive mapping defined on a closed, convex with nonempty interior subset of a uniformly convex Banach space to have a fixed point are given. We close this section by establishing the following result in this direction.

Corollary 5.9. *Let X be a uniformly convex Banach space and A a nonempty, open and convex subset of X. If $f : \overline{A} \to X$ is a nonexpansive mapping satisfying conditions (a) and (b) in* Theorem 5.1*. Then, f has a fixed point in $\overline{A}$.*

Proof. To get the result let us see that f is under the conditions of Theorem 5.7. Indeed, we may notice that f is α-1-set contractive mapping since it is nonexpansive. Moreover, $I - f$ is of demiclosed-type at zero since $\overline{A}$ is a closed and convex subset of X and $f : \overline{A} \to X$ is a nonexpansive mapping (for instance, see [15, Theorem 10.4]). $\square$

6. Examples

Example 1. Let $f : [0, +\infty) \to \mathbb{R}$ be the mapping defined by $f(x) = \cos(x) + \frac{1}{2}$. Clearly, f is a nonexpansive mapping. Moreover, f fulfills conditions (1) and (2) of Theorem 3.1. Indeed, if we take $x_0 = 0$ and $R = \frac{\pi}{2}$ we obtain condition (1). On the other hand, $\{0\} = \partial[0, +\infty)$ and $f(0) \neq 0$, moreover $f(0) - 0 > 0$. It means that $\cos(0) + \frac{1}{2} - 0 > 0$ therefore, taking $\xi_0 > 0$, we obtain that $(f(0) - 0)(\xi_0 - 0) > 0$. Finally, it is worth noting that, by Proposition 3.5 given an arbitrary $x_1 \in [0, +\infty)$ the recurrence sequence

$$x_{n+1} = \frac{1}{n} x_1 + \frac{n-1}{n} P_{[0,+\infty)}(f(x_n))$$

converges to a fixed point of f. As well as, by Proposition 3.6 the recurrence sequence

$$x_{n+1} = \frac{1}{2} x_n + \frac{1}{2} P_{[0,+\infty)}(f(x_n))$$

converges to a fixed point of f.

Example 2. Condition (1) in Theorem 3.1 is not a sufficient condition. Consider the mapping $f : [1, +\infty) \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{3} x + \frac{1}{2},$$

This nonexpansive mapping (in fact, it is contractive) is fixed point free in $[1, \infty)$ and it satisfies condition (1). To see this, it is enough to consider $x_0 = 1$ and $R = \frac{1}{3}$. In this case, we may notice that if we take $x_1 = 1$, then

$$x_{n+1} = \frac{1}{n} x_1 + \frac{n-1}{n} P_{[1,+\infty)}(f(x_n)) = 1.$$

Example 3. $f : [1, +\infty) \to \mathbb{R}$ defined by $f(x) = x + \frac{1}{x}$. Since $\{1\} = \partial[1, +\infty)$, This nonexpansive mapping satisfies condition (2) of Theorem 3.1 since taking $\xi_1 = 2$ we obtain

$$(f(1) - 1)(\xi_1 - 1) = 1 > 0.$$

However, it is fixed point free. In this example, it is also interesting to note that taken $z \in (1, 2)$ f fulfills condition (a) of Theorem 5.1.

Example 4. Let $f : [1, +\infty) \to \mathbb{R}$ be the mapping defined by $f(x) = x^2 - 1$. It is clear that f has a fixed point and f verifies condition (1) of Theorem 3.7 (it is enough to take $R = \frac{1}{3}$ and $x_0 = 1$), but f does not satisfy condition (2) of such theorem. Therefore, in Theorem 3.7 condition (2) is sufficient but not necessary. Now, it is also interesting to notice that f does not satisfy condition (a) in Theorem 5.1. Indeed, $\{1\} = \partial[1, +\infty[$ and given $z \in (1, +\infty)$ we have that

$$-z = f(1) - z = \lambda(1 - z), \text{ where } \lambda = \frac{z}{z-1} > 1.$$

Example 5. Let $f : \mathbb{R}\backslash(-1, 0) \to \mathbb{R}$ be the mapping defined as $f(x) = \frac{x+1}{cos(x)+2x}$. If we call $A = \mathbb{R}\backslash[-1, 0]$, we have that A is an open subset of $\mathbb{R}$ which is not convex; moreover, $\overline{A} = \mathbb{R}\backslash(-1, 0)$ and $\partial A = \{-1, 0\}$. Let us see that f is under the conditions of Theorem 5.1. In this sense, it is enough to take $z = 1 \in A$ and $R = \frac{3}{2}$.

Example 6. Define $A := \{0\} \times [0, 1] \subseteq \mathbb{R}^2$, clearly A is a bounded, closed and convex subset with $(0, 0) \in A$. If we consider the mapping $g : A \to \mathbb{R}^2$ by $g(0, x) = (-x, 1)$, this mapping is continuous and it does not have any zero on A. Let us see that f satisfies condition (a) in Corollary 5.4. Indeed, if we define $a(0, x) = (1, 0)$ it is easy to see that $a(0, x) \in N_A(0, x)$ and $\|a(0, x)\| = 1$. Moreover,

$$\langle a(0, x), g(0, x)\rangle = \langle (1, 0), (-x, 1)\rangle = -x \leq 0.$$

This example shows that the condition $(0, 0) \in \text{int}(A)$ in Corollary 5.4 cannot be omitted.

Example 7. On the Banach space $(C([0, 1]); \|\cdot\|_\infty)$ consider for some $r > 0$, the open set $A := \{x \in C([0, 1]) : \ x(0) > r\}$ and define the mapping $f : \overline{A} \to \overline{A}$ by

$$f(x)(t) = (1 - t)x(t).$$

Clearly f is a α-1 set contractive mapping since $|f(x)(t) - f(y)(t)| = (1 - t)|x(t) - y(t)|$ and this yields that $\|f(x) - f(y)\|_\infty \leq \|x - y\|_\infty$.

Suppose $z \in A$ then there are two possibilities either $z(1) = 0$ or $z(1) \neq 0$. If $x \in \overline{A}$ such that there exists $\lambda > 1$ such that $f(x) - z = \lambda(x - z)$, in particular we have

In the first case,

$$f(x)(1) - z(1) = \lambda(x(1) - z(1)) \Rightarrow 0 = \lambda x(1).$$

which is not possible whenever $x(1) \neq 0$.

In the second case,

$$-z(1) = \lambda(x(1) - z(1)) \Rightarrow (\lambda - 1)z(1) = \lambda x(1),$$

this does not hold whenever $x(1) = z(1)$.

The above argument shows that f satisfies conditions (a) and (b) in Theorem 5.7.

Suppose that f has a fixed point in $\overline{A}$, then there exists $x \in \overline{A}$ such that $x(t) = (1 - t)x(t)$ for all $t \in [0, 1]$. Which is a contradiction, because in this case $x(t) = 0$ for any $t \in (0, 1]$ and $x(0) \geq r > 0$, i.e.; $x \notin C([0, 1])$. This example shows that conditions (a) and (b) in Theorem 5.7 are not sufficient to guarantee the existence of a fixed point.

7. An application to differential equations

In this section we study the existence of at least a classical solution to the following differential equation:

$$\left.\begin{array}{l} u'(t) + \frac{(-1)^n}{u^n(t)} = p(t), \quad t \in (0,T) \\ u(0) = u(T). \end{array}\right\}, \tag{7.1}$$

where $T > 0$, $n \in \mathbb{N}$, and $p : [0,T] \to \mathbb{R}$ is a continuous function. To study the existence of solution, we argue as follows:

For each $n \in \mathbb{N}$, we define: $A_n : D(A_n) \subseteq \mathbb{R} \to \mathbb{R}$ by $A_n(x) = \frac{(-1)^n}{x^n}$, where

$$D(A_n) = \begin{cases} (-\infty, 0) \text{ if } n \textit{ is even} \\ (0, +\infty) \text{ if } n \textit{ is odd.} \end{cases} \tag{7.2}$$

It is not difficult to show that $A_n : D(A_n) \subseteq \mathbb{R} \to \mathbb{R}$ is an m-accretive operator for every $n \in \mathbb{N}$. Thus, we may apply [5, Proposition 3.8] (also see [34, Theorem 1.9.1]) and we obtain that there exists a unique classical solution $u_x : [0,T] \to \overline{D(A_n)}$ to

$$\left.\begin{array}{l} u'(t) + A_n(u(t)) = p(t), \quad t \in (0,T) \\ u(0) = x \in \overline{D(A_n)}. \end{array}\right\}, \tag{7.3}$$

Proposition 7.1. *If n is even and $p(t) > 0$ for all $t \in [0,T]$,* Eq. (7.1) *has at least a negative solution. If n is odd and $p(t) < 0$ for all $t \in [0,T]$,* Eq. (7.1) *has at least a positive solution.*

Proof. Consider the Poincaré mapping: $P : \overline{D(A_n)} \to \overline{D(A_n)}$ defined by $P(x) := u_x(T)$, where $u_x(\cdot)$ is the unique solution of Problem (7.3). Since $A_n : D(A_n) \to \mathbb{R}$ is an m-accretive operator, we may guarantee that P is a nonexpansive mapping. Therefore, to get the result we only need to see that P has a fixed point. To this end, we will show that P satisfies the condition of Proposition 3.2. In this sense, since $0 \in \overline{D(A_n)}$, it will be enough to find a $R > 0$ such that $P(x) \cdot x \leq R^2$ for all $x \in \overline{D(A_n)}$ with $|x| = R$.

Suppose that $n \in \mathbb{N}$ is even and $p(t) > 0$ for all $t \in [0,T]$ (if $n \in \mathbb{N}$ becomes odd and $p(t) < 0$ for all $t \in [0,T]$ the argument would be similar). In this case, since $\overline{D(A_n)} = (-\infty, 0]$, if $x \in (-\infty, 0]$ and $|x| = r > 0$, hence $x = -r$.

If $x \in (-\infty, 0)$, we have that

$$(p(t) - A_n(x))x = p(t)x - \frac{1}{x^{n-1}},$$

and since $p(t) > 0$ for all $t \in [0,T]$ there exists $R > 0$ such that if $x \leq -R$ then $(p(t) - A_n(x))x = p(t)x - \frac{1}{x^{n-1}} < 0$.

To get a contradiction, suppose that $u_{-R}(T) < -R$. Since u_{-R} is a continuous function and $u_{-R}(0) = -R$ there exists $\delta > 0$ such that $u_R(t) < -R$ if $t \in (T-\delta, T]$ and $u_{-R}(T-\delta) = -R$.

On the other hand, since u_{-R} is a classical solution, we have that

$$p(t) - u'_{-R}(t) = A_n(u_{-R}(t)), \quad \forall t \in (T-\delta, T).$$

Moreover,

$$(p(t) + u'_{-R}(t) - p(t))u_{-R}(t) = u'_{-R}(t)u_{-R}(t) = \frac{1}{2}(u^2_{-R}(t))', \quad \forall t \in (T-\delta, T).$$

The above argument shows that $\frac{1}{2}(u^2_{-R}(t))' \leq 0$, $\forall t \in (T-\delta, T)$. Which means that

$$u^2_{-R}(T) - u^2_{-R}(T-\delta) \leq 0 \Rightarrow u^2_{-R}(T) \leq u^2_{-R}(T_\delta) = R^2 \Rightarrow u_{-R}(T) \geq -R,$$

and the last inequality is a contradiction. □

Example 8. Consider the equation

$$\left.\begin{array}{l} u'(t) + \frac{1}{u^2(t)} = 0, \quad t \in (0,T) \\ u(0) = u(T). \end{array}\right\}, \tag{7.4}$$

It is easy to see that the solutions have the form $u(t) = \sqrt[3]{u(0)^3 - 3t}$. Therefore, Eq. (7.4) does not admit any periodic solution.

Example 9. Consider the equation

$$\left.\begin{array}{l} u'(t) + \frac{1}{u^2(t)} = -1, \quad t \in (0,T) \\ u(0) = u(T). \end{array}\right\}, \tag{7.5}$$

It is easy to see that the solutions have the form $u(t) - \arctan(u(t)) = -t + C$ Therefore, Eq. (7.5) does not admit any periodic solution.

Examples (8) and (9) show that if we drop the condition on function $p(\cdot)$ in Proposition 7.1, in general, such result is false.

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