# Hedgehogs and their widths in elliptic and hyperbolic planes

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**Abstract**

In this paper the notion of a hedgehog in non-Euclidean geometry is introduced. More specifically, the case of 2-dimensional spaces of constant curvature is considered and, in particular, spherical hedgehogs are defined in elliptic geometry and two different notions of hedgehogs in hyperbolic geometry are studied, namely, $g$-hedgehogs and $h$-hedgehogs, which extend, respectively, the corresponding well-known notions of $g$-convex and $h$-convex curves. The construction of these curves by support functions is discussed and some relations and properties are derived. Moreover, an extended notion of a width is provided for non-Euclidean hedgehogs.

## 1 Introduction

Classical real hedgehogs can be regarded as geometrical realizations of formal differences of arbitrary convex bodies in the Euclidean vector space $\mathbb{R}^{n+1}$ [25, 26]. Many classical notions for convex bodies extend to hedgehogs and quite a number of classical results find their counterparts. Of course, a few adaptations can be necessary. The notion of a hedgehog has in particular proved useful in the study of convex bodies and to geometrize certain analytical problems by considering functions as support functions [27]. The usefulness of hedgehogs in the study of convex bodies is mainly due to the fact that they often offer the possibility to provide appropriate decompositions of the convex bodies under consideration into sums of hedgehogs [27]. Hedgehogs are also useful for the study of convex bodies of constant width, a popular and fascinating topic which is now a well-established field of classical convex geometry [28].

Euclidean hedgehogs can be interpreted as envelopes of hyperplanes parameterized by their inverse Gauss map, in the sense that they can be parameterized by a parameter $u \in \mathbb{S}^n$ that, at regular points, is a normal vector to the hedgehog.

Let us consider the case $n = 1$. Given any $h \in \mathcal{C}^2(\mathbb{S}^1, \mathbb{R})$, consider the envelope $\mathcal{H}_h$ of a family of lines

$$x\cos(t) + y\sin(t) = p(t), \tag{1}$$

where $p(t) = (h \circ u)(t)$, with $u(t) = \big(\cos(t), \sin(t)\big)$. The curve $\mathcal{H}_h$ is said to be a *hedgehog with support function* $h$ and the lines of the family are called *supporting lines*. When wrapping the real line onto the unit circle through the map $u$, by abuse of language we will also say that $p$ is the support function of $\mathcal{H}_h$ and we will also write the hedgehog as $\mathcal{H}_p$.

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The envelope $\mathcal{H}_h$ is described analytically by the system of equations formed by (1) and its partial differentiation with respect to $t$:

$$-x\sin(t) + y\cos(t) = p'(t). \tag{2}$$

The unique solution of this system of equations yields the parametric expression of $\mathcal{H}_h$:

$$x_h(t) = p(t)\,\big(\cos(t),\, \sin(t)\big) + p'(t)\,\big(-\sin(t),\, \cos(t)\big).$$

Thus, given a unit vector $u(t)$ in $\mathbb{R}^2$, planar hedgehogs have exactly one cooriented supporting line that is cooriented by such $u(t)$, and exactly two cooriented supporting lines orthogonal to $u(t)$. The signed distance between these two supporting lines, given by $p(t) + p(t+\pi)$, is the width of the hedgehog in the direction $u(t)$, see Figure 1.

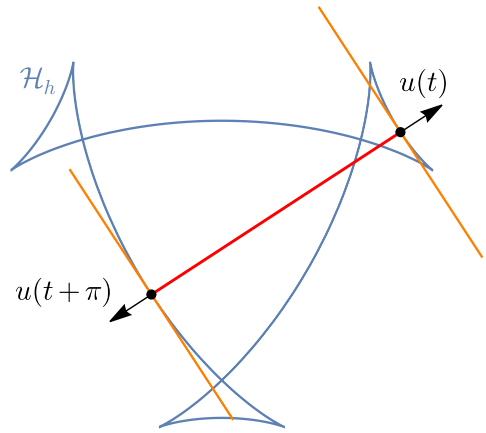


Figure 1: A planar hedgehog $\mathcal{H}_h$ of constant (positive) width.

In this paper we aim to introduce a notion of a hedgehog in non-Euclidean spaces. In particular we will work with a 2-dimensional space $\mathbb{M}^2(K)$ of constant curvature $K \neq 0$ that will serve as a model for elliptic and hyperbolic geometry. A brief introduction to non-Euclidean geometry is provided in Section 2.

In Section 3 we show that the notion of a hedgehog is affine, so that we can easily give a definition of a hedgehog in the real projective plane $\mathbb{P}^2$. As a consequence, a notion of a hedgehog in $\mathbb{M}^2(K)$ will be naturally induced.

We will see that the hedgehog structure in elliptic and hyperbolic geometry will be transferred by the planar Euclidean structure and will satisfy similar properties. In addition, in Section 4.1 we propose a definition of a width for hedgehogs in $\mathbb{M}^2(K)$ that generalizes the known definition of a width of non-Euclidean convex curves proposed by Santaló [31, 33] and studied by several authors, see e.g. [18, 14, 6, 13].

Later, in Section 4.2, we study several ways to describe hedgehogs in $\mathbb{M}^2(K)$ using support functions. From the Euclidean planar setting, a naturally induced support function is given for hedgehogs in $\mathbb{M}^2(K)$ that extends a known notion of a support function for non-Euclidean convex curves, studied by authors such as Besau, Schuster, Werner, Borisenko and Drach in several papers [3, 5, 7]. The explicit expression of hedgehogs in $\mathbb{M}^2(K)$ using this support function is given in Proposition 2. However, the value of the support function is not easily related to the concept of a width in non-Euclidean geometry. In order to ease the construction of hedgehogs of constant width in $\mathbb{M}^2(K)$ we consider the notion of a support function for non-Euclidean convex curves due to Leichtweiss [21] and we generalize it to hedgehogs (Proposition 3).

If we focus on the hyperbolic plane, it is well known that there are two notions of convexity for convex bodies in $\mathbb{H}^2$, giving rise to geodesically convex sets (or $g$-convex sets) and horocyclically convex sets (or $h$-convex sets). These notions differ in the kind of arcs used for joining any two points in the set and that are fully contained inside the set, if either geodesics or horocycles are taken. The boundaries of these sets are called, respectively, $g$-convex curves or $h$-convex curves. It is known that any $h$-convex curve is a $g$-convex curve, but the converse is not true.

The notion of a hedgehog we have outlined above, in the particular case of the hyperbolic plane, corresponds to a generalization of the notion of a $g$-convex curve, and so we also call it a $g$-hedgehog. Finally, in Section 5 we show that it is possible to extend the notion of $h$-convex curves to what we call $h$-hedgehogs. We prove in Theorem 2 that, in the same way it happens in the convex case, any $h$-hedgehog is also a $g$-hedgehog (with the converse being false). We derive some properties of $h$-hedgehogs and we introduce another notion of a width for $h$-hedgehogs, called $h$-width (Definition 8). Finally, we explore the relation of $h$-hedgehogs of constant $h$-width with the width of these curves seen as $g$-hedgehogs.

## 2 The 2-dimensional non-Euclidean space

In this section we will briefly introduce a 2-dimensional Riemannian manifold $\mathbb{M}^2(K)$ of constant curvature $K \neq 0$ that will serve as a model for elliptic and hyperbolic geometry. The interested reader can find a similar introduction to non-Euclidean geometry for instance in [29] or [34]. For a detailed treatment, see [36] or [30].

Given $K \neq 0$, consider the subset of $\mathbb{R}^3$ defined by

$$\mathbb{M}^2(K) = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : \ \operatorname{sgn}(K)\, x_1^2 + x_2^2 + x_3^2 = \frac{1}{K}, \quad x_1 \in D_K \right\},$$

where $\operatorname{sgn}(\cdot)$ is the sign function and

$$D_K = \begin{cases} ]0, +\infty[ & \text{if } K < 0, \\ \mathbb{R} & \text{if } K > 0. \end{cases}$$

Notice that if $K < 0$, then $x_1 \geq 1/\sqrt{|K|}$. We will further assume that $\mathbb{R}^3$ has the usual orientation by requiring the standard basis to be positively oriented.

The set $\mathbb{M}^2(K)$ is naturally endowed with the structure of a smooth 2-dimensional submanifold of $\mathbb{R}^3$ that we are going to equip with a metric of constant curvature $K$. Thus, $\mathbb{M}^2(K)$ will be either a 2-sphere $\mathbb{S}^2(K)$ or a hyperbolic plane $\mathbb{H}^2(K)$, with constant curvature $K$, depending on whether $K$ is positive or negative, respectively.

The polarization of the quadratic form that defines $\mathbb{M}^2(K)$ is a bilinear form $\mathcal{B} : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ whose expression is

$$\mathcal{B}(p, q) = \operatorname{sgn}(K)\, p_1\, q_1 + p_2\, q_2 + p_3\, q_3,$$

where $p = (p_1, p_2, p_3)$ and $q = (q_1, q_2, q_3)$. The norm associated with $\mathcal{B}$ is the function $\|\cdot\| : \mathbb{R}^3 \to \mathbb{C}$ defined by

$$\|\mathbf{u}\| = \sqrt{\mathcal{B}(\mathbf{u}, \mathbf{u})}.$$

The value of this norm is either positive, zero or purely imaginary which corresponds, respectively, to a space-like, a light-like or a time-like vector.

The tangent vector space of $\mathbb{M}^2(K)$ at a point $p \in \mathbb{M}^2(K)$ is the 2-dimensional vector space given by

$$T_p\mathbb{M}^2(K) = \left\{ \mathbf{v} \in \mathbb{R}^3 \ : \ \mathcal{B}(p, \mathbf{v}) = 0 \right\}.$$

The restriction of the polar form $\mathcal{B}$ to the tangent vector space of $\mathbb{M}^2(K)$ at any point $p \in \mathbb{M}^2(K)$ provides a Riemannian metric $g_p$ on $\mathbb{M}^2(K)$. It can be checked that the Gauss curvature of $\mathbb{M}^2(K)$ is in fact $K$. Therefore, $\mathbb{M}^2(K)$ is a 2-dimensional Riemannian manifold of constant curvature $K$.

Geodesics in $\mathbb{M}^2(K)$ are exactly intersections of planes through the origin with $\mathbb{M}^2(K)$. More precisely, if $p \in \mathbb{M}^2(K)$ and $\mathbf{v} \in T_p\mathbb{M}^2(K)$ with $\|\mathbf{v}\| = 1$, the geodesic that goes through $p$ in the direction $\mathbf{v}$ can be parameterized by arc length as

$$g(s) = \cos\bigl(\sqrt{K}\, s\bigr)\, p + \frac{\sin\bigl(\sqrt{K}\, s\bigr)}{\sqrt{K}}\, \mathbf{v}.$$

Note that if $K < 0$, $\sqrt{K}$ denotes $\mathbf{i}\sqrt{-K}$ and we can make use of the relations $\sin(\mathbf{i}\, x) = \mathbf{i}\sinh(x)$ and $\cos(\mathbf{i}\, x) = \cosh(x)$, for any $x \in \mathbb{R}$.

We recall that the distance between two points $p$ and $q$ in $\mathbb{M}^2(K)$ is given by

$$\mathrm{d}(p, q) = \frac{1}{\sqrt{K}} \arccos\bigl(K\, \mathcal{B}(p, q)\bigr).$$

Finally, given a regular parametric curve $\alpha$ in $\mathbb{M}^2(K)$, the (signed) geodesic curvature of $\alpha$ can be computed with the expression

$$\kappa_g(t) = \sqrt{|K|}\, \frac{\det\bigl(\alpha(t), \alpha'(t), \alpha''(t)\bigr)}{\bigl\|\alpha'(t)\bigr\|^3}.$$

# 3 Hedgehogs with respect to geodesics

## 3.1 Hedgehogs in the real projective plane

Like convexity, the notion of a hedgehog is affine, and this allows us to define a hedgehog of the real projective space $\mathbb{P}^{n+1} := \mathbb{R}P^{n+1}$ as a hedgehog of $\mathbb{R}^{n+1}$, regarded as the complement in $\mathbb{P}^{n+1}$ of any projective hyperplane (viewed as the hyperplane at infinity). This will provide us a natural extension of the concept of a hedgehog over $\mathbb{S}^{n+1}$ and $\mathbb{H}^{n+1}$ by making use of gnomonic projections. Here we will restrict ourselves to the case $n = 1$ (the interested reader can find a detailed treatment of the case $n > 1$ in [27]).

Thus, the first remark is to note that the notion of a hedgehog in $\mathbb{R}^2$ is affine (i.e., invariant under affine transformations) and can therefore be introduced in the absence of any metric structure. This can be done thanks to the set of all cooriented lines of $\mathbb{R}^2$.

**Definition 1** (Cooriented line)**.** A *cooriented line* of the affine plane $\mathbb{R}^2$ is an affine line of $\mathbb{R}^2$ together with a transverse orientation given by a crossing transverse direction indicated by a vector (or by the half-plane determined by the line and indicated by the vector).

Let $\mathcal{S}$ be the set of all cooriented line directions of $\mathbb{R}^2$ (i.e. of all cooriented vector lines if $\mathbb{R}^2$ is seen as a vector plane). Fixing an orientation of $\mathbb{R}^2$, $\mathcal{S}$ can be identified with the set of all oriented vector lines of $\mathbb{R}^2$, which is naturally equipped with the structure of a compact smooth 1-manifold, that is $\mathcal{C}^\infty$-diffeomorphic to $\mathbb{S}^1$.

**Definition 2** (Affine definition of a hedgehog of $\mathbb{R}^2$)**.** A *hedgehog of* $\mathbb{R}^2$ is the envelope of a $\mathcal{C}^2$-family of cooriented lines $\{L_l\}_{l\in\mathcal{S}}$, where $L_l$ is an affine line of $\mathbb{R}^2$ with the same coorientation as $l$.

Now we can give a definition of a hedgehog in the real projective space that extends, naturally, a known definition of a convex curve in $\mathbb{P}^2$ (see e.g. [1, 37, 9]).

**Definition 3** (Hedgehog of $\mathbb{P}^2$)**.** A curve $\mathcal{H}$ in the real projective plane $\mathbb{P}^2$ is said to be a *hedgehog of* $\mathbb{P}^2$ if $\mathcal{H}$ is a hedgehog of the real affine plane $\mathbb{R}^2 \cong \mathbb{P}^2 \setminus L$ for any projective line $L$ that does not meet $\mathcal{H}$.

If we see $\mathbb{P}^2$ as the sphere $\mathbb{S}^2$ identifying pairs of antipodal points, then a hedgehog of $\mathbb{P}^2$ can be included in an open hemisphere of $\mathbb{S}^2$ (and be regarded as a spherical hedgehog), with the quotient of the boundary of this hemisphere by $\{\mathrm{id}, -\mathrm{id}\}$ being a projective line $L$ that does not meet the hedgehog in $\mathbb{P}^2$. An important observation is that a change of this projective line (which represents the points at infinity) by another projective line that does not meet the curve won't change the hedgehog nature of the curve. In fact, we can relax the definition as follows.

**Proposition 1.** *A curve $\mathcal{H}$ in the real projective plane $\mathbb{P}^2$ is a hedgehog of $\mathbb{P}^2$ if and only if $\mathcal{H}$ is a hedgehog of the real affine plane $\mathbb{R}^2 \cong \mathbb{P}^2 \setminus L$ for some projective line $L$ that does not meet $\mathcal{H}$.*

*Proof.* Suppose that, given a projective line $L$ that does not meet $\mathcal{H}$, the curve $\mathcal{H}$ is a hedgehog of $\mathbb{R}^2 \cong \mathbb{P}^2 \setminus L$. Let $L'$ be another projective line that does not meet $\mathcal{H}$, which can be seen as an affine line of $\mathbb{R}^2$ that does not intersect $\mathcal{H}$ as a hedgehog of $\mathbb{R}^2$. Given any point $P$ of this line $L'$, we know by Theorem 1 of [24] that the number of supporting lines to the hegdehog $\mathcal{H}$ passing through $P$ is exactly 2. By sending the line $L'$ to infinity, any pair of these two supporting lines become parallel supporting lines when looking $\mathcal{H}$ as a curve of $\mathbb{P}^2 \setminus L'$. Therefore, since the curve has exactly two parallel supporting lines for each direction, it is also a hedgehog of $\mathbb{P}^2 \setminus L'$. $\square$

Finally, note that the condition that the projective line does not meet the hedgehog is necessary because otherwise the hedgehog would be an unbounded curve.

### 3.2 Geodesically hedgehog curves in elliptic and hyperbolic geometry

Once we have the notion of a hedgehog in $\mathbb{P}^2$, it can be transferred via projections to different spaces. In particular, we can use the gnomonic projection which, given $p \in \mathbb{M}^2(K)$, sends each point $m \in \mathbb{M}^2(K)$ onto the intersection of the vector line $\mathbb{R}m$ with the tangent hyperplane $p + T_p\mathbb{M}^2(K)$. This hyperplane can be identified with $T_p\mathbb{M}^2(K)$ and also with $\mathbb{R}^2$.

Explicitly, given $p \in \mathbb{M}^2(K)$, the *$p$-gnomonic projection* is the function $\rho_{p,K} : S_p(K) \to p + T_p\mathbb{M}^2(K)$ given by

$$\rho_{p,K}(q) = \frac{1}{K\,\mathcal{B}(p,q)}\, q, \tag{3}$$

where $S_p(K) = \left\{q \in \mathbb{M}^2(K) \ : \ \mathcal{B}(p,q) \neq 0\right\} \subset \mathbb{M}^2(K)$.

Given the affine nature of a hedgehog, we can use a gnomonic projection to define *hedgehogs* in $\mathbb{S}^2(K)$ and in $\mathbb{H}^2(K)$ from the planar setting and study the properties of these hedgehogs when we add the ambient space metric. In addition, *supporting lines* to planar hedgehogs are transformed into *supporting geodesics* to the corresponding hedgehogs in $\mathbb{S}^2(K)$ and in $\mathbb{H}^2(K)$. For this reason, the hedgehogs that we obtain can also be called *hedgehogs with respect to geodesics* or *geodesically hedgehog curves.* In $\mathbb{H}^2(K)$, we also call them *$g$-hedgehogs* and we will see later that they constitute a generalization of geodesically convex curves (or $g$-convex curves).

From the construction, we can parameterize hedgehogs of $\mathbb{M}^2(K)$ via the support function of the associated planar Euclidean hedgehog. However, note that metric properties (such as being of constant width) will not be preserved through this transformation.

**Definition 4** (Hedgehog in $\mathbb{M}^2(K)$)**.** A curve contained in an open hemisphere of $\mathbb{S}^2(K)$ is said to be a *hedgehog of* $\mathbb{S}^2(K)$ or a *spherical hedgehog* if its $p$-gnomonic projection from $\mathbb{S}^2(K)$ onto $p + T_p\mathbb{S}^2(K)$ is a hedgehog of $p + T_p\mathbb{S}^2(K) \cong \mathbb{R}^2$ for all $p \in \mathbb{S}^2(K)$ such that the curve is fully contained in the open hemisphere of $\mathbb{S}^2(K)$ with center $p$.

A curve of $\mathbb{H}^2(K)$ is said to be a *hedgehog of* $\mathbb{H}^2(K)$ or a *g-hedgehog* if its $p$-gnomonic projection from $\mathbb{H}^2(K)$ onto $p + T_p\mathbb{H}^2(K)$ is a hedgehog of $p + T_p\mathbb{H}^2(K) \cong \mathbb{R}^2$ for all $p \in \mathbb{H}^2(K)$.

In either case, it can also be called a *hedgehog with respect to geodesics* or a *geodesically hedgehog curve.*

The *inverse p-gnomonic projection* is the function $\rho_{p,K}^{-1} : B_p(K) \to \mathbb{M}^2(K) \subset \mathbb{R}^3$ given by

$$\rho_{p,K}^{-1}(x) = \frac{1}{\sqrt{K\,\mathcal{B}(x,x)}}\, x, \tag{4}$$

where $B_p(K) = p + T_p\mathbb{M}^2(K)$ if $K > 0$ or $B_p(K)$ is an open ball of radius $1/\sqrt{|K|}$ (with the Lorentz metric) centered at $p$ in the affine plane $p + T_p\mathbb{M}^2(K)$ if $K < 0$. Indeed, if $K > 0$, the function is well defined for any $x \in p + T_p\mathbb{M}^2(K)$. However, if $K < 0$, given $x \in p+T_p\mathbb{M}^2(K)$, the value $\rho_{p,K}^{-1}(x)$ is well defined if and only if $\mathcal{B}(x,x) < 0$. Now, $x = p+u$, where $u \in T_p\mathbb{M}^2(K)$. Thus, since $\mathcal{B}(p,u) = 0$, we have that

$$\mathcal{B}(x,x) = \mathcal{B}(p,p) + \mathcal{B}(u,u) = \frac{1}{K} + \|u\|^2 < 0$$

is equivalent to

$$\|u\|^2 < -\frac{1}{K} = \frac{1}{|K|}.$$

Notice that a choice of $p \in \mathbb{M}^2(K)$ in Definition 4 above is equivalent to a choice of the projective line $L$ of points at infinity in the definition of a hedgehog of $\mathbb{P}^2$. We have seen in Proposition 1 that it suffices to fix a projective line $L$ because the hedgehog nature is preserved if the points at infinity are replaced by another projective line that does not meet the hedgehog. Therefore, this means that it suffices to write Definition 4 "for some $p \in \mathbb{M}^2(K)$" instead of "for all $p \in \mathbb{M}^2(K)$". Thus, without loss of generality, we can consider the point $p = \big(1/\sqrt{|K|},\, 0,\, 0\big)$ so that $p+T_p\mathbb{M}^2(K)$ is the plane $x = 1/\sqrt{|K|}$. In such a case, the *p-gnomonic projection* is the function $\rho_K := \rho_{p,K} : S \to \big\{1/\sqrt{|K|}\big\} \times \mathbb{R}^2$ given by

$$\rho_K(x,y,z) = \frac{1}{\sqrt{|K|}}\left(1,\, \frac{y}{x},\, \frac{z}{x}\right),$$

where $S = \{q \in \mathbb{M}^2(K) \ : \ q_1 \neq 0\} \subset \mathbb{M}^2(K)$.

The *inverse p-gnomonic projection* is the function $\rho_K^{-1} := \rho_{p,K}^{-1} : \big\{1/\sqrt{|K|}\big\} \times B(K) \to \mathbb{M}^2(K) \subset \mathbb{R}^3$ given by (4), where $B(K) = \mathbb{R}^2$ if $K > 0$ or $B(K) \subset \mathbb{R}^2$ is the open ball of radius $1/\sqrt{|K|}$ centered at $(0,0)$ if $K < 0$.

See in Figure 2 an example of a hedgehog in $\mathbb{S}^2$ and in $\mathbb{H}^2$.

**Definition 5** (Projective hedgehog in $\mathbb{M}^2(K)$)**.** Let $p \in \mathbb{M}^2(K)$ and let $\alpha$ be a hedgehog of $\mathbb{M}^2(K)$ such that if $K > 0$, $\alpha$ is contained in the open hemisphere with center $p$. The hedgehog $\alpha$ is said to be *projective* if its $p$-gnomonic projection from $\mathbb{M}^2(K)$ onto $p+T_p\mathbb{M}^2(K)$ is a projective hedgehog of $p + T_p\mathbb{M}^2(K) \cong \mathbb{R}^2$.

Analogously as discussed above in Definition 4, the definition of a projective hedgehog in $\mathbb{M}^2(K)$ does not depend on the choice of the point $p \in \mathbb{M}^2(K)$ as long as it satisfies the said condition if $K > 0$.

Similarly as in the planar case, projective hedgehogs in $\mathbb{M}^2(K)$ are those for which through each point of the hedgehog we have two different cooriented supporting geodesics that are coincident except for their coorientation (the precise definition of a cooriented geodesic is given below in Section 4.1).

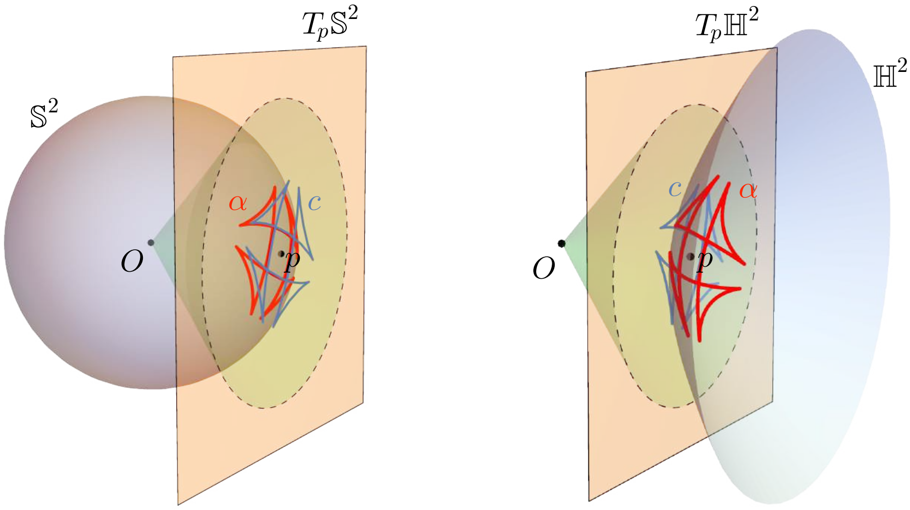


Figure 2: A hedgehog $\alpha$ of $\mathbb{S}^2$ (left) and of $\mathbb{H}^2$ (right) coming from the inverse $p$-gnomonic projection of the same planar hedgehog $c$ located at the corresponding tangent plane at $p = (1, 0, 0)$.

**Remark 1.** The gnomonic projection sends convex bodies in an open hemisphere of the sphere onto Euclidean convex bodies of $\mathbb{R}^2$ bijectively. Analogously, it sends convex bodies of the hyperboloid onto Euclidean convex bodies inside an open ball of $\mathbb{R}^2$, with the corresponding metric, bijectively (see [23] and [4]). Therefore, by construction, $\mathcal{C}^2$-hedgehogs in $\mathbb{M}^2(K)$ constitute, indeed, a generalization of non-Euclidean $\mathcal{C}^2$-convex curves of positive geodesic curvature. Notice that the notion of hyperbolic convexity taken here is by geodesics. The name *g-hedgehog* introduced in Definition 4 has been chosen to differentiate these hedgehogs from another class of hedgehogs that will be introduced in Section 5, namely, *h-hedgehogs*.

# 4 Widths and support functions for non-Euclidean hedgehogs

There are several definitions of a width or a diameter for non-Euclidean convex curves in the literature, as well as several definitions of support functions, which should not be confused as some of them are in general not equivalent (see [14, 8]).

The most natural and geometrical definition of a width in a non-Euclidean setting is due to Blaschke and Santaló. Santaló introduced his concept of a width in the sphere [31, 32] and in the hyperbolic plane [33] as follows: given a point $A_1$ of the curve, consider the supporting geodesic $G_1$ that goes through this point. Now, consider the orthogonal geodesic $G$ to $G_1$ at $A_1$ and take the other unique supporting geodesic $G_2$ to $\alpha$ that is orthogonal to $G$. If we denote by $A_2$ the intersection of $G_2$ with $G$, then the *width of the curve corresponding to the point* $A_1$ is the geodesic distance from $A_1$ to $A_2$ (see Figure 3). The notion of Santaló coincides with Blaschke's definition on self-parallel curves. Other authors, such as Lassak (see [18, 16, 17, 15, 19]) defined a width of a spherical curve via the notion of a narrowest lune (see also [6] and [13]). This definition agrees with Santaló's definition (this is a consequence of Lemma in [17]).

In this section we will study how we can extend the known notion of a width for convex curves and their support functions to hedgehogs in $\mathbb{M}^2(K)$.

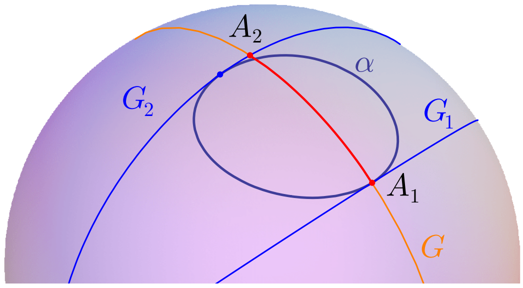


Figure 3: Definition of the width of $\alpha$ at a point $A_1$.

### 4.1 Width of hedgehogs in $\mathbb{M}^2(K)$

A *cooriented geodesic of* $\mathbb{M}^2(K)$ is a geodesic of $\mathbb{M}^2(K)$ together with a transverse direction indicated by the choice of a normal vector to the plane that contains such a geodesic and the origin of coordinates. Notice that given a hedgehog $\alpha$ in $\mathbb{M}^2(K)$, the supporting geodesics to $\alpha$ are naturally cooriented by the induced coorientation of the supporting lines of the associated planar Euclidean hedgehog via a gnomonic projection.

Given a point $A$ of a hedgehog $\alpha$ in $\mathbb{M}^2(K)$, the *oriented normal geodesic to* $\alpha$ *at* $A$ is the oriented geodesic that goes through $A$ and that is orthogonal to the supporting geodesic $G_1$ to $\alpha$ at $A$, with an orientation according to the coorientation of the supporting geodesic $G_1$.

**Definition 6** (Well-behaved hedgehog and its width)**.** A hedgehog $\alpha$ in $\mathbb{M}^2(K)$ is said to be *well-behaved* if for each cooriented supporting geodesic $G_1$ to $\alpha$ at some point $A$ of $\alpha$, there exists a unique cooriented supporting geodesic $G_2$ to $\alpha$, $G_2 \neq G_1$, that is orthogonal to the oriented normal geodesic $G$ to $\alpha$ at $A$.

In such a case, the *width of* $\alpha$ *at* $A$ is defined as the signed length of the path of $G$ that joins $G_1$ with $G_2$ without cutting any geodesic that does not meet $\alpha$, where the length is positive if the tangent vector of $G$ at $A$ is pointing towards the geodesics that do not encounter $\alpha$, and negative otherwise.

Not all hedgehogs of $\mathbb{M}^2(K)$ are well-behaved and thus not every hedgehog of $\mathbb{M}^2(K)$ admits a definition of a width as above for all its points. Let us show an example.

**Example 1.** Let $K = 1$, $p = (1, 0, 0)$ and consider the planar Euclidean hedgehog $\mathcal{H}_p$ defined by a support function $p(t) = \sin(3\,t)$ at $p + T_p\mathbb{S}^2$. The inverse $p$-gnomonic projection $\alpha$ of $\mathcal{H}_p$ onto $\mathbb{S}^2$ is a projective hedgehog in $\mathbb{S}^2$. However, $\alpha$ is not well-behaved. There are points of $\alpha$ for which there are more than two cooriented supporting geodesics to $\alpha$ which are orthogonal to the normal geodesic to $\alpha$ at such points. See an example in Figure 4.

Next we aim to provide a sufficient condition in order to get easily well-behaved hedgehogs in $\mathbb{M}^2(K)$. For this purpose we will need first two lemmas.

**Lemma 1.** *Let* $p \in \mathbb{M}^2(K)$ *and let* $G$ *be a geodesic in* $\mathbb{M}^2(K)$ *passing through* $p$*. Let* $L$ *be the straight line found as the image of* $G$ *under the* $p$*-gnomonic projection onto* $p + T_p\mathbb{M}^2(K)$*. Then there is a bijection between orthogonal geodesics in* $\mathbb{M}^2(K)$ *to* $G$ *and orthogonal lines in* $p + T_p\mathbb{M}^2(K)$ *to* $L$*.*

*Proof.* Suppose that $\mathbf{v}$ is a unit vector in $T_p\mathbb{M}^2(K)$ that directs the line $L$ and the geodesic $G$.

First, we must show that given any orthogonal line $\tilde{L}$ to $L$ in $p + T_p\mathbb{M}^2(K)$, its image under the inverse $p$-gnomonic projection $\rho_{p,K}^{-1}$ is a geodesic that is orthogonal to $G$. The line $L$ in $p + T_p\mathbb{M}^2(K)$ can be parameterized as

$$\ell(\lambda) = p + \lambda\,\mathbf{v}.$$

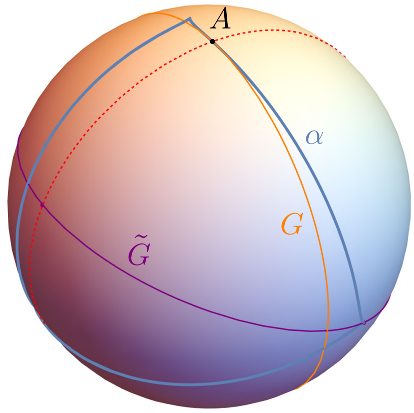


Figure 4: A not well-behaved projective spherical hedgehog $\alpha$. The supporting geodesic $G$ to $\alpha$ at $A$ is made of two cooriented supporting geodesics with opposite coorientations. There exists another cooriented supporting geodesic $\tilde{G}$ to $\alpha$ that is orthogonal to the normal geodesic to $\alpha$ at $A$.

Suppose that $\tilde{L}$ cuts $L$ at $\ell(\lambda_0)$ for some $\lambda_0 \in \mathbb{R}$. Thus, the line $\tilde{L}$ can be parameterized as

$$\tilde{\ell}(\mu) = \ell(\lambda_0) + \mu\, J\mathbf{v},$$

where $J\mathbf{v}$ is a unit vector in $T_p\mathbb{M}^2(K)$ that is orthogonal to $\mathbf{v}$.

With the expression of the inverse $p$-gnomonic projection, (4), we have

$$(\rho_{p,K}^{-1} \circ \ell)(\lambda) = \frac{1}{\sqrt{1 + K\,\lambda^2}}\, \ell(\lambda),$$

and therefore,

$$\begin{aligned}(\rho_{p,K}^{-1} \circ \ell)'(\lambda) &= -\frac{K\,\lambda}{(1 + K\,\lambda^2)^{3/2}}\, \ell(\lambda) + \frac{1}{\sqrt{1 + K\,\lambda^2}}\, \ell'(\lambda) \\ &= -\frac{K\,\lambda}{(1 + K\,\lambda^2)^{3/2}}\, p + \left( \frac{-K\,\lambda^2}{(1 + K\,\lambda^2)^{3/2}} + \frac{1}{\sqrt{1 + K\,\lambda^2}} \right) \mathbf{v}.\end{aligned}$$

Similarly, since $\mathcal{B}\big(\tilde{\ell}(\mu), \tilde{\ell}(\mu)\big) = \frac{1}{K} + \lambda_0^2 + \mu^2$, we have

$$(\rho_{p,K}^{-1} \circ \tilde{\ell})(\mu) = \frac{1}{\sqrt{1 + K\,(\lambda_0^2 + \mu^2)}}\, \tilde{\ell}(\mu),$$

and thus

$$(\rho_{p,K}^{-1} \circ \tilde{\ell})'(\mu) = -\frac{K\,\mu}{\big(1 + K\,(\lambda_0^2 + \mu^2)\big)^{3/2}}\, \tilde{\ell}(\mu) + \frac{1}{\sqrt{1 + K\,(\lambda_0^2 + \mu^2)}}\, \tilde{\ell}'(\mu).$$

From

$$(\rho_{p,K}^{-1} \circ \tilde{\ell})'(0) = \frac{1}{\sqrt{1 + K\,\lambda_0^2}}\, \tilde{\ell}'(0) = \frac{1}{\sqrt{1 + K\,\lambda_0^2}}\, J\mathbf{v},$$

we can finally deduce that

$$\mathcal{B}\big((\rho_{p,K}^{-1} \circ \tilde{\ell})'(0),\, (\rho_{p,K}^{-1} \circ \ell)'(\lambda_0)\big) = 0,$$

so the geodesic $\rho_{p,K}^{-1} \circ \tilde{\ell}$ is orthogonal to the geodesic $\rho_{p,K}^{-1} \circ \ell$ (which is $G$).

Now, conversely, given an orthogonal geodesic $\tilde{G}$ in $\mathbb{M}^2(K)$ to $G$, we must show that its $p$-gnomonic projection is a line that is orthogonal to $L$. The geodesic $G$ can be parameterized by

$$g(t) = \cos\bigl(\sqrt{K}\, t\bigr)\, p + \frac{\sin\bigl(\sqrt{K}\, t\bigr)}{\sqrt{K}}\, \mathbf{v}.$$

Suppose that $\tilde{G}$ cuts $G$ at $g(t_0)$ for some $t_0 \in \mathbb{R}$, so a parameterization of $\tilde{G}$ is

$$\tilde{g}(s) = \cos\bigl(\sqrt{K}\, s\bigr)\, g(t_0) + \frac{\sin\bigl(\sqrt{K}\, s\bigr)}{\sqrt{K}}\, \mathbf{w},$$

where $\mathbf{w} \in T_{g(t_0)}\mathbb{M}^2(K)$. More precisely, if $\times$ denotes the usual cross product if $K > 0$ or the Lorentz cross product if $K < 0$, we have that the vector $\mathbf{w}$ is given by

$$\mathbf{w} = \frac{g(t_0) \times g'(t_0)}{\bigl\| g(t_0) \times g'(t_0) \bigr\|}.$$

Since

$$\begin{aligned} g(t_0) \times g'(t_0) &= \left( \cos\bigl(\sqrt{K}\, t_0\bigr)\, p + \frac{\sin\bigl(\sqrt{K}\, t_0\bigr)}{\sqrt{K}}\, \mathbf{v} \right) \times \Bigl( -\sqrt{K} \sin\bigl(\sqrt{K}\, t_0\bigr)\, p + \cos\bigl(\sqrt{K}\, t_0\bigr)\, \mathbf{v} \Bigr) \\ &= p \times \mathbf{v}, \end{aligned}$$

we have that the vector $\mathbf{w}$ is orthogonal to $p$ and $\mathbf{v}$.

With the expression (3) of the $p$-gnomonic projection we can compute

$$\bigl(\rho_{p,K} \circ g\bigr)(t) = \frac{1}{\cos\bigl(\sqrt{K}\, t\bigr)}\, g(t)$$

and

$$\bigl(\rho_{p,K} \circ \tilde{g}\bigr)(s) = \frac{1}{\cos\bigl(\sqrt{K}\, s\bigr) \cos\bigl(\sqrt{K}\, t_0\bigr)}\, \tilde{g}(s).$$

With a straightforward computation we can see that

$$\bigl(\rho_{p,K} \circ g\bigr)'(t) = \frac{1}{\cos^2\bigl(\sqrt{K}\, t\bigr)}\, \mathbf{v}$$

and

$$\bigl(\rho_{p,K} \circ \tilde{g}\bigr)'(s) = \frac{1}{\cos^2\bigl(\sqrt{K}\, s\bigr) \cos\bigl(\sqrt{K}\, t_0\bigr)}\, \mathbf{w},$$

which are orthogonal. Therefore, the straight lines $\rho_{p,K} \circ \tilde{g}$ and $\rho_{p,K} \circ g$ (which is $L$) are orthogonal. $\square$

**Lemma 2.** *Let $\alpha$ be a hedgehog of $\mathbb{M}^2(K)$. Given any point $p \in \mathbb{M}^2(K)$, such that $\alpha$ is contained in the open hemisphere with center $p$ if $K > 0$, and a geodesic $G$ in $\mathbb{M}^2(K)$ passing through $p$, then there exist precisely two cooriented supporting geodesics to $\alpha$ that are orthogonal to $G$ and that have opposite coorientation.*

*Proof.* Since $\alpha$ is a hedgehog of $\mathbb{M}^2(K)$, we know that its $p$-gnomonic projection for such a $p \in \mathbb{M}^2(K)$ is a planar Euclidean hedgehog $c$ at $p + T_p\mathbb{M}^2(K)$.

The $p$-gnomonic projection of the geodesic $G$ provides a line $L$ in $p + T_p\mathbb{M}^2(K)$ passing through $p$. By Lemma 1 there is a bijection between orthogonal geodesics to $G$ and orthogonal lines to $L$ via the $p$-gnomonic projection. Since $c$ is a planar Euclidean hedgehog, we know that there are precisely two cooriented supporting lines orthogonal to $L$ and with opposite coorientation. Therefore, the inverse $p$-gnomonic projection of these two cooriented supporting lines will correspond to two cooriented supporting geodesics to $\alpha$ with opposite coorientation that are orthogonal to $G$ (and by the bijection there are only these two cooriented supporting geodesics satisfying this property). $\square$

Now we are ready to present a property of hedgehogs in $\mathbb{M}^2(K)$ related to the notion of a width, under the assumption that the curve lie in the interior of a spherical cap subtended by a half-angle $\pi/4$ from the center of the sphere $\mathbb{S}^2(K)$ if $K > 0$ (the following theorem can also be visualized in Figure 3).

**Theorem 1.** *Let $\alpha$ be a hedgehog of $\mathbb{M}^2(K)$ such that, if $K > 0$, $\alpha$ lies on an open spherical cap subtended by a half-angle $\pi/4$. Then, $\alpha$ is the envelope of a $\mathcal{C}^2$-family $\mathcal{F}$ of cooriented supporting geodesics to $\alpha$ such that for each cooriented supporting geodesic $G_1 \in \mathcal{F}$ to $\alpha$ at a point $A_1$, there is a unique cooriented supporting geodesic $G_2 \in \mathcal{F}$, $G_2 \neq G_1$, that is orthogonal to the normal geodesic to $G_1$ at $A_1$.*

*Proof.* Let $p \in \mathbb{M}^2(K)$ be such that if $K > 0$, $p$ is the center of the open spherical cap subtended by a half-angle $\pi/4$ that contains $\alpha$. Let $B_p(K)$ be the open ball of radius $1/\sqrt{|K|}$ (with the corresponding metric) lying on the tangent plane $p + T_p\mathbb{M}^2(K)$ and centered at $p$.

If $\alpha$ is a hedgehog of $\mathbb{M}^2(K)$, then its $p$-gnomonic projection is a planar hedgehog lying in $B_p(K)$. Moreover, cooriented supporting lines to this planar hedgehog are bijectively sent to cooriented supporting geodesics to $\alpha$. Therefore, the curve $\alpha$ can be generated as an envelope of a $\mathcal{C}^2$-family of cooriented supporting geodesics, say $\mathcal{F}$.

Let $A_1$ be a point of $\alpha$ and let $G_1 \in \mathcal{F}$ be a cooriented supporting geodesic at this point. Consider now the normal geodesic $G$ to $G_1$ at $A_1$.

Now we can use Lemma 2 applied to the point $A_1$ and the geodesic $G$. Notice that this can be done in the spherical case as well because $\alpha$ is contained in the open hemisphere with center $A_1$ (by the fact that $\alpha$ was contained in a open spherical cap subtended by a half-angle $\pi/4$). Therefore, we find that there exist precisely two cooriented supporting geodesics to $\alpha$ that are orthogonal to $G$ and that have opposite coorientation. One of these is $G_1$, let $G_2 \in \mathcal{F}$ be the second one, that is such that $G_2 \neq G_1$ because it has opposite coorientation. $\square$

As a direct consequence of Theorem 1 we have the following result.

**Corollary 1.**

1. *Any hedgehog of $\mathbb{S}^2(K)$ that lies on an open spherical cap subtended by a half-angle $\pi/4$ is well-behaved.*
2. *Any hedgehog of $\mathbb{H}^2(K)$, i.e. any g-hedgehog, is well-behaved.*

Thus, a not-well behaved hedgehog such as the one given in Example 1, which is represented in Figure 4, cannot be contained in any open spherical cap subtended by a half-angle $\pi/4$.

**Remark 2.** The condition of Corollary 1 regarding spherical hedgehogs is sufficient but not necessary, that is, there are well-behaved hedgehogs that are not fully contained in an open spherical cap subtended by a half-angle $\pi/4$.

### 4.2 A support function for hedgehogs in $\mathbb{M}^2(K)$

Henceforth, if $K > 0$, we will assume without loss of generality that the hedgehog in $\mathbb{M}^2(K)$ is contained in the open hemisphere with center $O = \left(1/\sqrt{|K|}, 0, 0\right)$.

We already mentioned that a hedgehog of $\mathbb{M}^2(K)$ can be parameterized via the support function of the associated planar Euclidean hedgehog. But in fact a support function over $\mathbb{M}^2(K)$ is induced by the planar setting. Besau and Schuster introduced in [3] this concept of a support function in the sphere via the inverse gnomonic projection (see also [5] by Besau and Werner). The same can actually be done in the hyperbolic case, so that we can make the following definition.

Given $O = \left(1/\sqrt{|K|}, 0, 0\right)$, let $p : \mathbb{R} \to \mathbb{R}$ be the support function of a planar Euclidean hedgehog $\mathcal{H}_p$ in $O + T_O\mathbb{M}^2(K)$. The inverse $O$-gnomonic projection of $\mathcal{H}_p$ over $\mathbb{M}^2(K)$ gives us a hedgehog of $\mathbb{M}^2(K)$ that can be parameterized by the induced support function in $\mathbb{M}^2(K)$ given by

$$h(t) = \frac{1}{\sqrt{K}} \arctan\left(\sqrt{K}\, p(t)\right). \tag{5}$$

We will say that $h$ is the *support function of the hedgehog* $\alpha$ (based on the point $O$). The value of the support function corresponds geometrically to a signed geodesic distance from the origin $O$ to the corresponding supporting geodesic to the hedgehog of $\mathbb{M}^2(K)$.

Note that if $K < 0$, the expression (5) reads

$$h(t) = \frac{1}{\sqrt{-K}} \operatorname{arctanh}\left(\sqrt{-K}\, p(t)\right). \tag{6}$$

In this case, since the associated planar Euclidean hedgehog is inside an open ball of radius $1/\sqrt{-K}$ centered at the origin, we have that $\left|p(t)\right| < \frac{1}{\sqrt{-K}}$. This is equivalent to have $\left|\sqrt{-K}\, p(t)\right| < 1$, which makes (6) well defined.

We must remark that the function $h$ is exactly the same support function that Borisenko and Drach used in [7] for strictly convex curves in the sphere $\mathbb{S}^2(K)$. Actually, their construction can be reproduced in the hyperbolic case as well or, more generally, for hedgehogs in $\mathbb{M}^2(K)$ as follows (see Figure 5). Let $\alpha$ be a hedgehog of $\mathbb{M}^2(K)$ and let $O \in \mathbb{M}^2(K)$ such that $\alpha$ is contained in the open hemisphere with center $O$ if $K > 0$. For each oriented geodesic $G$ passing though $O$, Lemma 2 ensures that there exists exactly one cooriented supporting geodesic $G_1$ to $\alpha$ that is orthogonal to $G$ and whose coorientation is given by the orientation of $G$. The value of the support function (relative to the origin $O$) in the direction of $G$ can be defined as a signed distance from $O$ to the supporting cooriented geodesic $G_1$. Each oriented geodesic $G$ can be described by an angle $t \in [0, 2\pi[$ from a fixed oriented geodesic $G_0$ passing through $O$ according to an orientation of $O + T_O\mathbb{M}^2(K)$. So, the support function $h$ can be described by the parameter $t \in \mathbb{R}$ and, since the curve $\alpha$ is closed, $h$ must be $2\pi$-periodic.

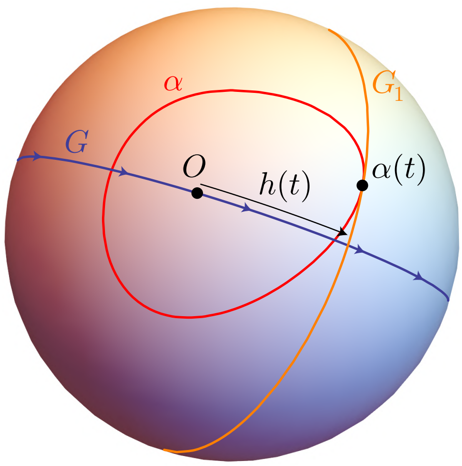


Figure 5: A hedgehog $\alpha$ in $\mathbb{M}^2(K)$ parameterized by a support function $h$.

As a remark, the support function $p$ of the associated planar Euclidean hedgehog is what the authors of [7] call the *contact radius of curvature*.

In the following result we show the explicit expression of a hedgehog in $\mathbb{M}^2(K)$ parameterized by a support function $h$.

**Proposition 2.** *Any hedgehog $\alpha = (x,\, y,\, z)$ of $\mathbb{M}^2(K)$, assumed to be in the open hemisphere of $\mathbb{M}^2(K)$ with center $O = \big(1/\sqrt{|K|},\, 0,\, 0\big)$ if $K > 0$, can be parameterized by a ($2\pi$-periodic) $\mathcal{C}^2$-support function $h$ (based on the point $O$) as*

$$x(t) = \frac{\cos\big(\sqrt{K}\, h(t)\big)}{\sqrt{|K|}\,\sqrt{1 + K\, h'(t)^2\, \sec^2\big(\sqrt{K}\, h(t)\big)}},$$

$$y(t) = \frac{\cos(t)\sin\big(\sqrt{K}\, h(t)\big) - \sqrt{K}\sin(t)\, h'(t)\sec\big(\sqrt{K}\, h(t)\big)}{\sqrt{K}\,\sqrt{1 + K\, h'(t)^2 \sec^2\big(\sqrt{K}\, h(t)\big)}},$$

$$z(t) = \frac{\sin(t)\sin\big(\sqrt{K}\, h(t)\big) + \sqrt{K}\cos(t)\, h'(t)\sec\big(\sqrt{K}\, h(t)\big)}{\sqrt{K}\sqrt{1 + K\, h'(t)^2 \sec^2\big(\sqrt{K}\, h(t)\big)}},$$

*for $t \in [0, 2\pi[$. In such a case, $\big|h(t)\big| < \frac{\pi}{2\sqrt{K}}$ if $K > 0$.*

*Proof.* The hedgehog $\alpha$ of $\mathbb{M}^2(K)$ can be parameterized as

$$\frac{1}{\sqrt{1 + K\,\big(p(t)^2 + p'(t)^2\big)}}\left(\frac{1}{\sqrt{|K|}},\, p(t)\cos(t) - \sin(t)\, p'(t),\, p(t)\sin(t) + \cos(t)\, p'(t)\right), \quad (7)$$

where $p : \mathbb{R} \to \mathbb{R}$ is the $\mathcal{C}^2$-support function of the associated planar Euclidean hedgehog at $O + T_O\mathbb{M}^2(K)$. From (5), we have

$$p(t) = \frac{1}{\sqrt{K}} \tan\big(\sqrt{K}\, h(t)\big).$$

Substituting this expression of $p$ and its derivative into (7) we find the expression of the statement after a simplification. We must notice that $\cos\big(\sqrt{K}\, h(t)\big) > 0$. Indeed, if $K > 0$ we have $\big|h(t)\big| < \frac{\pi}{2\sqrt{K}}$ by the fact that $\alpha$ is contained in the open hemisphere with center $O$. □

The main drawback of the introduced support function $h$ is that its value cannot be easily related to the concept of a width as it happens in the plane. This is because the value of $h(t)$ is a signed distance coming from a fixed point $O$ and, in non-Euclidean geometry, the width is in general not governed by a signed distance over geodesics passing though a fixed point. Next, we propose another definition of a support function in $\mathbb{M}^2(K)$ that allows a better relationship with the width of a hedgehog.

### 4.3 A Leichtweiss support function in $\mathbb{M}^2(K)$

In this section we are going to extend the support function presented by Leichtweiss in [20, 21, 22] for non-Euclidean convex curves to hedgehogs in $\mathbb{M}^2(K)$. The advantage of this support function over the one given above is that this will be related to the fact of having a constant width. However, we will see that not all the curves that we can parameterize through our Leichtweiss support function are hedgehogs, so an additional condition must be set.

Let $O = \big(1/\sqrt{|K|},\, 0,\, 0\big) \in \mathbb{M}^2(K)$ and let $\tilde{B}(K)$ be the open ball at $O + T_O\mathbb{M}^2(K)$ centered at $O$ and with radius $1/\sqrt{|K|}$ if $K > 0$ or $\tilde{B}(K) = O + T_O\mathbb{M}^2(K)$ otherwise. Suppose that we have a planar Euclidean hedgehog $c$ in $B_K$. This planar hedgehog can be parameterized by a support function $p : \mathbb{R} \to \mathbb{R}$ as

$$c(t) = \left(\frac{1}{\sqrt{|K|}},\, p(t)\cos(t) - \sin(t)\, p'(t),\, p(t)\sin(t) + \cos(t)\, p'(t)\right).$$

Consider now the orthogonal projection $\pi_K : \tilde{B}(K) \to \mathbb{M}^2(K)$ defined by

$$\pi_K(x, y, z) = \left( \frac{\sqrt{1 - K\left(y^2 + z^2\right)}}{\sqrt{|K|}},\, y,\, z \right).$$

The orthogonal projection of $c$ onto $\mathbb{M}^2(K)$ gives us a curve $\alpha$ in $\mathbb{M}^2(K)$ which we call a *Leichtweiss curve*. The curve $\alpha$ is not necessarily a hedgehog in $\mathbb{M}^2(K)$.

Let $G_t$ be the geodesic that goes through $O$ in a direction $\big(0, -\sin(t), \cos(t)\big)$, namely a *leading geodesic*, and let $\alpha(t)$ be the orthogonal projection of $c(t)$ onto $\mathbb{M}^2(K)$. Define the *Leichtweiss support function* $h$ *of* $\alpha$ as follows: for each $t \in \mathbb{R}$, $h(t)$ is the signed distance from the leading geodesic $G_t$ to $\alpha(t)$.

The orthogonal projection of the line in $O + T_O\mathbb{M}^2(K)$ that goes through $c(t)$ in the direction $\big(0, -\sin(t), \cos(t)\big)$ is an equidistant curve to the leading geodesic in $\mathbb{M}^2(K)$ (see Figure 6). Therefore, the relation between $h$ and $p$ is given by

$$p(t) = \frac{\sin\!\big(\sqrt{K}\, h(t)\big)}{\sqrt{K}}. \tag{8}$$

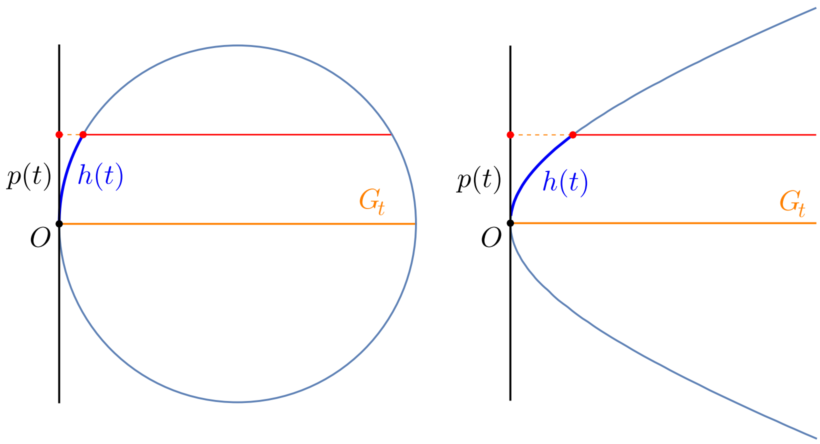


Figure 6: The leading geodesic $G_t$ and the orthogonal projection of a supporting line at $O + T_O\mathbb{M}^2(K)$ onto $\mathbb{M}^2(K)$.

If $K > 0$, any Leichtweiss curve lies in the open hemisphere of $\mathbb{M}^2(K)$ with center $O = \big(1/\sqrt{|K|}, 0, 0\big)$. As a consequence, in this case $h(t) < \frac{\pi}{2\sqrt{K}}$.

**Proposition 3.** *Any Leichtweiss curve $\alpha = (x, y, z)$ in $\mathbb{M}^2(K)$ can be parameterized by a $2\pi$-periodic (Leichtweiss) $\mathcal{C}^2$-support function $h$ such that $1 - K\, h'(t)^2 > 0$ as*

$$\begin{aligned} x(t) &= \frac{\sqrt{1 - K\, h'(t)^2}\cos\!\big(\sqrt{K}\, h(t)\big)}{\sqrt{|K|}}, \\ y(t) &= \frac{\cos(t)\sin\!\big(\sqrt{K}\, h(t)\big)}{\sqrt{K}} - \sin(t)\, h'(t)\cos\!\big(\sqrt{K}\, h(t)\big), \\ z(t) &= \frac{\sin(t)\sin\!\big(\sqrt{K}\, h(t)\big)}{\sqrt{K}} + \cos(t)\, h'(t)\cos\!\big(\sqrt{K}\, h(t)\big), \end{aligned}$$

*for $t \in [0, 2\pi[$. Moreover, $\alpha$ is a hedgehog in $\mathbb{M}^2(K)$ if and only if the support function $h$ satisfies that*

$$1 - K\, h'(t)^2 - \sqrt{K}\tan\!\big(\sqrt{K}\, h(t)\big)\, h''(t) \tag{9}$$

*has no zeros.*

*Proof.* Once we have the relation (8) between the $\mathcal{C}^2$-support function $p$ of the planar Euclidean hedgehog $c$ at $O + T_O\mathbb{M}^2(K)$ and the Leichtweiss support function $h$, we only must write the expression of $\alpha = \pi_K \circ c$, namely,

$$\alpha(t) = \left( \frac{\sqrt{1 - K\left(p(t)^2 + p'(t)^2\right)}}{\sqrt{|K|}},\, p(t)\cos(t) - \sin(t)\, p'(t),\, p(t)\sin(t) + \cos(t)\, p'(t) \right),$$

where $1 - K\left(p(t)^2 + p'(t)^2\right) > 0$, in terms of $h$ by a substitution of $p$ and its derivative $p'$. By doing so, a simplification yields the expression of the statement of $\alpha$, for which we must notice that $\cos\bigl(\sqrt{K}\, h(t)\bigr) > 0$. Further, notice that

$$1 - K\left(p(t)^2 + p'(t)^2\right) = \cos\bigl(\sqrt{K}\, h(t)\bigr)^2 \bigl(1 - K\, h'(t)^2\bigr) > 0,$$

where we have $1 - K\, h'(t)^2 > 0$ by hypothesis.

Now, it is left to find under which conditions the curve $\alpha$ is a hedgehog of $\mathbb{M}^2(K)$. Given the expression of $\alpha(t)$, its $O$-gnomonic projection onto the tangent plane of $\mathbb{M}^2(K)$ at $O$, identified with $\mathbb{R}^2$, is

$$\beta(t) = \frac{\tan\bigl(\sqrt{K}\, h(t)\bigr)}{\sqrt{K}\,\sqrt{1 - K\, h'(t)^2}}\, u(t) + \frac{h'(t)}{\sqrt{1 - K\, h'(t)^2}}\, u'(t), \tag{10}$$

where $u(t) = \bigl(\cos(t), \sin(t)\bigr)$. Now, it is a matter of an exercise to show that $\beta$ is the envelope of the family of lines with equation

$$\bigl\langle (x, y),\, \mathbf{w}(t) \bigr\rangle = \frac{\tan\bigl(\sqrt{K}\, h(t)\bigr)\sqrt{1 - K\, h'(t)^2}}{\sqrt{K}},$$

where

$$\mathbf{w}(t) = u(t) - \sqrt{K}\tan\bigl(\sqrt{K}\, h(t)\bigr)\, h'(t)\, u'(t).$$

The curve $\beta$ is a planar Euclidean hedgehog if and only if the angle function $\gamma : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}/2\pi\mathbb{Z}$ such that $\mathbf{w}(t) = \bigl\|\mathbf{w}(t)\bigr\| \bigl(\cos(\gamma(t)), \sin(\gamma(t))\bigr)$ is a reparameterization of $u$ (i.e., a diffeomorphism, which has non-zero derivative). We can compute $\gamma(t)$ as the argument of $\mathbf{w}(t)$ seen as a complex number. Since $u(t) = e^{\mathbf{i}\,t}$ and $u'(t) = \mathbf{i}\, e^{\mathbf{i}\,t}$, we have

$$\mathbf{w}(t) = \bigl\|\mathbf{w}(t)\bigr\|\, e^{\mathbf{i}\,t}\, z(t),$$

where $z(t) = 1 - \sqrt{K}\, \tan\bigl(\sqrt{K}\, h(t)\bigr)\, h'(t)\,\mathbf{i}$. Thus, up to a multiple of $2\pi$, we have

$$\gamma(t) = t + \arctan\Bigl(-\sqrt{K}\, \tan\Bigl(\sqrt{K}\, h(t)\Bigr)\, h'(t)\Bigr).$$

A straightforward computation shows that

$$\gamma'(t) = \frac{1 - K\, h'(t)^2 - \sqrt{K}\tan\bigl(\sqrt{K}\, h(t)\bigr)\, h''(t)}{1 + K\tan^2\bigl(\sqrt{K}\, h(t)\bigr)\, h'(t)^2},$$

where notice that $1 + K\tan^2\bigl(\sqrt{K}\, h(t)\bigr)\, h'(t)^2 > 0$ for all $t \in [0, 2\pi[$. Hence, $\alpha$ is a hedgehog of $\mathbb{M}^2(K)$ if and only if

$$1 - K\, h'(t)^2 - \sqrt{K}\tan\bigl(\sqrt{K}\, h(t)\bigr)\, h''(t)$$

is never zero. $\square$

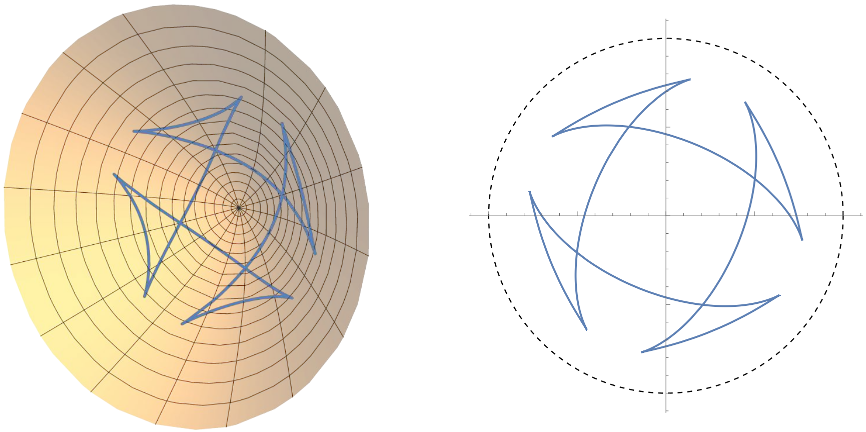

Figure 7: On the left, a curve in $\mathbb{H}^2$ parameterized by a Leichtweiss support function that is not a $g$-hedgehog. On the right, the planar curve found by its gnomonic projection (inside the open unit ball) onto the tangent plane to $\mathbb{H}^2$ at $(1, 0, 0)$, which indeed is not a planar hedgehog.

**Example 2.** As a consequence of Proposition 3, a curve parameterized by a Leichtweiss $\mathcal{C}^2$-support function is not necessarily a hedgehog. For instance, the Leichtweiss support function $h(t) = \frac{2}{3} + \frac{1}{5}\sin(4\,t)$ satisfies $1 - K\,h'(t)^2 > 0$ but (9) has zeros for $K = \pm 1$ (see the resulting curve for $K = -1$ in Figure 7-left). Indeed, the planar curve found by the $O$-gnomonic projection of the curve is not a planar Euclidean hedgehog (see Figure 7-right for $K = -1$).

The advantage of a Leichtweiss support function to describe hedgehogs is that its value is a signed distance in the geodesic that measures a width for each point of the hedgehog. The geometric meaning of a Leichtweiss support function in terms of supporting geodesics is shown in Figure 8.

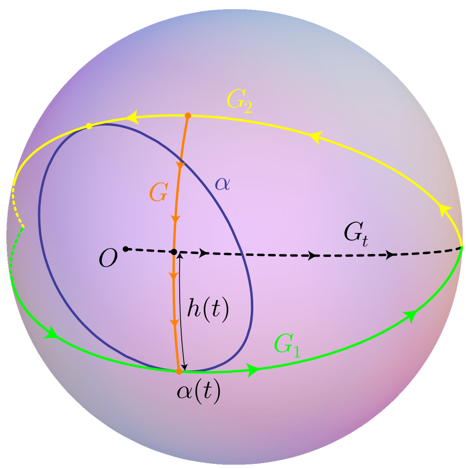


Figure 8: A hedgehog $\alpha$ parameterized by a Leichtweiss support function $h$.

**Proposition 4.** *If the Leichtweiss $\mathcal{C}^2$-support function $h$ of a Leichtweiss curve $\alpha$ satisfies that $h(t) + h(t + \pi) = d$ is constant for all $t \in [0, 2\pi[$, then the curve $\alpha$ is self-parallel at a distance $d$.*

*Proof.* The expression given in Proposition 3 of a Leichtweiss curve $\alpha$ can be written as

$$\alpha(t) = \cos\bigl(\sqrt{K}\, h(t)\bigr)\, v(t) + \frac{\sin\bigl(\sqrt{K}\, h(t)\bigr)}{\sqrt{K}}\, u(t),$$

where

$$\begin{aligned} v(t) &= \left( \frac{\sqrt{1 - K\, h'(t)^2}}{\sqrt{|K|}},\ -h'(t)\sin(t),\ h'(t)\cos(t) \right), \\ u(t) &= \bigl(0,\ \cos(t),\ \sin(t)\bigr). \end{aligned}$$

Notice that $\mathcal{B}\bigl(u(t), v(t)\bigr) = 0$, $\mathcal{B}\bigl(u(t), u(t)\bigr) = 1$ and $\mathcal{B}\bigl(v(t), v(t)\bigr) = \frac{1}{K}$. From this, we also have $\mathcal{B}\bigl(v(t), v'(t)\bigr) = \mathcal{B}\bigl(u(t), u'(t)\bigr) = 0$ and $\mathcal{B}\bigl(u(t), v'(t)\bigr) = -\mathcal{B}\bigl(u'(t), v(t)\bigr) = -h'(t)$. Since $h(t+\pi) = d - h(t)$, $v(t+\pi) = v(t)$ and $u(t+\pi) = -u(t)$, we have

$$\alpha(t+\pi) = \cos\Bigl(\sqrt{K}\,\bigl(d - h(t)\bigr)\Bigr)\, v(t) - \frac{\sin\Bigl(\sqrt{K}\,\bigl(d - h(t)\bigr)\Bigr)}{\sqrt{K}}\, u(t).$$

Therefore,

$$\mathcal{B}\bigl(\alpha(t),\, \alpha(t+\pi)\bigr) = \frac{1}{K}\cos\bigl(\sqrt{K}\, d\bigr),$$

which means that the geodesic distance between $\alpha(t)$ and $\alpha(t+\pi)$ is $d$. Now,

$$\begin{aligned} \alpha'(t) = &-\sqrt{K}\, h'(t)\sin\bigl(\sqrt{K}\, h(t)\bigr)\, v(t) + \cos\bigl(\sqrt{K}\, h(t)\bigr)\, v'(t) \\ &+ \cos\bigl(\sqrt{K}\, h(t)\bigr)\, h'(t)\, u(t) + \frac{\sin\bigl(\sqrt{K}\, h(t)\bigr)}{\sqrt{K}}\, u'(t). \end{aligned}$$

It can be easily checked that

$$\mathcal{B}\bigl(\alpha'(t),\, \alpha(t+\pi)\bigr) = 0.$$

And of course we have $\mathcal{B}\bigl(\alpha'(t), \alpha(t)\bigr) = 0$. The geodesic $G$ that goes from $\alpha(t)$ to $\alpha(t+\pi)$ is the intersection of $\mathbb{M}^2(K)$ with a plane containing the origin of coordinates and the two points $\alpha(t)$ and $\alpha(t+\pi)$. The vector $\mathbf{v} \in T_{\alpha(t)}\mathbb{M}^2(K)$ that provides the tangent direction of $G$ at $\alpha(t)$ is spanned by $\alpha(t)$ and $\alpha(t+\pi)$. Thus, we have proved that $\alpha'(t)$ is orthogonal to $\mathbf{v}$, which means that $G$ is the normal geodesic to $\alpha$ at $\alpha(t)$. As a consequence, $\alpha$ is self-parallel at a distance $d$. $\square$

**Remark 3.** Since a Leichtweiss curve is not necessarily a hedgehog, we can find curves that are self-parallel at a constant distance $d$ that are not hedgehogs (and thus, not hedgehogs of constant width $d$). For example, take a Leichtweiss support function $h(t) = \frac{2}{3} + \frac{1}{4}\sin(3\,t)$, which satisfies that $h(t) + h(t+\pi) = \frac{4}{3}$ and $1 - K\, h'(t)^2 > 0$ for $K = \pm 1$. However, the function (9) has some zeros, for $K = \pm 1$. See these curves in Figure 9.

If a $2\pi$-periodic Leichtweiss $\mathcal{C}^2$-support function $h$ satisfies that $1 - K\, h'(t)^2 > 0$, $h(t) + h(t+\pi) = d$ is constant and that (9) has no zeros, then the corresponding Leichtweiss curve is a hedgehog of constant width $d$. This provides a simple way to construct hedgehogs of constant width, similar to that in the plane.

However, notice that a width of a well-behaved hedgehog constructed by a Leichtweiss support function is in general not given by $h(t) + h(t+\pi)$. This is only the case if the hedgehog is of constant width.

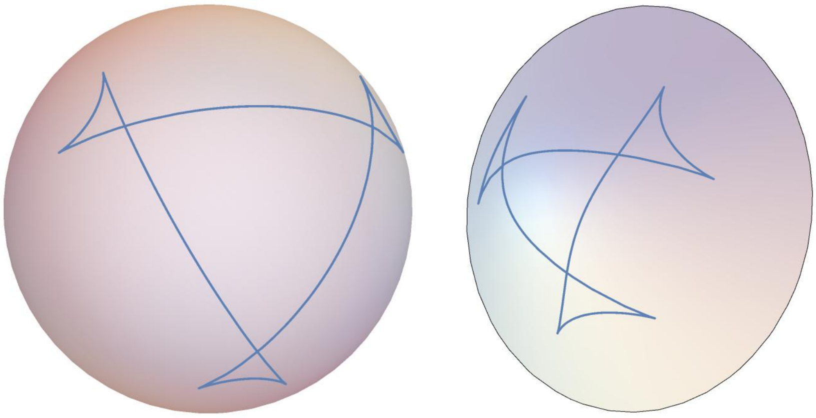

Figure 9: A (Leichtweiss) self-parallel curve in $\mathbb{M}^2(K)$ at a distance $\frac{4}{3}$ that is not a hedgehog, both for $K = 1$ (left) and $K = -1$ (right).

# 5 Hedgehogs with respect to horocycles

We can define two types of convex bodies in the hyperbolic space: geodesically convex bodies (or $g$-convex bodies) and horospherically convex bodies (or $h$-convex bodies). It is well-known that any $h$-convex body is also $g$-convex, but the converse is not true.

The definition of a non-Euclidean hedgehog we have provided so far in the hyperbolic space corresponds to a generalization of $g$-convex bodies and, because of that, we have called them $g$-hedgehogs. In this section we will provide the corresponding natural extension of $h$-convex bodies and we will call them $h$-hedgehogs. This extension follows closely the notion defined by Epstein and studied in several papers [10, 11, 12] as a hypersurface enveloped by a family of horospheres, but an additional condition is needed. Again, we will restrict ourselves to the 2-dimensional case although it is generalizable to a higher dimension (see the cited paper [10] by Epstein or [27] for more details).

We will focus on the case $K = -1$. In the Poincaré disk model of hyperbolic geometry, ideal points are those points that lie in the boundary circle at infinity, which is identified with $\mathbb{S}^1$. Recall that a horocycle is a curve in the hyperbolic plane whose normal geodesics all converge asymptotically to the same ideal point. It can be interpreted as a "circle with infinite radius", the center of which is precisely the ideal point where all normal geodesics asymptotically converge. Parallel horocycles can be seen as those Euclidean circles that are tangent at the same ideal point $u \in \mathbb{S}^1$.

## 5.1 Definition of horocyclically hedgehog curves

Consider the Poincaré disk model of the hyperbolic plane with the usual metric

$$ds^2 = \frac{4\,(dx^2 + dy^2)}{\left(1 - x^2 - y^2\right)^2}.$$

Let $O = (0, 0)$ be the center (origin) of the Poincaré disk, which is represented as the open unit ball with boundary $\mathbb{S}^1$.

Given $u \in \mathbb{S}^1$, let $H(u, s)$ be the horocycle centered at the ideal point $u$ and that is at a signed hyperbolic distance $s$ from the origin $O$ (the sign is positive if $O$ is exterior to the horocycle and negative otherwise).

Let $h : \mathbb{S}^1 \to \mathbb{R}$ be a $\mathcal{C}^2$-function. If $u : \mathbb{R} \to \mathbb{S}^1$ is given by $u(t) = \big(\cos(t), \sin(t)\big)$, then we define $s = h \circ u$. Consider the envelope $\mathcal{H}_h$ of the family of horocycles

$$\mathcal{F} = \big\{H\big(u, h(u)\big)\big\}_{u\in\mathbb{S}^1}.$$

The horocycles from this family are called *supporting horocycles*, which will be cooriented towards their interior (see Figure 10) and the function $h$ is said to be a *support function of* $\mathcal{H}_h$. By taking a parameter in the real line, we will also say that $s$ is the support function of $\mathcal{H}_h$ and we will also write $\mathcal{H}_s$ for such an envelope.

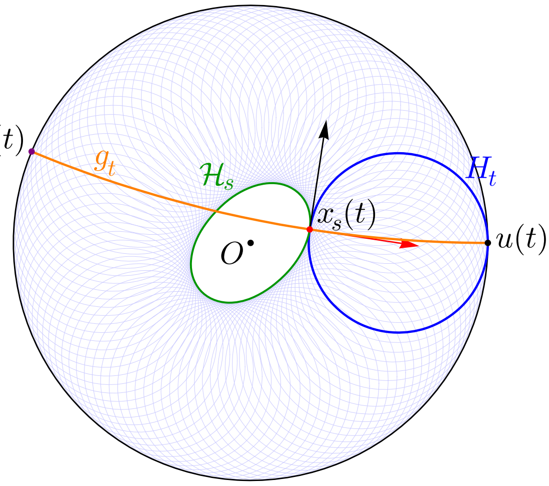


Figure 10: An envelope $\mathcal{H}_s$ of a family of cooriented horocycles $\{H_t\}_t$.

The explicit parameterization of the envelope $\mathcal{H}_s$ inside the open unit ball can be written as

$$x_s(t) = \frac{s'(t)^2 + e^{2s(t)} - 1}{s'(t)^2 + \left(1 + e^{s(t)}\right)^2}\, u(t) + \frac{2\, s'(t)}{s'(t)^2 + \left(1 + e^{s(t)}\right)^2}\, u'(t). \tag{11}$$

The function $s(t)$ represents the signed distance from the origin $O$ to the corresponding supporting horocycle at $x_s(t)$ (with the sign chosen according to the coorientation of such a horocycle).

The supporting horocycle at each point $x_s(t)$ can be parameterized as

$$H_t(w) = \frac{1}{2}\left(\left(1 + \tanh\frac{s(t)}{2}\right)\cos(t), \left(1 + \tanh\frac{s(t)}{2}\right)\sin(t)\right) + \frac{1}{1 + e^{s(t)}}\, u(w).$$

Given a point $x_s(t)$ of $\mathcal{H}_s$, we have exactly one cooriented supporting horocycle from the family $\mathcal{F}$ to $\mathcal{H}_s$ at $x_s(t)$, with an associated ideal point $u(t)$. If $n(t)$ is the unit vector defining the coorientation of the supporting horocycle $H_t$, pointing towards its interior, then the geodesic $g_t$ from $x_s(t)$ in the direction $n(t)$ converges asymptotically to $u(t)$. Therefore, we have a bijection between points of $\mathcal{H}_s$ and ideal points $u \in \mathbb{S}^1$, which can be interpreted as a (positive) hyperbolic Gauss map. The same geodesic $g_t$, which we call the *normal geodesic to* $\mathcal{H}_s$ *at* $x_s(t)$ converges asymptotically to another ideal point $v(t)$ in the opposite direction $-n(t)$ (see Figure 10). The point $v(t)$ can be understood as a negative hyperbolic Gauss map.

Envelopes as in (11) are the kind of curves considered by Epstein in the cited works. However, if we aim to define a notion of a $h$-hedgehog as a curve parameterized by the inverse hyperbolic Gauss map, we must impose additionally that the function $\Lambda : \mathbb{S}^1 \to \mathbb{S}^1$, which sends each ideal point $u \in \mathbb{S}^1$ to the other ideal point $v_u \in \mathbb{S}^1$ defined by the normal geodesic to $\mathcal{H}_s$ at the point associated to $u$, is a diffeomorphism.

**Definition 7** ($h$-hedgehog)**.** A curve in the hyperbolic space is said to be a *hedgehog with respect to horocycles*, a *horocyclically hedgehog curve* or a *$h$-hedgehog* if it is the envelope of a family of horocycles as in (11) such that the function $\Lambda$ is a diffeomorphism.

**Remark 4.** In the definition of a $h$-hedgehog, that the function $\Lambda$ is one-to-one is equivalent to the fact that there is no pair of distinct oriented normal geodesic lines that are parallel (i.e., that share the same ideal point).

The function $v : \mathbb{R} \to \mathbb{S}^1$ defined by $v = \Lambda \circ u$ provides a parameterization $v(t)$ of the second ideal point. Now, if we define $\theta : \mathbb{R} \to \mathbb{R}$ such that $v(t) = u\big(\theta(t)\big)$, then the condition on $\Lambda$ can be translated to $\theta$ being a diffeomorphism in $\mathbb{R}/2\pi\mathbb{Z}$.

The normal geodesic to $\mathcal{H}_s$ at $x_s(t)$ can be parameterized as

$$g_t(w) = \frac{-1 + e^{2\,(w+s(t))} + s'(t)^2}{\big(1 + e^{w+s(t)}\big)^2 + s'(t)^2}\, u(t) + \frac{2\, s'(t)}{\big(1 + e^{w+s(t)}\big)^2 + s'(t)^2}\, u'(t). \tag{12}$$

The associated ideal points can be found via limits. So,

$$\lim_{w \to +\infty} g_t(w) = u(t)$$

and

$$\lim_{w \to -\infty} g_t(w) = v(t).$$

A computation of the second limit gives

$$v(t) = \frac{s'(t)^2 - 1}{s'(t)^2 + 1}\, u(t) + \frac{2\, s'(t)}{s'(t)^2 + 1}\, u'(t). \tag{13}$$

Thus, notice that $v(t)$ can be seen as a stereographic projection in the basis $\big\{u(t), u'(t)\big\}$ evaluated at $s'(t)$.

The function $\theta(t)$ can be computed simply as the counterclockwise oriented angle from $(1, 0)$ to $v(t)$. This is equivalent to compute the argument $\theta(t)$ of $v(t)$ seen as a complex number. In complex notation, since $u(t) = e^{\mathbf{i}\, t}$ and $u'(t) = \mathbf{i}\, e^{\mathbf{i}\, t}$ we have

$$v(t) = \frac{s'(t) + \mathbf{i}}{s'(t) - \mathbf{i}}\, e^{\mathbf{i}\, t} = \left(\frac{s'(t)^2 - 1}{s'(t)^2 + 1} + \frac{2\, s'(t)}{s'(t)^2 + 1}\, \mathbf{i}\right) e^{\mathbf{i}\, t}.$$

We know that if $y \neq 0$, then

$$\arg(x + \mathbf{i}\, y) = 2 \arctan\left(\frac{\sqrt{x^2 + y^2} - x}{y}\right). \tag{14}$$

Therefore, if $s'(t) \neq 0$, we have, up to a multiple of $2\pi$, that

$$\arg\big(v(t)\big) = \arg\left(\frac{s'(t) + \mathbf{i}}{s'(t) - \mathbf{i}}\right) + \arg\big(e^{\mathbf{i}\, t}\big) = t + 2 \arctan\left(\frac{1}{s'(t)}\right).$$

Now, since for all $x \neq 0$,

$$\arctan(x) + \arctan\left(\frac{1}{x}\right) = \operatorname{sgn}(x)\, \frac{\pi}{2}, \tag{15}$$

we can write

$$\begin{aligned} \arg\big(v(t)\big) &= t + \operatorname{sgn}\big(s'(t)\big)\, \pi - 2 \arctan\big(s'(t)\big) \pmod{2\pi} \\ &= t + \pi - 2 \arctan\big(s'(t)\big) \pmod{2\pi}, \end{aligned}$$

which holds true even if $s'(t) = 0$. Therefore, we can conclude that

$$\theta(t) = t + \pi - 2 \arctan\big(s'(t)\big). \tag{16}$$

Now, differentiating this expression we get

$$\theta'(t) = \frac{1 + s'(t)^2 - 2\,s''(t)}{1 + s'(t)^2}. \tag{17}$$

Moreover, notice that $\theta$ cannot be decreasing because in such a case it would exist $t_0 \in \mathbb{R}/2\pi\mathbb{Z}$ such that $u\big(\theta(t_0)\big) = u(t_0)$, which is impossible. Thus, we deduce the following result.

**Proposition 5.** *If $u(t) = \big(\cos(t), \sin(t)\big)$ is a hyperbolic Gauss map of a curve $x_s$ defined by (11) for a $\mathcal{C}^2$-support function $s$, then the function $\Lambda : \mathbb{S}^1 \to \mathbb{S}^1$ is a diffeomorphism if and only if $1 + s'(t)^2 - 2\,s''(t) > 0$ for all $t \in \mathbb{R}/2\pi\mathbb{Z}$.*

Therefore, a $h$-hedgehog is the envelope of a family of horocycles as in (11) with a $\mathcal{C}^2$-support function $s$ such that $1 + s'(t)^2 - 2\,s''(t) > 0$ for all $t \in \mathbb{R}/2\pi\mathbb{Z}$.

From the definition of a $h$-hedgehog, since we have a diffeomorphism $\Lambda$ between pairs of ideal points $u(t)$ and $v(t)$, the family of supporting horocycles $\mathcal{F}$ centered at points $u(t)$ has naturally associated another family of horocycles $\widetilde{\mathcal{F}}$ centered at points $v(t)$ that also envelope the same $h$-hedgehog $\mathcal{H}_s$ (see Figure 11). The condition that $\Lambda$ is a diffeomorphism is necessary in order that the envelope of $\widetilde{\mathcal{F}}$ generates the same curve (a bijection is not enough). We will refer to the horocycles from $\widetilde{\mathcal{F}}$ as the *complementary supporting horocycles.* The conversion from one family of horocycles to the other can be achieved by reversing the orientation of the normal lines to the $h$-hedgehog. If $H_t$ is the horocycle from $\mathcal{F}$ centered at the ideal point $u(t)$, we denote by $\widetilde{H}_t$ the horocycle from $\widetilde{\mathcal{F}}$ with the same ideal point $u(t)$. If $s$ is the support function associated with $\mathcal{F}$, denote by $\tilde{s}$ the support function associated with $\widetilde{\mathcal{F}}$.

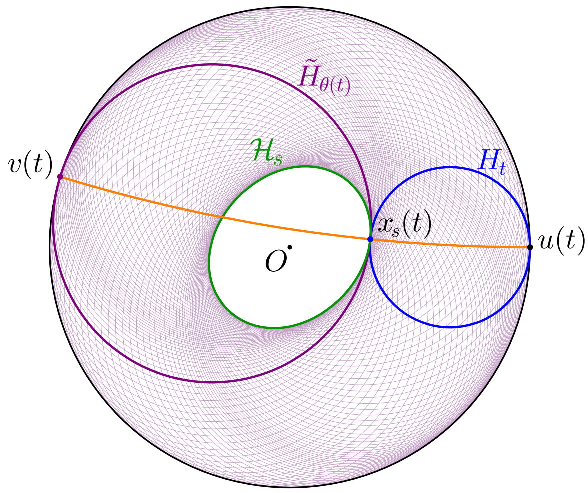


Figure 11: A $h$-hedgehog can be enveloped by either its family of supporting horocycles $\{H_t\}_t$ or its family of complementary supporting horocycles $\{\widetilde{H}_t\}_t$.

The value of the support function $\tilde{s}(t)$ is the signed distance from the origin $O$ to the horocycle $\widetilde{H}_t$ (which is cooriented towards its interior). Next, we relate the support functions $\tilde{s}$ and $s$.

**Proposition 6.** *Given a $h$-hedgehog enveloped by a family of horocycles $\{H_t\}_t$ defined by a support function $s$, then the support function $\tilde{s}$ for the family of complementary horocycles $\{\widetilde{H}_t\}_t$ satisfies*

$$\tilde{s}\big(\theta(t)\big) = \ln\big(1 + s'(t)^2\big) - s(t).$$

*Proof.* The complementary horocycle $\widetilde{H}_{\theta(t)}$ is centered at the ideal point $v(t) = u\big(\theta(t)\big)$. Let us compute now the signed hyperbolic distance from $O$ to this horocycle in terms of $s$. We have that the center of this circle $\widetilde{H}_{\theta(t)}$ (seen as an Euclidean circle inside the unit ball) is given by

$$c(t) = \big(1 - r(t)\big)\, u\big(\theta(t)\big),$$

where $r(t)$ is its Euclidean radius. Moreover, we know that

$$\big\|c(t) - x_s(t)\big\|^2 = r(t)^2.$$

This equation has a unique solution for $r(t)$, which can be found using the expression (13) of $u\big(\theta(t)\big) = v(t)$. After a simplication, we get

$$r(t) = \frac{e^{s(t)}}{1 + e^{s(t)} + s'(t)^2}.$$

With that, we also know the center $c(t)$. The Euclidean distance between $O$ and $\widetilde{H}_{\theta(t)}$ is $2r(t) - 1 > 0$ if $O$ is interior to $\widetilde{H}_{\theta(t)}$ or $1 - 2r(t) > 0$ otherwise. Taking into account the sign according to the coorientation of $\widetilde{H}_{\theta(t)}$, we decuce that the signed Euclidean distance from $O$ to $\widetilde{H}_{\theta(t)}$ is

$$1 - 2r(t) = \frac{1 - e^{s(t)} + s'(t)^2}{1 + e^{s(t)} + s'(t)^2}.$$

Therefore, the associated signed hyperbolic distance from $O$ to $\widetilde{H}_{\theta(t)}$ is

$$2\,\mathrm{arctanh}\left(\frac{1 - e^{s(t)} + s'(t)^2}{1 + e^{s(t)} + s'(t)^2}\right),$$

which, using the property $\mathrm{arctanh}(z) = \frac{\ln(1+z)}{2} - \frac{\ln(1-z)}{2}$, can also be rewritten as

$$\ln\big(1 + s'(t)^2\big) - s(t). \qquad \square$$

It is known that any $h$-convex body is a $g$-convex body, but the converse is not true. With our definitions, we have that such a property has an extension to hedgehogs.

**Theorem 2.** *Any $h$-hedgehog is a $g$-hedgehog.*

*Proof.* Let $\mathcal{H}_s$ be a $h$-hedgehog. The expression (11) can be translated to the hyperboloid $\mathbb{H}^2$ via the function $f : \mathbb{R}^2 \to \mathbb{H}^2$ defined by

$$f(x, y) = \left(\frac{1 + x^2 + y^2}{1 - x^2 - y^2}, \frac{2\,x}{1 - x^2 - y^2}, \frac{2\,y}{1 - x^2 - y^2}\right).$$

The associated planar curve at the tangent plane to $\mathbb{H}^2$ at $(1, 0, 0)$ can be computed via the gnomonic projection, which yields

$$c(t) = \frac{-1 + s'(t)^2 + e^{2s(t)}}{1 + s'(t)^2 + e^{2s(t)}}\, u(t) + \frac{2\, s'(t)}{1 + s'(t)^2 + e^{2s(t)}}\, u'(t).$$

where $u(t) = \big(\cos(t), \sin(t)\big)$. Now, it is easy to check that $c$ is the envelope of the family of lines with equation

$$\big\langle (x, y),\, \mathbf{w}(t)\big\rangle = 1 - e^{2s(t)} + s'(t)^2,$$

where

$$\mathbf{w}(t) = -\big(1 + e^{2s(t)} - s'(t)^2\big)\, u(t) + 2\, s'(t)\, u'(t).$$

It is only left to prove that the angle function $\gamma : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}/2\pi\mathbb{Z}$ such that $\mathbf{w}(t) = \|\mathbf{w}(t)\| \left(\cos(\gamma(t)), \sin(\gamma(t))\right)$ is a reparameterization of $u$ (i.e., a diffeomorphism). We can compute $\gamma(t)$ as the argument of $\mathbf{w}(t)$ seen as a complex number. Thus, since $u(t) = e^{\mathbf{i}\,t}$ and $u'(t) = \mathbf{i}\, e^{\mathbf{i}\,t}$, we have

$$\gamma(t) = t + \arg\Big(-\big(1 + e^{2s(t)} - s'(t)^2\big) + 2\, s'(t)\, \mathbf{i}\Big) \pmod{2\pi}.$$

If $s'(t) \neq 0$, we can use (14) and (15) to get

$$\gamma(t) = t + \pi - 2\arctan\left(\frac{2\, s'(t)}{1 + e^{2s(t)} - s'(t)^2 + \sqrt{4s'(t)^2 + \big(1 + e^{2s(t)} - s'(t)^2\big)^2}}\right) \pmod{2\pi}.$$

Notice that this expression holds true also if $s'(t) = 0$, because in such a case, $-\big(1 + e^{2s(t)} - s'(t)^2\big) < 0$. Also note that the denominator inside the arctangent is never zero. Therefore, from this we deduce

$$\gamma'(t) = \frac{\big(1 + s'(t)^2 + e^{2s(t)}\big)\,\big(1 + e^{2s(t)} + s'(t)^2 - 2\, s''(t)\big)}{\big(1 + e^{2s(t)} - s'(t)^2\big)^2 + 4\, s'(t)^2}.$$

By Proposition 5 we have that $1 + s'(t)^2 - 2\, s''(t) > 0$. Therefore, $\gamma'(t) > 0$ for all $t \in \mathbb{R}/2\pi\mathbb{Z}$, which implies that the curve $c$ is indeed a planar Euclidean hedgehog and so $\mathcal{H}_s$ is a $g$-hedgehog. □

**Remark 5.** The converse of Theorem 2 is not true. A $g$-hedgehog in the Poincaré disk model can be parameterized by

$$x_p(t) = \left(\frac{p(t)\cos(t) - p'(t)\sin(t)}{1 + \sqrt{1 - p(t)^2 - p'(t)^2}}, \frac{p(t)\sin(t) + p'(t)\cos(t)}{1 + \sqrt{1 - p(t)^2 - p'(t)^2}}\right),$$

where $p$ is the support function of the associated planar Euclidean hedgehog in the tangent plane to $\mathbb{H}^2$ at $(1, 0, 0)$. Notice that $1 - p(t)^2 - p'(t)^2 > 0$.

If we take $p(t) = \frac{1}{4}\sin(3\,t)$, we have a $g$-hedgehog that is not a $h$-hedgehog because the function $\Lambda$ that associates pairs of ideal points is not even a bijection (see Figure 12).

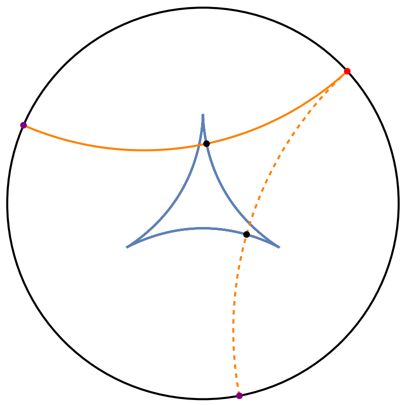

Figure 12: A $g$-hedgehog in the Poincaré disk model that is not a $h$-hedgehog.

### 5.2 Curvature and relation to $h$-convexity

If we differentiate the expression (11) and use the basis $\{u(t), u'(t)\}$, we can write

$$x_s'(t) = \frac{1}{2}\, e^{-s(t)} \left(-1 + e^{2s(t)} - s'(t)^2 + 2\, s''(t)\right) \mathbf{v}(t),$$

where

$$\mathbf{v}(t) = \frac{4\, e^{s(t)} \left(1 + e^{s(t)}\right) s'(t)}{\left(s'(t)^2 + \left(1 + e^{s(t)}\right)^2\right)^2}\, u(t) + \frac{2\, e^{s(t)} \left(\left(1 + e^{s(t)}\right)^2 - s'(t)^2\right)}{\left(s'(t)^2 + \left(1 + e^{s(t)}\right)^2\right)^2}\, u'(t)$$

has unit norm (in the Poincaré metric).

The function

$$R_s(t) = \frac{1}{2}\, e^{-s(t)} \left(-1 + e^{2s(t)} - s'(t)^2 + 2\, s''(t)\right),$$

which we can call *curvature function*, is such that

$$\int_0^{2\pi} R_s(t)\, \mathrm{d}t$$

gives the signed length of the curve $x_s$, with sign according to the sign of $R_s$ (like the analogous function in the case of Euclidean planar hedgehogs [25]). Moreover, the roots of $R_s(t)$ are the values for which the curve $x_s$ is singular.

If $x_s$ is regular at a point $t \in [0, 2\pi[$, then a computation of the geodesic curvature leads to

$$\kappa_g(t) = \frac{1 + e^{2s(t)} + s'(t)^2 - 2\, s''(t)}{\left|1 - e^{2s(t)} + s'(t)^2 - 2\, s''(t)\right|}.$$

**Proposition 7.** *If $\mathcal{H}_s$ is a $h$-hedgehog, then $\kappa_g > 1$ at regular points.*

*Proof.* Since $\mathcal{H}_s$ is a $h$-hedgehog, we have that $1 + s'(t)^2 - 2\, s''(t) > 0$. Now, by the triangle inequality,

$$\left|\left(1 + s'(t)^2 - 2\, s''(t)\right) - e^{2s(t)}\right| \le \left|1 + s'(t)^2 - 2\, s''(t)\right| + e^{2s(t)} = 1 + s'(t)^2 - 2\, s''(t) + e^{2s(t)}.$$

Therefore, $\kappa_g(t) \ge 1$.

Now, from the expression of the geodesic curvature, if $\kappa_g(t) = 1$ for some $t$, then notice that $1 - e^{2s(t)} + s'(t)^2 - 2\, s''(t) < 0$ because otherwise we get $e^{2s(t)} = 0$, which is a contradiction. Thus, if $\kappa_g(t) = 1$, we can deduce that $1 + s'(t)^2 - 2\, s''(t) = 0$. In conclusion, $\kappa_g > 1$. $\square$

As a direct consequence, we have the following result if the curve is regular.

**Corollary 2.** *If $\mathcal{H}_s$ is a singularity-free $h$-hedgehog, then $\mathcal{H}_s$ is a strictly $h$-convex curve.*

And we also have the converse result.

**Proposition 8.** *Any strictly $h$-convex regular smooth curve is a $h$-hedgehog.*

*Proof.* Any strictly $h$-convex regular smooth curve can be enveloped by a family of supporting horocycles with a smooth support function $h : \mathbb{S}^1 \to \mathbb{R}$ (see e.g. [2]). Let $s = h \circ u$, where $u : \mathbb{R} \to \mathbb{S}^1$ is given by $u(t) = \left(\cos(t), \sin(t)\right)$ and denote by $\mathcal{H}_s$ such an envelope.

Since $\mathcal{H}_s$ is a strictly $h$-convex regular curve we have $\kappa_g > 1$. If there exists $t_0$ such that $1 + s'(t_0)^2 - 2\, s''(t_0) \le 0$, then from the expression of the geodesic curvature and $\kappa_g > 1$, since $-e^{2s(t_0)} + 1 + s'(t_0)^2 - 2\, s''(t_0) < 0$, we deduce $1 + s'(t_0)^2 - 2\, s''(t_0) > 0$, which is a contradiction. Therefore, we have $1 + s'(t)^2 - 2\, s''(t) > 0$ for all $t$. As a consequence $\mathcal{H}_s$ is a $h$-hedgehog. $\square$

**Remark 6.** Any point in $\mathbb{H}^2$ is a $h$-hedgehog. Indeed, we want a $\mathcal{C}^2$-support function $s$ such that

$$x_s(t) = (a, b),$$

where $(a, b)$ is a point inside the open unit ball. Differentiating this expression we get $x_s'(t) = (0, 0)$, and this happens if and only if $s(t)$ is a solution of the differential equation $R_s(t) = 0$. A 2-parametric set of solutions of

$$-1 + e^{2s(t)} - s'(t)^2 + 2\, s''(t) = 0$$

is given by

$$s(t) = \ln\left(\frac{2\left(\sqrt{c_1^2-4}\,\sin(t+c_2)+c_1\right)}{c_1^2\cos^2(t+c_2)+4\sin^2(t+c_2)}\right),$$

where $c_1, c_2 \in \mathbb{R}$. The parameterization of the corresponding $h$-hedgehog is

$$x_s(t) = \left(\frac{(c_1-2)\sin(c_2)}{\sqrt{c_1^2-4}}, \frac{(c_1-2)\cos(c_2)}{\sqrt{c_1^2-4}}\right),$$

which is indeed a point. Moreover, notice that points in $\mathbb{H}^2$ are those $h$-hedgehogs such that "$\kappa_g$ is infinity everywhere".

### 5.3 The notion of a $h$-width for $h$-hedgehogs

In the hyperbolic space, Santaló provided a natural notion of a width for $h$-convex curves, called $h$-width, as the distance between parallel horocycles enclosing the curve [35]. More specifically, given a point $A$ of a $h$-convex curve $C$, there are two horocycles that "support" $C$ at $A$. Let $H^-$ be the one which has the same direction of convexity at $A$ as $C$, and let $H^+$ be the other one.

The $h^-$-width of $C$ at $A$ is the distance between the horocycle $H^-$ and the parallel horocycle with the same ideal point that supports the curve $C$ and such that the curve $C$ is enclosed in between (Figure 13-right). The $h^+$-width of $C$ at $A$ is defined analogously but taking the horocycle $H^+$ instead (Figure 13-left).

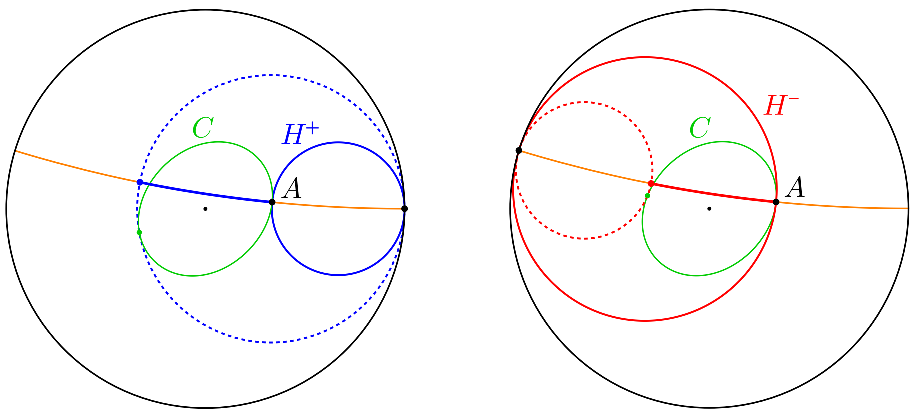


Figure 13: A $h^+$-width (left) and a $h^-$-width (right) of a $h$-convex curve $C$ at a point $A$.

Now we will introduce a definition of a $h$-width for $h$-hedgehogs which, although related to that of Santaló, will be presented differently as it will come naturally from the family of cooriented supporting horocycles that envelopes the $h$-hedgehog (see Figure 14).

Notice, in fact, that the signed distance between two parallel horocycles can be computed simply as the difference of the signed distances from the origin $O$ to each horocycle (with the sign according to the coorientation of the horocycles towards their interior).

**Definition 8** ($h$-width of a $h$-hedgehog)**.** Let $\mathcal{H}_s$ be a $h$-hedgehog. The *$h$-width of $\mathcal{H}_s$ in the direction* $u(t) \in \mathbb{S}^1$ is defined as the signed distance between the cooriented horocycles $H_t$ and $\widetilde{H}_t$, namely,

$$\omega_s(t) := s(t) - \tilde{s}(t).$$

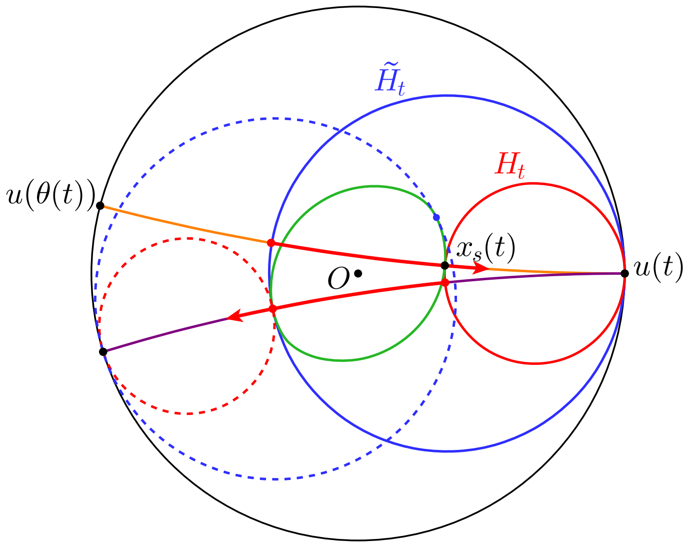


Figure 14: The $h$-width of a $h$-hedgehog in the direction $u(t) \in \mathbb{S}^1$ is the signed distance between the parallel horocycles $H_t$ and $\widetilde{H}_t$.

We have seen in Proposition 6 that

$$\tilde{s}\big(\theta(t)\big) = \ln\big(1 + s'(t)^2\big) - s(t).$$

Therefore,

$$\omega_s\big(\theta(t)\big) = s\big(\theta(t)\big) + s(t) - \ln\big(1 + s'(t)^2\big). \tag{18}$$

Thus, the explicit expression of the $h$-width of $\mathcal{H}_s$ in the direction $u(t)$ can be written as

$$\omega_s(t) = s(t) + s\big(\theta^{-1}(t)\big) - \ln\big(1 + s'\big(\theta^{-1}(t)\big)^2\big). \tag{19}$$

However, notice that we cannot compute a general expression of $\theta^{-1}$ because $\theta(t)$ depends on $s'(t)$.

**Remark 7.** With the terminology of Santaló for $h$-convex curves [35], the expressions (18) and (19), taken in absolute value, correspond to the notions of the $h^-$-width of $\mathcal{H}_s$ at $x_s(t)$ and the $h^+$-width of $\mathcal{H}_s$ at $x_s(t)$, respectively. As a direct consequence, the $h^-$-width at $x_s(t)$ is the $h^+$-width at $x_s\big(\theta(t)\big)$.

If we have a $h$-hedgehog $\mathcal{H}_s$ of constant $h$-width $d$, we will have that $\omega_s(t) = \omega_s\big(\theta(t)\big) = d$, and so the following characterization in terms of the support function $s$.

**Corollary 3.** *If $\mathcal{H}_s$ is a $h$-hedgehog of constant $h$-width $d$, with $d \in \mathbb{R}$, then*

$$s\big(\theta(t)\big) + s(t) - \ln\big(1 + s'(t)^2\big) = d.$$

Given a point $x_s(t)$ of a $h$-hedgehog $\mathcal{H}_s$, supported by a horocycle $H_t$ at $x_s(t)$, we say that $A$ is an *opposite point of* $x_s(t)$ if $A$ is the contact point of $\widetilde{H}_t$ (see Figure 14)

**Proposition 9.** *A $h$-hedgehog of constant $h$-width satisfies that the geodesic chord that joins each pair of opposite points is orthogonal to the curve at these points. More specifically, the map $\Lambda$ associates pairs of opposite points.*

*Proof.* Let $x_s$ be a parameterization of a $h$-hedgehog of constant $h$-width $d \in \mathbb{R}$ with a $\mathcal{C}^2$-support function $s$. By definition of the map $\theta$, we can write $\theta(t) = t + a(t)$, where $a(t)$ is the counterclockwise oriented angle from $u(t)$ to $v(t)$. Explicitly, from the expression (16) of $\theta(t)$, we have

$$a(t) = \pi - 2\arctan\big(s'(t)\big).$$

To prove the result it is enough to show that

$$a\big(\theta(t)\big) + a(t) = 2\pi.$$

Since the $h$-hedgehog is of constant $h$-width $d$ we know by Corollary 3 that

$$s\big(\theta(t)\big) = d - s(t) + \ln\big(1 + s'(t)^2\big).$$

In addition, differentiating this expression and using (17) we find

$$s'\big(\theta(t)\big)\,\theta'(t) = -s'(t)\,\theta'(t),$$

so that $s'\big(\theta(t)\big) = -s'(t)$, because $\theta'(t) \neq 0$ as a consequence of Proposition 5. Therefore,

$$a\big(\theta(t)\big) = \pi - 2\arctan\Big(s'\big(\theta(t)\big)\Big) = \pi + 2\arctan\big(s'(t)\big),$$

and so we can conclude $a\big(\theta(t)\big) + a(t) = 2\pi$ as desired. □

As a consequence of Proposition 9, the notion of a $h$-width coincides with the usual notion of a width, because supporting horocycles can be replaced by supporting geodesics (the curve is also a $g$-hedgehog by Theorem 2 and is well-behaved by Corollary 1). This extends the analogous known property for $h$-convex bodies of constant $h$-width (see [35]) to $h$-hedgehogs as follows.

**Theorem 3.** *Any $h$-hedgehog of constant $h$-width is also of constant width.*

**Remark 8.** In addition, in the convex case we have that any convex body of constant width is also $h$-convex (see [35] or [21]). For hedgehogs this relation is not true. In Remark 5 we saw an example of a projective $g$-hedgehog ($g$-hedgehog of constant width 0) that is not a $h$-hedgehog. See in Figure 15 another example for a non-zero constant width generated with a Leichtweiss support function $h(t) = \frac{1}{2} + \frac{1}{7}\sin(3\,t)$.

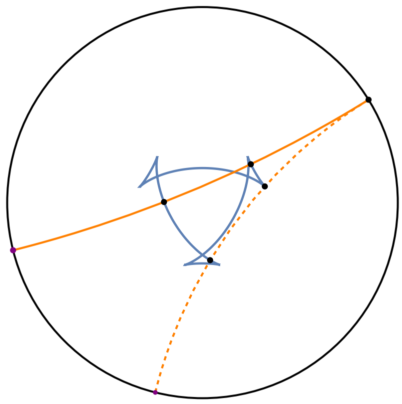

Figure 15: A $g$-hedgehog of constant width that is not a $h$-hedgehog.

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