AN ANALYTIC STUDY ON AVERAGED MAPPINGS

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To the memory of Professors Kazimierz Goebel and W. Art Kirk

Abstract. The purpose of this paper is to study the relationship between a mapping T defined on a nonempty subset C of a Banach space $(X, \|\cdot\|)$ and its averaged mappings $T_\lambda := (1-\lambda)I + \lambda T$, where $0 < \lambda < 1$ and I is the identity operator. In particular, we consider the case in which T or some T_λ are a generalized nonexpansive selfmapping on C.

1. Introduction

Let C be a nonempty closed and convex subset of a Banach space $(X, \|.\|)$, and let $T : C \to X$ be a mapping. For every $\lambda \in (0,1)$ the mapping $T_\lambda : C \to X$ defined as

$$T_\lambda(x) = (1-\lambda)x + \lambda T(x)$$

is called an *averaged mapping* generated by the mapping T, (and the identity operator).

It is easy to see that $\mathrm{Fix}(T) = \mathrm{Fix}(T_\lambda)$, that is, both mappings T and T_λ have the same set of fixed points. Therefore, the study of the existence of fixed points of the averaged mappings T_λ can help in the study of existence of fixed points of the generating mapping T.

Under the assumption that the mapping T is nonexpansive (i.e. 1-Lipschitz), averaged mappings implicitly appeared in the seminal Krasnoselskii's work [17], The term 'averaged' was coined in 1977 (see [5]). As a matter of fact, under this assumption, the averaged mappings mappings enjoy several remarkable properties which are not shared by all the nonexpansive mappings. For instance, they are asymptotically regular whenever C is also bounded (Ishikawa [13]). Over the past decades many authors have studied the asymptotic behaviour of the averaged mappings T_λ generated by a nonexpansive mapping T. (See [1, 9, 20, 21, 27] among many others). For instance, in uniformly convex Banach spaces such that their dual has the KK-property (i.e., if whenever $w - \lim_n x_n = x$ with $\|x_n\| \to \|x\|$, it follows that $\lim_n x_n = x$ stronglty), the Picard iterates of the averaged mappings T_λ converge weakly to a fixed point of T provided such mapping is nonexpansive and not fixed point free (see [7, Theorem 6.4]). An interesting report about the main features of these averaged mappings can be read in [11, Section 1].

The study of the fixed point theory for nonexpansive mappings has been extended to cover larger families of mappings. This line of thinking can be traced back to 1967 when F. Browder introduced and studied the so-called pseudo-contractive mappings (See [3]). Later, in 1973 Goebel, Kirk, and Shimi (see [12]), enlarged the previous research on generalized contractions by Hardy and Rogers. Since then,

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over 40 classes of generalized nonexpansive mappings (referred to as GNE) have been introduced and investigated.

It is obvious that if T is nonexpansive, then each T_λ is also nonexpansive. However, if $T : C \to C$ belongs to a class of GNE mappings, say $\mathcal{F}$, it is not clear whether T_λ belongs to $\mathcal{F}$. In these notes, first we will analyze, through various examples, the permanence of T_λ in $\mathcal{F}$, whenever $T \in \mathcal{F}$, when $\mathcal{F}$ is one of the four well-known classes of generalized nonexpansive mappings (GNE), specifically those defined by Browder in 1967 [3], Goebel, Kirk, and Shimi in 1971 [12], Jaggi in 1983 [15]; and Suzuki in 2008 [24].

We will be also concerned with the converse problem: If $T_\lambda \in \mathcal{F}$, what can be said about T? In particular, we will give some examples in which $T_\lambda \in \mathcal{F}$ with $T \notin \mathcal{F}$. To the best of the authors' knowledge, the consideration of averaged mappings generated from mappings other than nonexpansive ones has only been explored in a few known paper authored by Ishikawa (see [14]). However, this line of thinking has not been further pursued.

Finally, in the last section, we will focus mainly on the scenario where T_λ is nonexpansive, including results of existence of fixed point and asymptotic behaviour of such mappings in unbounded domains.

2. Preliminaires

Throughout this paper we suppose that $(X, \|\cdot\|)$ is a real Banach space and that X^* is its topological dual. We use $S_r(x_0)$ to denote the sphere centered at $x_0 \in X$ with radius $r > 0$.

If $x \in X$, we will denote by $J(x)$ the normalized duality mapping at x defined by $J(x) := \{j \in X^* : j(x) = \|x\|^2, \|j\| = \|x\|\}$. We will often use the mappings $\langle \cdot, \cdot \rangle_+ : X \times X \to \mathbb{R}$ defined by $\langle y, x\rangle_+ := \max\{j(y) : j \in J(x)\}$ and $\langle \cdot, \cdot \rangle_- : X \times X \to \mathbb{R}$ defined by $\langle y, x\rangle_- := \min\{j(y) : j \in J(x)\}$

From now on, C stands for a given nonempty, closed and convex subset of X. A mapping $T : C \to X$ is *nonexpansive* if for all $x, y \in C$, $\|T(x) - T(y)\| \leq \|x - y\|$. We will denote as $\mathcal{NE}(C)$ the class of all nonexpansive self mappings of C, or simply as $\mathcal{NE}$ if no confusion arises.

Recall that a mapping $T : C \to C$ is said to be *asymptotically regular* provided that for every $x \in C$, $T^{n+1}(x) - T^n(x) \to 0_X$. A mapping $T : C \to C$ is said to be *quasi-nonexpansive*, (QNE in short), provided that T has at least one fixed point $p \in C$, and for every fixed point p of T and for all $x \in C$, $\|p - T(x)\| \leq \|p - x\|$. Of course, nonexpansive selfmappings of C with some fixed point are quasi-nonexpansive, but the converse does not hold.

Let $(X, \|\cdot\|)$ be a Banach space. we will say that X has the *the FPP for $\mathfrak{F}$-mappings* if every $T \in \mathfrak{F}$ which leaves invariant to a bounded, closed and convex subset C of X has a fixed point. If the same holds for the nonempty weakly compact convex subsets C of X, we will use wFPP instead of FPP. For example, after a seminal paper due to Kirk [16], the reflexive Banach spaces enjoying the so called normal structure have the FPP for $\mathcal{NE}$-mappings.

We recall now the definitions of GNE mappings what we will be concerned here.

If $T : C \to X$ is a nonexpansive mapping, it is not hard to see that T satisfies the following inequality:

$$\|x - y\| \leq \|(1 + r)(x - y) + r(Ty - Tx)\| \quad \forall r > 0, \text{and } \forall x, y \in C. \tag{1}$$

This motivates the following.

Definition 2.1. *A mapping $T : C \to X$ satisfying (1) is called pseudo-contractive.* (Browder, 1968, [3]).

The class of the pseudo-contractive mappings is easily seen to be properly more general than the class of nonexpansive mappings. For instance, the pseudo-contractive mapping $T : [0, 1] \to [0, 1]$ defined as

$$T(x) = (1 - (x)^{2/3})^{3/2}.$$

fails to be nonexpansive. Interest in these mappings stems also from the fact that they are firmly connected to the well known class of accretive mappings; specifically, $T : C \to X$ is pseudo-contractive if and only if $A := I - T$ is accretive (see [8]), where I stands for the identity mapping. A mapping $A : C \to X$ is said to be accretive if the inequality

$$\|x - y\| \leq \|x - y + r(Ax - Ay)\| \tag{2}$$

holds $\forall r > 0$ and $\forall x, y \in C$.

We now recall some important facts regarding accretive operators which will be used in our paper (see for example [8]).

Proposition 2.1. *Let $A : D(A) \to X$ be a mapping. The following conditions are equivalent.*

- *A is an accretive mapping,*
- *The inequality $\langle Ax - Ay, x - y\rangle_+ \geq 0$ holds, for all $x, y \in D(A)$.*
- *For each $\lambda > 0$ the resolvent $J_\lambda := (I + \lambda A)^{-1} : R(I + \lambda A) \to D(A)$ is a single-valued nonexpansive mapping.*

Moreover, the following conditions are satisfied

- *$\lim_{\lambda \to 0^+} J_\lambda x = x$ for all $x \in \cap_{\lambda > 0} R(I + \lambda A)$.*
- *Given $x \in \cap_{\lambda>0} R(I + \lambda A)$, the function $\lambda \to J_\lambda x$ is continuous.*

If, in addition, $R(I+\lambda A)$ (i.e., the range of the mapping $I+\lambda A$) is for one, hence for all, $\lambda > 0$, precisely X, then A is called m-accretive. We say that A satisfies the range condition if $\overline{D(A)} \subset R(I + \lambda A)$ for all $\lambda > 0$. In this sense, it is well known that if $T : C \to C$ is a continuous pseudocontractive mapping and C is a closed convex subset, then $A := I - T$ enjoys the range condition (see [19]).

As a consequence of Proposition 2.1 the following result is fulfilled:

Proposition 2.2. *Let $T : C \to X$ a mapping. The following conditions are equivalent.*

- *T is a pseudo-contractive mapping,*
- *The inequality $\langle x - Tx - (y - Ty), x - y\rangle_+ \geq 0$ holds, for all $x, y \in C$.*
- *The inequality $\langle Tx - Ty, x - y\rangle_- \leq \|x - y\|^2$ holds, for all $x, y \in C$*

We will denote as $\mathcal{P}$ the class of all pseudo-contractive selfmappings of C.

Definition 2.2. *We say that a mapping $T : C \to X$ is strong pseudocontractive if for all $x, y \in C$ we have that*

$$\langle\, (I-T)(x) - (I-T)(y), x - y \,\rangle_- \geq 0. \tag{3}$$

Inequality (3) is equivalent to

$$\langle Tx - Ty, j \rangle \leq \|x-y\|^2 \;\text{ for all } j \in J(x-y). \tag{4}$$

It is clear that every nonexpansive mapping is strong pseudocontractive and every strong pseudocontractive mapping is pseudocontractive. If in addition X is a smooth Banach space then $\langle \cdot, \cdot \rangle_+ = \langle \cdot, \cdot \rangle_-$ and consequently strong pseudocontractive mappings and pseudocontractive mappings defined in X coincide. We will denote as $\mathcal{SP}$ the class of all strong pseudo-contractive selfmappings of C.

Other classes of GNE mappings that we will be concerned are: Jaggi nonexpansive, Goebel-Kirk-Shimi GNE and Suzuki C-type mappings. The three classes share the wFPP-property in Banach spaces enjoying normal structure. (See [15, 12, 18]).

Definition 2.3. *A mapping $T : C \to C$ is said to be Jaggi nonexpansive (JNE in short), if for every T-invariant, closed, convex, bounded subset E of C with at least two points, for every $x \in E$,*

$$\sup\{\|T(x) - T(y)\| : y \in E\} \leq \sup\{\|x-y\| : y \in E\}. \tag{5}$$

(Jaggi, 1983, [3]).

Of course, every nonexpansive mapping is JNE, but the converse fails because in [15] an easy example of a noncontinuous JNE is given. We will denote as $\mathcal{J}$ the class of all JNE selfmappings of C.

Definition 2.4. *Let $T : C \to C$. If there exists nonnegative constants a, b, c with $a + 2b + 2c = 1$, such that for every $x, y \in C$*

$$\|T(x)-T(y)\| \leq a\,\|x-y\| + b\,(\|x-T(x)\| + \|y-T(y)\|) + c\,(\|x-T(y)|\| + \|y-T(x)\|), \tag{6}$$

then we say that T is a generalized nonexpansive mapping in the sense of Goebel Kirk and Shimi, (GKS) in short. (Goebel, Kirk and Shimi 1973, [12]).

We will denote as $\mathcal{N}_{a,b,c}$ the class of all selfmappings of C satisfying the above condition (6), and as $\mathcal{N} := \bigcup_{a+2b+2c=1} \mathcal{N}_{a,b,c}$. Of course, $\mathcal{NE} = \mathcal{N}_{1,0,0}$. It is well known that if $T \in \mathcal{N}$ and it has some fixed point, then T is QNE.

Definition 2.5. *We say that a mapping $T : C \to X$ satisfies* condition *(C) on C if for all $x, y \in C$,*

$$\frac{1}{2}\|x - T(x)\| \leq \|x-y\| \;\text{ implies }\; \|T(x) - T(y)\| \leq \|x-y\|. \tag{7}$$

(Suzuki, 2008, [3]).

Of course, every nonexpansive mapping $T : C \to X$ satisfies condition (C) on C, but in [24, Example 1] a non continuous mapping satisfying condition (C) is given. Every mapping $T : C \to C$ which satisfies condition (C) on C and has some fixed point, is quasi-nonexpansive. We will denote as $\mathcal{C}$ the class of all selfmappings of C satisfying the above condition (C).

3. Permanence in $\mathcal{P}, \mathcal{J}, \mathcal{N}_{abc}, \mathcal{C}$ after averaging

It is easy to check that, for every nonexpansive mapping $T : C \to C$ and for every $\lambda \in (0,1)$ the averaged map T_λ is nonexpansive too. In symbols, $\forall \lambda \in (0,1)$, $T \in \mathcal{NE}(C) \Rightarrow T_\lambda \in \mathcal{NE}(C)$. In this section we first will realize that for the classes $\mathcal{J}$, $\mathcal{N}$ and $\mathcal{C}$, the situation is very different. Then we will see the nice behaviour of the class $\mathcal{P}$ with respect to the averaging process.

3.1. **Averaging JNE mappings.** In [4, Proposition 3.5.] a mapping $T : [0,1] \to [0,1]$ such that $T \in \mathcal{J}$ but $T_{\frac{1}{2}} \notin \mathcal{J}$ is given. Such a mapping T is defined as follows.

$$T(x) = \begin{cases} x & x \in \mathbb{Q} \\ 1 - x & x \notin \mathbb{Q}. \end{cases}$$

.

3.2. **Averaging GKS-type mappings.**

Example 1. *Consider the mapping $T : [0,1] \to [0,1]$ defined as*

$$T(x) = \begin{cases} 0, & 0 \le x < 1 \\ \frac{1}{2}, & x = 1. \end{cases}$$

Claim 3.1. *For every $x, y \in [0,1]$,*

$$|T(x) - T(y)| \le \frac{1}{3}|x-y| + \frac{1}{6}(|x - T(x)| + |y - T(y)| + |x - T(y)| + |y - T(x)|) \quad (8)$$

(That is, $T \in \mathcal{N}_{\frac{1}{3},\frac{1}{6}\frac{1}{6}}$).

We only need to check the case $0 \le x < y = 1$. Suppose that $x \ge \frac{1}{2}$. Then, $|T(x) - T(1)| = \frac{1}{2}$ and

$$\begin{aligned} &\tfrac{1}{3}|x-y| + \tfrac{1}{6}(|x - T(x)| + |y - T(y)| + |x - T(y)| + |y - T(x)|) \\ &= \tfrac{1}{3}(1-x) + \tfrac{1}{6}(x + 1 - \tfrac{1}{2} + x - \tfrac{1}{2} + 1) \\ &= \tfrac{2}{3} - \tfrac{1}{6} = \tfrac{1}{2}. \end{aligned}$$

Thus, inequality (8) holds in this case.

Suppose now that $0 \le x < \frac{1}{2}$.

$$\begin{aligned} &\tfrac{1}{3}|x-y| + \tfrac{1}{6}(|x - T(x)| + |y - T(y)| + |x - T(y)| + |y - T(x)|) \\ &= \tfrac{1}{3}(1-x) + \tfrac{1}{6}(x + 1 - \tfrac{1}{2} + \tfrac{1}{2} - x + 1) \\ &= \tfrac{2}{3} - \tfrac{x}{3} \end{aligned}$$

Since $\frac{1}{2} = 2 - \frac{3}{2}$, then, $x < \frac{1}{2} = 2 - \frac{3}{2}$, and therefore $\frac{3}{2} < 2 - x$, which implies that

$$|T(x) - T(y)| = \frac{1}{2} < \frac{2-x}{3}$$

that is, inequality (8) holds again in this case.

Claim 3.2. *If $a, b, c \ge 0$ with $a + 2b + 2c = 1$ then $T_{\frac{1}{2}} \notin \mathcal{N}_{a,b,c}$.*

$$T_{\frac{1}{2}}(x) = \begin{cases} \frac{x}{2}, & 0 \le x < 1 \\ \frac{3}{4}, & x = 1. \end{cases}$$

Let us suppose, for a contradiction, that $T_{\frac{1}{2}} \in \mathcal{N}_{a,b,c}$. Then, for $x = 1, y = \frac{1}{2}$ we have

$$|T_{\frac{1}{2}}(1) - T_{\frac{1}{2}}(\frac{1}{2})| \le a|1 - \frac{1}{2}| + b(|\frac{1}{2} - T_{\frac{1}{2}}(\frac{1}{2})| + |1 - T_{\frac{1}{2}}(1)|) + c(|1 - T_{\frac{1}{2}}(\frac{1}{2})| + |\frac{1}{2} - T_{\frac{1}{2}}(1)|)$$

that is,

$$\frac{1}{2} \le \frac{a}{2} + \frac{b}{2} + c \Rightarrow 1 \le a + b + 2c = 1 - b.$$

Thus, $b = 0$.

On the other hand, for $x = 1, y = \frac{5}{8}$, bearing in mind that $b = 0$, and that $a + 2c = 1$ we have

$$|T_{\frac{1}{2}}(1) - T_{\frac{1}{2}}(\frac{5}{8})| \le a|1 - \frac{5}{8}| + \frac{1-a}{2}\left(|1 - T_{\frac{5}{8}}| + |\frac{5}{8} - T_{\frac{1}{2}}(1)|\right)$$

that is,

$$\frac{3}{4} - \frac{5}{16} \le \frac{3a}{8} + \frac{1-a}{2}(\frac{11}{16} + \frac{1}{8}).$$

Therefore,

$$\frac{7}{16} \le \frac{3a}{8} + \frac{13(1-a)}{32} \Rightarrow 14 \le 12a + 13(1-a) = 13 - a,$$

which is absurd. Thus, $T_{\frac{1}{2}} \notin \mathcal{N}_{a,b,c}$ as we claimed. □

Notice that $T_{\frac{1}{2}}$ is QNE. ♢

3.3. **Averaging C-type mappings.** A (C)-type mapping may generate a non (C)-type averaged mapping, as in the following example.

Example 2. *Let $T : [0,1] \to [0,1]$ be the mapping defined by*

$$T(x) = \begin{cases} 1 - x, & 0 \le x \le \frac{3}{4} \\ \frac{1}{2}, & \frac{3}{4} < x \le 1. \end{cases}$$

Claim 3.3. *$T \in \mathcal{C}$.*

Case 1. Suppose that $x \le \frac{1}{2}$, $y \in (\frac{3}{4}, 1]$. Then,

$$|T(x) - T(y)| = |1 - x - \frac{1}{2}| = |\frac{1}{2} - x| = \frac{1}{2} - x \le y - x = |y - x|.$$

Case 2. Suppose that $\frac{1}{2} < x \le \frac{3}{4}$, $y \in (\frac{3}{4}, 1]$. In this case, $\frac{1}{2}|x - T(x)| = \frac{1}{2}|2x - 1| = x - \frac{1}{2}$.

If $\frac{1}{2}|x - T(x)| \le |x - y|$, then

$$|T(x) - T(y)| = |1 - x - \frac{1}{2}| = x - \frac{1}{2} = \frac{1}{2}|x - T(x)| \le |x - y|.$$

If $\frac{1}{2}|y-T(y)| \leq |x-y|$, then $\frac{1}{2}|y-\frac{1}{2}| \leq y-x$, that is,

$$\begin{aligned}&\tfrac{1}{2}y-\tfrac{1}{4} \leq y-x\\ &\Leftrightarrow 2x-\tfrac{1}{2} \leq y\\ &\Leftrightarrow x-\tfrac{1}{2} \leq y-x.\end{aligned}$$

Therefore,

$$|T(x)-T(y)| = |1-x-\frac{1}{2}| = x-\frac{1}{2} \leq y-x = |y-x|.$$

Thus, T is a (C)-type mapping. □

Claim 3.4. *$T_{\frac{1}{2}} \notin \mathcal{C}$.*

Note that

$$T_{\frac{1}{2}}(x) = \begin{cases} \frac{1}{2}x+\frac{1}{2}(1-x), & 0 \leq x \leq \frac{3}{4} \\ \frac{1}{2}x+\frac{1}{4} & \frac{3}{4} < x \leq 1 \end{cases} = \begin{cases} \frac{1}{2}, & 0 \leq x \leq \frac{3}{4} \\ \frac{1}{2}x+\frac{1}{4}, & \frac{3}{4} < x \leq 1. \end{cases}$$

One has that

$$\frac{1}{2}\left|\frac{3}{4}-T_{\frac{1}{2}}(\frac{3}{4})\right| = \frac{1}{2}\left|\frac{3}{4}-\frac{1}{2}\right| = \frac{1}{8} \leq \left|\frac{7}{8}-\frac{3}{4}\right|,$$

while

$$\left|T_{\frac{1}{2}}(\frac{7}{8})-T_{\frac{1}{2}}(\frac{3}{4})\right| = \left|\frac{7}{16}+\frac{1}{4}-\frac{1}{2}\right| = \frac{3}{16} > \frac{7}{8}-\frac{3}{4}.$$

□

Thus, we have seen that the average of the identity with a (C)-type mapping may fail to be a (C)-type mapping.

3.4. **Averaging pseudo-contractive mappings.** The class $\mathcal{P}$ has a very singular feature with respect to the averaged with the identity mapping. This averaged process can be regarded as stable in $\mathcal{P}$. With more precision.

Theorem 3.1. *A mapping $T : C \to X$ is pseudo-contractive if and only if any averaged mapping T_λ is pseudo-contractive.*

Proof. Fix $\lambda \in (0,1)$. For every $r > 0$ and $x, y \in C$ we have.

$$\begin{aligned}(1+r)(x-y)+r(T_\lambda(x)-T_\lambda(y)) &= (1+r)(x-y)+r(1-\lambda)(x-y)\\ &\quad +r\lambda(T(x)-T(y)) \qquad (9)\\ &= (1+r\lambda)(x-y)+r\lambda(T(x)-T(y)).\end{aligned}$$

Therefore, if the mapping T is PSC, for $x, y \in C$ and $r > 0$

$$\|x-y\| \leq \|(1+r\lambda)(x-y)+r\lambda(T(x)-T(y))\| = \|(1+r)(x-y)+r(T_\lambda(x)-T_\lambda(y))\|,$$

which means that the mapping $T_\lambda \in \mathcal{P}$.

Conversely, if the mapping $T_\lambda \in \mathcal{P}$, for $x, y \in C$, $r > 0$,

$$\|x-y\| \leq \|(1+\frac{r}{\lambda})(x-y)+\frac{r}{\lambda}(T_\lambda(x)-T_\lambda(y))\| = \|(1+r)(x-y)+r(T(x)-T(y))\|.$$

Thus, $T \in \mathcal{P}$. □

Corollary 3.1. *If $T : C \to C$ satisfies that T_λ is nonexpansive for some $\lambda \in (0,1)$, then $T \in \mathcal{P}$.*

Indeed, if $T_\lambda \in \mathcal{NE}$, then, $T_\lambda \in \mathcal{P}$. Thus, $T \in \mathcal{P}$. □

Remark 3.1. Consider again the mapping $T : [0,1] \to [0,1]$ introduced in Example (1). It is easy to see that T is not pseudo-contractive. From the above result, any averaged mapping T_λ cannot be pseudo-contractive. However, it is worthwhile to note that $T \in \mathcal{N}_{\frac{1}{3},\frac{1}{6}\frac{1}{6}} \bigcap \mathcal{J}$

4. Mappings with a nonexpansive averaged

It is well known that

$$\mathcal{A} := \bigcup_{\lambda \in (0,1)} \{T_\lambda : T \in \mathcal{NE}(C)\} \subset \mathcal{NE}(C).$$

However, not every nonexpansive selfmapping of C is an averaged mapping. For instance, if $T \in \mathcal{NE}(C)$ is not asymptotically regular, (for example $T : B_{\mathbb{R}^2} \to B_{\mathbb{R}^2}$ defined as $T(x,y) = (y, -x)$), according Ishikawa's Theorem, $T \neq U_\lambda$ for every $\lambda \in (0,1)$ and $U \in \mathcal{NE}(C)$. As a matter of fact, in a Hilbert space, if $T_{\frac{1}{2}} \in \mathcal{NE}(C)$, then T is called firmly nonexpansive (see [10, Proposition 11.2]). Nevertheless, it is not commented in the literature that an averaged mapping T_λ can be nonexpansive while the original mapping T is not.

Example 3. Let X be an arbitrary Banach space and $a \in X$ with $a \neq 0_X$. The mapping $T : X \to X$ defined by $Tx = -rx + a$ where r is a fixed real number in $(1,3]$ is an expansive mapping and $T_{\frac{1}{2}}$ is nonexpansive.

The following examples shows that this situation may occur even in the case of a mapping defined on a compact convex domain.

Example 4. *Let $C := [0,1] \times [0,1]$ endowed with the standard Euclidian norm. Let $T : C \to C$ be the mapping*

$$T(x,y) := \left(\frac{x}{1+\sqrt{2}}, x \right).$$

First we have.

Claim 4.1. *This mapping T fails to be QNE on C. Hence $T \notin \mathcal{N} \cup \mathcal{C}$.*

Indeed,

$$\|T(1,0) - T(0,0)\| = \| \left(\frac{1}{1+\sqrt{2}}, 1 \right) \| > \|(1,0) - (0,0)\|.$$

□

Claim 4.2. *$T \notin \mathcal{J}$.*

Indeed, the set $E := \{t \left(\frac{1}{1+\sqrt{2}}, 1 \right), t \in [0,1]\}$ is closed convex and T-invariant. Consider the point

$$P := T(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2} \left(\frac{1}{1+\sqrt{2}}, 1 \right) = \left(\frac{\sqrt{2}-1}{2}, \frac{1}{2} \right) \in E.$$

One has that

$$\begin{aligned}
&\sup\{\|P-(u,v)\| : (u,v)\in E\}\\
&=\sup\left\{\left\|\left(\tfrac{\sqrt{2}-1}{2},\tfrac{1}{2}\right)-(u,v)\right\| : (u,v)\in E\right\}\\
&=\sup\left\{\left\|\left(\tfrac{\sqrt{2}-1}{2},\tfrac{1}{2}\right)-t\left(\tfrac{1}{1+\sqrt{2}},1\right)\right\| : t\in[0,1]\right\}\\
&=\max\{\left\|\left(\tfrac{\sqrt{2}-1}{2},\tfrac{1}{2}\right)\right\|,\left\|\left(\tfrac{\sqrt{2}-1}{2},\tfrac{1}{2}\right)-\left(\tfrac{1}{1+\sqrt{2}},1\right)\right\|\}\\
&=\sqrt{\tfrac{4-2\sqrt{2}}{4}}\approx 0.54,
\end{aligned}$$

whereas

$$T(P)=\left(\frac{\frac{\sqrt{2}-1}{2}}{1+\sqrt{2}},\frac{\sqrt{2}-1}{2}\right)=\left(\frac{3-2\sqrt{2}}{2},\frac{\sqrt{2}-1}{2}\right)$$

and

$$\begin{aligned}
&\sup\{\|T(P)-T(u,v)\| : (u,v)\in E\}\\
&=\sup\left\{\left\|\left(\tfrac{3-2\sqrt{2}}{2},\tfrac{\sqrt{2}-1}{2}\right)-T(u,v)\right\| : (u,v)\in E\right\}\\
&=\sup\left\{\left\|\left(\tfrac{3-2\sqrt{2}}{2},\tfrac{\sqrt{2}-1}{2}\right)-t\left(\tfrac{1}{1+\sqrt{2}},1\right)\right\| : t\in[0,1]\right\}\\
&\geq\left\|\left(\tfrac{3-2\sqrt{2}}{2},\tfrac{\sqrt{2}-1}{2}\right)-\left(\tfrac{1}{1+\sqrt{2}},1\right)\right\|\\
&=\sqrt{\tfrac{34-23\sqrt{2}}{2}}\approx 0.858.
\end{aligned}$$

□

However, for the corresponding averaged mapping $T_{\frac{1}{2}}$ one has that

Claim 4.3. *The mapping $T_{\frac{1}{2}} : C\to C$ is nonexpansive. Hence, it is QNE, and $T_{\frac{1}{2}}\in\mathcal{J}\cup\mathcal{C}\cup\mathcal{N}$.*

$$T_{\frac{1}{2}}(x,y)=\left(\frac{x}{2}+\frac{x}{2+2\sqrt{2}},\frac{y}{2}+\frac{x}{2}\right)=\left(\frac{2+\sqrt{2}}{2+2\sqrt{2}}\,x,\frac{x+y}{2}\right).$$

Then, if $(x,y),(x',y')\in C$, one has that

$$\begin{aligned}
\|T_{\frac{1}{2}}(x,y)-T_{\frac{1}{2}}(x',y')\| &=\left\|\left(\tfrac{2+\sqrt{2}}{2+2\sqrt{2}}\,(x-x'),\tfrac{x-x'+y-y'}{2}\right)\right\|\\
&=\sqrt{\left(\tfrac{2+\sqrt{2}}{2+2\sqrt{2}}\right)^2(x-x')^2+\left(\tfrac{x-x'+y-y'}{2}\right)^2}\\
&\leq\sqrt{\left(\tfrac{6+4\sqrt{2}}{12+8\sqrt{2}}\right)(x-x')^2+\tfrac{(x-x')^2+(y-y')^2+2|x-x'||y-y'|}{4}}\\
&\leq\sqrt{\left(\tfrac{6+4\sqrt{2}}{12+8\sqrt{2}}\right)(x-x')^2+\tfrac{(x-x')^2+(y-y')^2+|x-x'|^2+|y-y'|^2}{4}}\\
&=\sqrt{\left(\tfrac{1}{2}+\tfrac{6+4\sqrt{2}}{12+8\sqrt{2}}\right)(x-x')^2+\tfrac{(y-y')^2}{2}}\\
&=\sqrt{(x-x')^2+\tfrac{(y-y')^2}{2}}\\
&\leq\|(x,y)-(x',y')\|.
\end{aligned}$$

□

Thus, $T\notin N\cup C\cup J$, while $T_{\frac{1}{2}}\in\mathcal{NE}\subset\mathcal{N}\cap\mathcal{C}\cap\mathcal{J}$.

♢

We conclude that a mapping failing to be nonexpansive may have a nonexpansive averaged. In other words, the class of those selfmappings of C which have a nonexpansive averaged mapping properly contains the class of the nonexpansive mappings, that is

$$\bigcup_{\lambda\in(0,1)} \{T_\lambda : C \to C : T \in \mathcal{NE}(C)\} \subsetneq \mathcal{NE}(C) \subsetneq \bigcup_{\lambda\in(0,1)} \{T : C \to C : T_\lambda \in \mathcal{NE}(C)\}.$$

5. Mappings in $\mathcal{J}\bigcup\mathcal{N}\bigcup\mathcal{C}$ after averaging

The process of averaging a selfmapping of C, that is the transformation $T \mapsto T_\lambda$, often yields a mapping T_λ enjoying nice metric properties, even if T lacks of these properties. The following examples can illustrate this situation.

Example 5. *Let $T : \left[\frac{1}{3}, 3\right] \to \left[\frac{1}{3}, 3\right]$ be the mapping defined as*

$$T(x) = \frac{1}{x}.$$

Claim 5.1. *This mapping T fails to be QNE. (Hence $T \notin \mathcal{N}\bigcup\mathcal{C}$).*

Indeed $p = 1$ is the unique fixed point of T and

$$\left|1 - T\left(\frac{1}{3}\right)\right| = 2 > \left|1 - \frac{1}{3}\right|.$$

□

Claim 5.2. *$T \notin \mathcal{J}$.*

The closed interval $C := \left[\frac{1}{3}, 3\right]$ is obviously T-invariant. For $x = \frac{5}{3} \in I$ we have that

$$\sup\{|T(x) - T(y)| : y \in C\} = \sup\{|\frac{3}{5} - T(y)| : y \in I\} = 3 - \frac{3}{5} = \frac{12}{5},$$

whereas

$$\sup\{|x - y| : y \in C\} = \frac{4}{3} < \frac{12}{5}.$$

□

Now we will see that, for the above mapping, the averaged mapping $T_{\frac{1}{2}}$ is QNE, and ever more, $T_{\frac{1}{2}} \in \mathcal{N}_{\frac{1}{3},\frac{1}{3},0}$

Claim 5.3. *Let $T_{\frac{1}{2}} : \left[\frac{1}{3}, 3\right] \to \left[\frac{1}{3}, 3\right]$ be the averaged mapping generated by T*

$$T_{\frac{1}{2}}(x) = \frac{x^2+1}{2x} = \frac{1}{2}x + \frac{1}{2}T(x). \tag{10}$$

Then, $T_{\frac{1}{2}} \in \mathcal{N}_{\frac{1}{3},\frac{1}{3},0}$.

Proof. We need to see that, for every $x, y \in \left[\frac{1}{3}, 3\right]$,

$$|T_{\frac{1}{2}}(x) - T_{\frac{1}{2}}(y)| \le \frac{1}{3}|x - y| + \frac{1}{3}\big(|x - T_{\frac{1}{2}}(x)| + |y - T_{\frac{1}{2}}(y)|\big). \tag{11}$$

Since

$$\left|\frac{x^2+1}{2x} - \frac{y^2+1}{2y}\right| = \frac{|xy(x-y) - (x-y)|}{2xy} = \frac{|x-y||xy-1|}{2xy},$$

and $|x - T_{\frac{1}{2}}(x)| = |x - \frac{x^2+1}{2x}| = \frac{|x^2-1|}{2x}$, inequality (11) is equivalent to

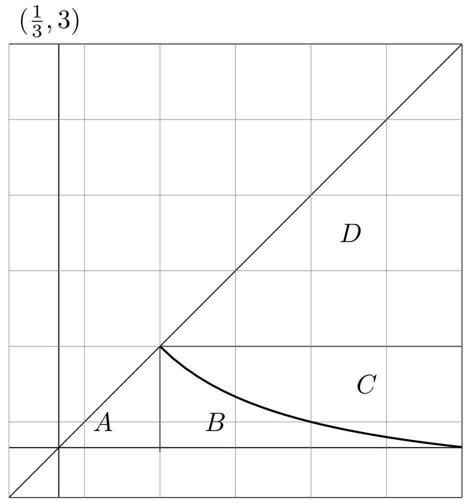
$$\frac{|x-y||xy-1|}{2xy} \leq \frac{1}{3}|x-y| + \frac{1}{3}\left(\frac{|x^2-1|}{2x} + \frac{y^2+1}{2y}\right),$$

which, in turn, is the same than

$$3|x-y||xy-1| \leq 2xy|x-y| + y|x^2-1| + x|y^2-1|. \tag{12}$$

To analyze inequality (12) we first note that it remains unchanged after a permutation between x and y. Then, to check that this inequality holds, it is enough to do this at the points (x, y) belonging to the triangle Δ whose vertices are $(\frac{1}{3}, \frac{1}{3}), (3, \frac{1}{3})$ and $(3, 3)$.

We now consider four subsets of Δ, namely



$$A := \{(x, y) \in \Delta : \tfrac{1}{3} \leq y \leq x \leq 1\}$$
$$B := \{(x, y) \in \Delta : 1 \leq x \leq 3, \tfrac{1}{3} \leq y \leq \tfrac{1}{x}\}$$
$$C := \{(x, y) \in \Delta : 1 \leq x \leq 3, \tfrac{1}{x} \leq y \leq 1\}$$
$$D := \{(x, y) \in \Delta : 1 \leq y \leq x \leq 3\}$$

Case 1. $(x, y) \in A$.

Then, $\frac{1}{3} \leq y \leq x \leq 1$. Therefore, Inequality (12) is

$$3(x-y)(1-xy) \leq 2xy(x-y) + y(1-x^2) + x(1-y^2), \tag{13}$$

which is equivalent to

$$(-3(x-y) - 2(x-y))xy \leq -3(x-y) - xy(x+y) + x + y \Leftrightarrow$$
$$(-4x+6y)xy \leq -2x + 4y \Leftrightarrow$$
$$0 \leq xy(2x-3y) - x + 2y$$

Consider now the real function $\psi : A \to \mathbb{R}$, defined as $\psi(x, y) = 2x^2y - 3xy^2 - x + 2y$. Since A is a compact set, then there exists a point $(x_0, y_0) \in A$ which is a global minimum of ψ in A. If (x_0, y_0) is an interior point of D, then it is a solution of the system of equations

$$\left.\begin{array}{l} 0 = \frac{\partial\phi}{\partial x}(x,y) = 4xy - 6y^2 - 1 \\ 0 = \frac{\partial\phi}{\partial y}(x,y) = 2x^2 - 6xy + 2 \end{array}\right\}$$

It is easy to check that this system has not real solutions. Consequently $(x_0, y_0) \in bd(A)$ where $bd(A)$ stands for the boundary of A.

Of course, $bd(A) = S_1 \cup S_2 \cup S_3$ where $S_1 := [(\frac{1}{3}, \frac{1}{3}), (1,1)]$, $S_2 := [(1,1), (1, \frac{1}{3})]$, and $S_3 := [(1, \frac{1}{3}), (\frac{1}{3}, \frac{1}{3})]$. If $(x,y) \in S_1$, then $x = y$. Since $\psi(x,x) = x - x^3 \geq 0$, we have that $\psi(x,y) \geq 0$ for every $(x,y) \in S_1$.

If $(x,y) \in S_2$ then $x = 1$ and $y \in [\frac{1}{3}, 1]$. Since $\psi(1,y) = 4y - 3y^2 - 1 = (1-y)(3y-1)$ then, for every $y \in [\frac{1}{3}, 1]$, $\psi(1,y) \geq 0$.

Finally, if $(x,y) \in S_3$ then $y = \frac{1}{3}$ and $x \in [\frac{1}{3}, 1]$. Since

$$\psi(x, \frac{1}{3}) = \frac{2}{3}x^2 - \frac{4}{3}x + \frac{2}{3} = \frac{2}{3}(x-1)^2 \geq 0$$

then, for every $x \in [\frac{1}{3}, 1]$, $\psi(x, \frac{1}{3}) \geq 0$. In summary, $(1,1)$ is a point of A where ψ attains its global minimum, that is for every $(x,y) \in D$, $\psi(x,y) \geq 0$.

Case 2. $(x,y) \in B$.

If $(x,y) \in B$ then $xy \leq 1$ and $\frac{1}{3} \leq y \leq 1 \leq x$. Therefore, Inequality (12) is

$$3(x-y)(1-xy) \leq 2xy(x-y) + y(x^2-1) + x(1-y^2). \tag{14}$$

Since $\frac{1}{3} \leq y \leq 1 \leq x$ then $1 \leq 3y \leq 3xy$ and therefore, if $0 < x - y$,

$$\begin{aligned} & 0 < x - y \leq 3xy(x-y) \Leftrightarrow \\ & 2x + 6xy^2 \leq 6x^2y + 2y \Leftrightarrow \\ & 3x - 3x^2y - 3y + 3xy^2 \leq 2x^2y - 2xy^2 + x^2y + x - y - xy^2 \Leftrightarrow \\ & 3(x-y)(1-xy) \leq 2xy(x-y) + y(x^2-1) + x(1-y^2). \end{aligned}$$

Thus, Inequality (14) holds for $0 < x - y$. For $0 = x - y$ the same inequality is obvious. Consequently, for $(x,y) \in B$, Inequality (12) holds in this case.

Case 3. $(x,y) \in C$.

If $(x,y) \in C$ then $xy \geq 1$. Therefore, Inequality (12) is

$$3(x-y)(xy-1) \leq 2xy(x-y) + y(x^2-1) + x(1-y^2), \tag{15}$$

which is equivalent to

$$\begin{aligned} & 3x^2y - 3x - 3xy^2 + 3y \leq 2x^2y - 2xy^2 + yx^2 - y + x - xy^2 \Leftrightarrow \\ & 3(y-x) \leq x - y. \end{aligned}$$

Since $x \geq y$, the above inequality is obvious. Consequently, for $(x,y) \in C$, Inequality (12) holds.

Case 4. $(x,y) \in D$.

Then, Inequality (12) is

$$3(x-y)(xy-1) \leq 2xy(x-y) + y(x^2-1) + x(y^2-1), \tag{16}$$

which is equivalent to

$$
\begin{aligned}
&(3(x-y)-2(x-y))xy \leq 3(x-y)+yx(x+y)-x-y \Leftrightarrow \\
&(x-y)xy \leq 2x-4y+yx(x+y) \Rightarrow \\
&-2xy^2 \leq 2x-4y \Rightarrow \\
&0 \leq xy^2+x-2y
\end{aligned}
$$

Consider now the real function $\phi : D \to \mathbb{R}$, defined as $\phi(x,y) = xy^2+x-2y$. Since D is a compact set, there exists a point $(x_0,y_0) \in D$ which is a global minimum of ϕ in D. If (x_0,y_0) is an interior point of D, then

$$y_0^2+1 = \frac{\partial \phi}{\partial x}(x_0,y_0) = 0,$$

which is absurd. Consequently $(x_0,y_0) \in bd(D)$ where $bd(D)$ stands for the boundary of D.

Of course, $bd(D) = S_1 \cup S_2 \cup S_3$ where $S_1 := [(1,1),(3,3)]$, $S_2 := [(3,3),(3,1)]$, and $S_3 := [(1,1),(3,1)]$. If $(x,y) \in S_1$, then $x = y$. Since $\phi(x,x) = x^3 - x = x(x^2-1) \geq \phi(1,1) = 0$, we have that $\phi(x,y) \geq 0$ for every $(x,y) \in S_1$.

If $(x,y) \in S_2$ then $x = 3$ and $y \in [1,3]$. Since $\phi(3,y) = 3y^2-2y+3 \geq \phi(3,\frac{1}{3})$ then, for every $y \in [1,3]$, $\phi(3,y) \geq \phi(3,1) = 5$.

Finally, if $(x,y) \in S_3$ then $y = 1$ and $x \in [1,3]$. Since $\phi(x,1) = 2x-2 \geq \phi(1,1) = 0$ then, for every $x \in [1,3]$, $\phi(x,1) \geq 0$. In summary, $(1,1)$ is a point of D where ϕ attains its global minimum, that is for every $(x,y) \in D$, $\phi(x,y) \geq \phi(1,1) = 0$.

□

Claim 5.4. *$T_{\frac{1}{2}} \notin \mathcal{C}$. (Hence, $T_{\frac{1}{2}} \notin \mathcal{NE}$).*

Indeed, take $x = \frac{2}{3}$ and $y = \frac{1}{3}$.

$$\frac{1}{2}|x - T_{\frac{1}{2}}(x)| = \frac{5}{24} < \frac{1}{3} = |x-y|,$$

while

$$|T_{\frac{1}{2}}(x) - T_{\frac{1}{2}}(y)| = |\frac{5}{3} - \frac{13}{12}| = \frac{7}{12} > \frac{1}{3} = |x-y|.$$

□

Claim 5.5. *$T_{\frac{1}{2}} \notin \mathcal{J}$.*

The closed interval $I := [\frac{1}{3},1]$ is $T_{\frac{1}{2}}$-invariant. For $x = \frac{2}{3} \in I$ we have that

$$\sup\{|T_{\frac{1}{2}}(x) - T_{\frac{1}{2}}(y)| : y \in I\} = \sup\{|\frac{13}{12} - T_{\frac{1}{2}}(y)| : y \in I\} = \frac{5}{3} - \frac{13}{12} = \frac{7}{12},$$

whereas

$$\sup\{|x-y| : y \in I\} = \frac{1}{3} < \frac{7}{12}.$$

□

Claim 5.6. $T \in \mathcal{P}$

It is easy to see that for every $x \in [\frac{1}{3}, 3]$,

$$(I - T)'(x) = 1 + \frac{1}{x^2} > 0$$

Therefore, $I - T$ is accretive, which is equivalent to $T \in \mathcal{P}$ and then, by Theorem 3.1, $T_\lambda \in \mathcal{P}$ for every $\lambda \in [0, 1]$. $\square$

Thus, $T \notin \mathcal{N} \cup \mathcal{C} \cup \mathcal{J}$, while $T_{\frac{1}{2}} \in \mathcal{N} \backslash (\mathcal{C} \cup \mathcal{J})$, whereas $T, T_{\frac{1}{2}} \in \mathcal{P}$.

$\diamondsuit$

Example 6. *Let* $T : [0, 1] \to [0, 1]$ *be the mapping defined by* $T(x) = \sqrt{1 - x^2}$.

Put $U = T_{\frac{1}{2}}$. It is easy so check that, for every $x \in [0, 1]$,

$$\frac{1}{2} \leq U(x) \leq \sqrt{\frac{1}{2}}.$$

In fact, the function U increases on the interval $[0, \frac{\sqrt{2}}{2}]$, and decreases on $[\frac{\sqrt{2}}{2}, 1]$. Hence it attains its maximum value at $\tau := \frac{\sqrt{2}}{2}$ which is just the unique fixed point of T and U in $[0, 1]$.

Claim 5.7. *The mapping* U *is QNE on* $[0, 1]$.

For $0 \leq x \leq \tau$, $T(x) \geq x$. Hence $U(x) \geq x$. Therefore, for $0 \leq x \leq \tau$,

$$|\tau - U(x)| = \tau - U(x) \leq \tau - x = |\tau - x|.$$

Let us suppose now that $\tau \leq x \leq 1$. Then $x \geq \tau \geq U(x)$. We need to see again that $|\tau - U(x)| \leq |\tau - x|$ that is, that the inequality

$$\tau - U(x) \leq x - \tau, \tag{17}$$

holds for every $x \in [\tau, 1]$. Note that it is equivalent to

$$2\tau \leq x + U(x).$$

One can check that the real function $\phi : [\tau, 1] \to [0, 2]$ defined as $\phi(x) := x + U(x)$ has a unique critical point at $\sigma := \frac{3\sqrt{10}}{10} \approx 0.94$, where ϕ attains its global maximum value on $[\tau, 1]$, namely $\phi(\sigma) = \frac{\sqrt{10}}{2} \approx 1.58$. Consequently, the global minimum of ϕ on this compact interval $[\tau, 1]$ is then attained at one of its end points. Since $\phi(1) = \frac{3}{2}$ and $\phi(\tau) = 2\tau = \sqrt{2}$, then this minimum is just attained at the point τ . Therefore, for every $x \in [\tau, 1]$,

$$2\tau = \phi(\tau) \leq \phi(x),$$

that is

$$2\tau \leq x + U(x).$$

Thus,

$$\tau - U(x) \leq x - \tau,$$

which is just inequality 17. $\square$

Claim 5.8. *The mapping* U *is JNE on* $[0, 1]$.

Proof. Let $E := [p,q]$ be a nonempty closed convex U-invariant subset of $[0,1]$. Notice that the restriction of U to the interval $P := [0,\sigma]$ is nonexpansive. Therefore, if $E \subset P$, for every $x \in E$

$$\sup\{|U(x) - U(y)| : y \in E\} \le \sup\{|x-y| : y \in E\}.$$

Otherwise, $\sigma < q \Rightarrow U(q) < U(\sigma) = \frac{\sqrt{10}}{5}$. Then, since E is a convex set,

$$\left[\frac{\sqrt{10}}{5}, \sigma\right] \subset [U(q), q] \subset E = [p,q].$$

In particular, $\tau := \frac{\sqrt{2}}{2} \in E \cap U(E)$. Consequently, the set $U(E)$ is an interval with the form $[\alpha, \tau]$, where

$$\alpha = \min\{U(p), U(q)\}.$$

A) If $x \in E$ with $x > \sigma$, then

$$\sup\{|x-y| : y \in E\} \ge x - \frac{\sqrt{10}}{5} \ge \sigma - \frac{\sqrt{10}}{5} = \frac{\sqrt{10}}{10} \approx 0.316,$$

while

$$\sup\{|U(x) - U(y)| : y \in E\} \le \operatorname{diam}(U(E)) \le \operatorname{diam}(U([0,1])) = \frac{\sqrt{2}}{2} - \frac{1}{2} \approx 0.207.$$

Thus, $\sup\{|U(x) - U(y)| : y \in E\} \le \sup\{|x-y| : y \in E\}$.

B) If $x \in E$ with $\tau \le x \le \sigma$, then we distinguish two subcases.

b1) $U(p) \ge U(q)$.
Then,

$$\sup\{|U(x) - U(y)| : y \in E\} \le \tau - U(q) \le x - U(q) = |x - U(q)| \le \sup\{|x-y| : y \in E\}.$$

b2) $U(p) < U(q)$.
Then,

$$\sup\{|U(x) - U(y)| : y \in E\} \le \tau - U(p) \le x - U(p) = |x - U(p)| \le \sup\{|x-y| : y \in E\}.$$

Thus, both subcases yield the inequality,

$$\sup\{|U(x) - U(y)| : y \in E\} \le \sup\{|x-y| : y \in E\}.$$

C) Finally, if $x \in [p,q]$ with $x < \tau$, bearing in mind that $\tau - \frac{1}{2} \approx 0.20 < \sigma - \tau \approx 0.24$, we have

$$\sup\{|U(x) - U(y)| : y \in E\} \le \tau - \frac{1}{2} < \sigma - \tau < \sigma - x \le \sup\{|x-y| : y \in E\}.$$

$\square$

Remark 5.1. Notice that the mapping $T : [0,1] \to [0,1]$ defined as $T(x) = \sqrt{1-x^2}$ fails to be JNE. Indeed,

$$\sup\{|T(\frac{1}{2}) - T(y)| : y \in [0,1]\} = \sqrt{\frac{3}{4}} > \sup\{|\frac{1}{2} - y| : y \in [0,1]\} = \frac{1}{2}.$$

However, a non JNE mapping (like T) may admit a JNE averaged mapping ($U = T_{\frac{1}{2}}$). In passing we point out that neither U nor T are nonexpansive.

On the other hand, it is obvious that T fails to be a Picard operator, because for every $x \in [0,1]$, $x \neq \tau$, the orbit starting at x is the sequence $(x, T(x), x, T(x), ...)$ which is not convergent. However, if $0 \le x < \tau$, then $0 \le x < U(x) < \tau$. Then, the bounded sequence $(x, U(x), U^2(x), \dots)$ increases, and it has a limit, say $\ell \in [0, \tau]$. Let $x_n := U^n(x)$. Then

$$x_{n+1} = U(x_n) = \frac{x_n + \sqrt{1 - x_n^2}}{2}.$$

Since $x_n \to \ell$,

$$\ell = U(\ell) = \frac{\ell + \sqrt{1 - \ell^2}}{2} \Rightarrow \ell = \tau = U(\tau).$$

Finally, if $1 \geq x > \tau$, then $0 \leq U(x) \leq \tau$, and then, we repeat the same argument of above, to see that the orbit $(U(x), U^2(x), \dots)$ is convergent to ℓ. This means that $U = T_{\frac{1}{2}}$ is a Picard operator while T is not. Even more, is is obvious that, for every $x \in [0, 1]$, $U^{n+1}(x) - U^n(x) \to 0$. This means that $U = T_{\frac{1}{2}}$ is asymptotically regular. From a celebrated result due to Ishikawa (see [13]), if $T : C \to C$ is a nonexpansive mapping, then the averaged mapping $T_{\frac{1}{2}}$ is asymptotically regular. This example shows that Ishikawas's holds beyond the set of the nonexpansive mappings.

$\diamondsuit$

5.1.

Definition 5.1. *(See Ullah e.a. [26]). For $b \in [0, \infty)$, a mapping $T : C \to C$ is said to be b-Enriched Suzuli nonexpansive, (b-ESNE in short), for all $x, y \in C$,*

$$\tfrac{1}{2}\|x - T(x)\| \leq (b+1)\|x - y\| \Rightarrow \|b(x - y) + T(x) - T(y)\| \leq (b+1)\|x - y\|.$$

Notice that 0-ESNE are the Suzuli C-type mappings.

Proposition 5.1. *(See [26, Theorem 4]). If, for $b > 0$, $T : C \to C$ is b-ESNE, then there exists $\mu \in (0, 1)$, such that the averaged mapping*

$$T_\mu := \mu T + (1 - \mu)Id$$

satisfies Suzuki's condition (C).

Indeed, there exists b such that

$$\frac{1}{2}\|x - T(x)\| \leq (b+1)\|x - y\| \Rightarrow \|b(x - y) + T(x) - T(y)\| \leq (b+1)\|x - y\|. \quad (18)$$

Choose μ such that $\frac{1}{\mu} = b + 1$, what is the same, such that $b = \frac{1-\mu}{\mu}$. Of course $\mu = \frac{1}{b+1} \in (0, 1)$. Then condition (18) becomes

$$\frac{1}{2}\|x - T(x)\| \leq \frac{1}{\mu}\|x - y\| \Rightarrow \|\frac{1-\mu}{\mu}(x - y) + T(x) - T(y)\| \leq \frac{1}{\mu}\|x - y\|, \quad (19)$$

that is

$$\frac{1}{2}\|\mu x - \mu T(x)\| \leq \|x - y\| \Rightarrow \|(1 - \mu)(x - y) + \mu(T(x) - T(y))\| \leq \|x - y\|,$$

or

$$\frac{1}{2}\|x - [(1-\mu)x + \mu T(x)]\| \leq \|x - y\| \Rightarrow \|(1-\mu)x + \mu T(x) - [(1-\mu)y + \mu T(y)]\| \leq \|x - y\|,$$

what is the same than

$$\frac{1}{2}\|x - T_\mu(x)\| \leq \|x - y\| \Rightarrow \|T_\mu(x) - T_\mu(y)\| \leq \frac{1}{\mu}\|x - y\|. \quad (20)$$

This means that the mapping T_μ satisfies condition (C). $\square$

Corollary 5.1. *Let C be a weakly compact convex subset of a Banach space enjoying normal structure. If $T : C \to C$ is b-ESNE, for $b \geq 0$, then T has a fixed point in C.*

If $b = 0$ then T satisfies condition (C), and the result immediately follows from Proposition 3.4 and Theorem 4.4. of [18].

If $b > 0$, from the above proposition there exists an averaged mapping T_μ with $\mu > 0$, satisfying condition (C). Again from [18], one follows that there exists $x_0 \in C$ such that $T_\mu(x_0) = x_0$. Therefore, $x_0 = T(x_0)$. $\square$

The examples of this section show that the following theorem has a real scope. It can be regarded as a generalization

Theorem 5.1. *Let $(X, \|\cdot\|)$ be a reflexive Banach space enjoying normal structure. Let C be a nonempty, closed, convex and bounded subset of X. Let $T : C \to C$ be a mapping such that there exists some $\lambda \in (0, 1)$ with $T_\lambda \in \mathcal{J} \cup \mathcal{N} \cup \mathcal{C}$. Then, T has a fixed point in C.*

Proof. If the averaged mapping $T_\lambda \in \mathcal{J}$ then it has a fixed point, according the original Jaggi's result [15, Theorem 1]. If $T_\lambda \in \mathcal{N}$, from Bogin result [2, Theorem 2], it has a fixed point. Finally, if $T_\lambda \in \mathcal{C}$ then it has a fixed point, from [18, Theorem 4.4.]. In any case, the mapping T_λ has a fixed point, say $x_0 \in C$. Therefore, x_0 is a fixed point of T. $\square$

An obvious question naturally arises: Characterize those mappings $T : C \to C$ such that $T_\lambda \in \mathcal{J} \cup \mathcal{N} \cup \mathcal{C}$.

6. Fixed point results for mappings with a nonexpansive averaged in unbounded domains

Proposition 6.1. *Let C be a nonempty, closed and convex subset of a Banach space X and $T : C \to X$ is a mapping such that there exists $\lambda \in (0, 1)$ so that $T_\lambda : C \to X$ is nonexpansive, then T is a continuous strong pseudo-contractive map.*

Proof.
Since $T = \frac{1}{1-\lambda}(T_\lambda - \lambda I)$ and T_λ is continuous because it is nonexpansive, we derive that T is a continuous mapping.
To see that T is strong pseudo-contractive, we argue as follows:
T_λ is nonexpansive, then it is strong pseudocontractive and this means that for all $j \in J(x - y)$ with $x, y \in C$ the inequality

$$\langle T_\lambda x - T_\lambda y, j\rangle \leq \|x - y\|^2 \tag{21}$$

holds. That is,

$$\langle \lambda(x - y) + (1 - \lambda)(Tx - Ty), j\rangle = \lambda\|x - y\|^2 + (1 - \lambda)\langle Tx - Ty, j\rangle \leq \|x - y\|^2,$$

consequently

$$\langle Tx - Ty, j\rangle \leq \|x - y\|^2.$$

$\square$

Theorem 6.1. *Let $(X, \|\cdot\|)$ be a Banach space with the FPP for the family $\mathcal{NE}$ and let T be a self defined mapping on a closed convex subset C of X. If*

(i) there exists $r \in (0,1)$ such that T_r is nonexpansive and
(ii) there exist $x_0 \in C$ and $R > 0$ such that either $S_R(x_0) \cap C = \emptyset$ or $Tx - x_0 \neq \mu(x - x_0)$ for all $\mu > 1$ and $x \in S_R(x_0) \cap C$.

Then

(1) T has a fixed point in C,
(2) Fix $u \in C$ the sequence (x_n) in C defined by

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T_\gamma x_n,$$

where $\gamma \in (r, 1)$ and (α_n) is a sequence in $[0,1]$ such that $\alpha_n \to 0$. Then (x_n) is a bounded a.f.p. sequence.

Proof. (1) We argue as follows. Proposition 6.1 shows that $T : C \to C$ is a continuous pseudocontractive mapping, then if we call $A := I - T$, we know that A is an accretive operator, moreover A satisfies the range condition because T is a continuous weakly inward pseudocontractive mapping. Thus for each $\lambda > 0$ there exists $x_\lambda \in C$ such that $x_0 = x_\lambda + \lambda A x_\lambda$. Denote the resolvent of A by $J_\lambda := (I + \lambda A)^{-1}$, thus $x_\lambda = J_\lambda x_0$. We will show that the set $\Big\{ J_\lambda x_0 : \lambda > 0 \Big\}$ is bounded. Indeed, otherwise there exists $\lambda_1 > 0$ such that $\|J_{\lambda_1} x_0 - x_0\| > R$. Since, by Proposition 2.1, the mapping $\lambda \to J_\lambda x_0$ is continuous and $\lim_{\lambda \to 0^+} J_\lambda x_0 = x_0$, we can find $\lambda_2 > 0$ such that $\|J_{\lambda_2} x_0 - x_0\| = R$. Nevertheless,

$$T(J_{\lambda_2} x_0) - x_0 = \frac{\lambda_2 + 1}{\lambda_2} (J_{\lambda_2} x_0 - x_0), \tag{22}$$

which is a contradiction since we are assuming that (ii) holds.

Now, if for each $n \in \mathbb{N}$, we call $x_n := J_n x_0$, then the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded. Moreover, since $A(x_n) = \frac{1}{n}(x_0 - x_n)$, we get that $\|Ax_n\| \to 0$, which means that $(x_n)_{n\in\mathbb{N}}$ is a bounded approximate fixed point sequence for T. Let us see that (x_n) is an a.f.p. sequence for J_1. Indeed,

$$\|x_n - J_1 x_n\| = \|J_1(x_n + Ax_n) - J_1 x_n\| \le \|x_n + Ax_n - x_n\| = \|Ax_n\| \mapsto^{n\to\infty} 0.$$

Now, consider $x_1 \in C$ and we call $r_0 := \limsup_n \|x_n - x_1\|$ and $K = \{x \in C : \limsup \|x_n - x\| \le r_0\}$. It is clear that $x_1 \in K$ and it is bounded closed convex and J_1-invariant. Thus, since X enjoys the (FPP), there exists $z \in K$ such that $z = J_1(z)$ and then $0_X = A(z) = z - T(z)$.

(2) Consider $\beta := \frac{\gamma - r}{1 - r} \in (0,1)$. In this case, we have that $T_\gamma = (T_r)_\beta = \beta I + (1 - \beta) T_r$.

Now, we may call

$$y_n := \frac{\alpha_n u + (1-\alpha_n)\beta T_r x_n}{\alpha_n + (1-\alpha_n)\beta},$$

and then we can write

$$x_{n+1} = (\alpha_n + (1-\alpha_n)\beta) y_n + (1 - \alpha_n - (1-\alpha_n)\beta) x_n.$$

Since z is a fixed point of T and thus it is also a fixed point of T_r arguing as in Suzuki [23, Theorem 3] we obtain that both sequences (x_n) and (y_n) are bounded and $\|x_{n+1} - x_n\| \to 0$. This fact implies that (x_n) is a bounded a.f.p. sequence for T_r and therefore for T.

$\square$

Theorem 6.2. *Let $(X, \|\cdot\|)$ be a uniformly smooth Banach space and let T be a self defined mapping on a closed convex subset C of X, such that there exists $r \in (0,1)$ such that T_r is nonexpansive.*

Fix $u \in C$, $\gamma \in (r,1)$ and (α_n), a sequence in $[0,1]$ such that $\alpha_n \to 0$. Consider the following iterative algorithm

$$x_{n+1} = \alpha_n u + (1-\alpha_n)T_\gamma x_n. \tag{23}$$

Then, the following conditions are equivalent:

(1) There exist $x_0 \in C$ and $R > 0$ such that either $S_R(x_0) \cap C = \emptyset$ or $Tx - x_0 \neq \mu(x - x_0)$ for all $\mu > 1$ and $x \in S_R(x_0) \cap C$,
(2) The mapping T has a fixed point in C,
(3) (x_n) is a bounded a.f.p. sequence.
(4) If,in addition, $\sum_{n=1}^{\infty} \alpha_n = +\infty$, then (x_n) converges strongly to a fixed point of T.

Proof.
(1) $\Rightarrow$ (2) It is well know that if X is a uniformly smooth Banach space, then it is reflexive and has normal structure, (see [25]), and therefore from the well known Kirk's result, X enjoys the FPP for nonexpansive mappings. Thus, we derive (2) from Theorem 6.1 (1).
(2) $\Rightarrow$ (3) It is a consequence of Theorem 6.1 (2).
(3) $\Rightarrow$ (4) If (x_n) is a bounded a.f.p. sequence for T, then it is also a bounded a.f.p. sequence for T_r. Therefore, if we call $r_0 := \limsup \|x_n - u\| \vee \limsup \|x_n - x_1\| < \infty$. It is clear that $K := \{x \in C : \limsup \|x_n - x\| \leq r_0\}$ is a nonempty, closed convex and bounded T_r-invariant subset of C. Moreover $(x_n) \subsetneq K$. This allows us to apply [23, Corollary 1] to conclude that if additionally $\sum_{n=1}^{\infty} \alpha_n = +\infty$, then (x_n) converges strongly to a fixed point of T_r and therefore to a fixed point of T.
(4) $\Rightarrow$ (1) Let x_0 be the fixed point for T in C (i.e., $T(x_0) = x_0$).
Define $G : X \times X \to \mathbb{R}$ by $G(x,y) := \langle x, y\rangle_-$ It is easy to prove that G satisfies:

(g1) $G(\lambda x, y) = \lambda G(x,y)$ for all $\lambda > 0$,
(g2) $G(x,x) > 0$ whenever $x \neq 0$.

Let us see that if $x \in C$ then $G(Tx - x_0, x - x_0) \leq G(x - x_0, x - x_0) = \|x - x_0\|^2$. Indeed, Proposition 6.1 yields that T is a continuous pseudocontractive mapping and by Proposition 2.2, we have:

$$\begin{aligned} G(Tx - x_0, x - x_0) &= \langle Tx - x_0, x - x_0\rangle_- \\ &= \langle Tx - Tx_0, x - x_0\rangle_- \\ &\leq \|x - x_0\|^2 \\ &= G(x - x_0, x - x_0) \end{aligned}. \tag{24}$$

To get a contradiction, suppose that x_0 does not satisfy condition (1), then given $R > 0$ there exist $\lambda_R > 1$ and $x_R \in C \cap S_R(x_0)$ such that $Tx_R - x_0 = \lambda_R(x_R - x_0)$, but in this case, we obtain:

$$\begin{aligned} G(Tx_R - x_0, x_R - x_0) &= G(\lambda_R(x_R - x_0), x_R - x_0) \\ &= \lambda_R G(x_R - x_0, x_R - x_0) \\ &> G(x_R - x_0, x_R - x_0), \end{aligned}$$

Which is a contradiction by (24).

□

7. Conclusions

We have studied the averaged mappings T_λ where T is a generalized nonexpansive selfmapping of a closed convex subset C of a Banach space X. Moreover, we have seen that it has effective relevance the following result.

Theorem 7.1. *If a class $\mathfrak{F}$ of selfmappings of C enjoys the fixed point property (that is, every $T : C \to C$ for $T \in \mathfrak{F}$, has a fixed point), then every mapping $T : C \to C$ such that $T_\lambda \in \mathfrak{F}$ for some $\lambda \in (0, 1)$, has a fixed point in C.*

The above examples show that the scope this theorem may go beyond of the class $\mathfrak{F}$. This arises the problem of characterize those mappings T such that $T_\lambda \in \mathfrak{F}$.

The particular case $\mathfrak{F} = \mathcal{NE}$ have been fully analyzed, without the assumption of boundedness of the domain C.

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