

Research Article

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Characterization of the subdifferential and minimizers for the anisotropic p -capacity

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Abstract: We obtain existence of minimizers for the p -capacity functional defined with respect to a centrally symmetric anisotropy for $1 < p < \infty$, including the case of a crystalline norm in $\mathbb{R}^N$. The result is obtained by a characterization of the corresponding subdifferential and it applies to unbounded domains of the form $\mathbb{R}^N \setminus \overline{\Omega}$ under mild regularity assumptions (Lipschitz-continuous boundary) and no convexity requirements on the bounded domain Ω . If we further assume an interior ball condition (where the Wulff shape plays the role of a ball), then any minimizer is shown to be Lipschitz continuous.

Keywords: p -capacity, anisotropy, crystalline, minimizers, Lipschitz regularity, Euler–Lagrange equation

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1 Introduction and statement of main results

1.1 The p -capacity and its anisotropic version

For $N \geq 2$, the p -capacity of a given compact set $K \subset \mathbb{R}^N$ is defined as

$$\text{Cap}_p(K) = \inf \left\{ \int_{\mathbb{R}^N} \|\nabla u\|^p dx : u \in C_c^\infty(\mathbb{R}^N), u \geq 1 \text{ in } K \right\},$$

where $\|\cdot\|$ denotes the Euclidean norm and C_c^∞ are smooth functions with compact support. The relevance of this quantity comes from its geometric meaning, since for $p > 1$ the usual geometric functionals like area or volume do not provide a satisfactory depiction of properties of domains. Here is where inequalities involving the p -capacity naturally arise.

From the viewpoint of partial differential equations, the search for minimizers of the p -capacity, called *p -capacitary functions* or equilibrium potentials, has attracted a lot of attention due to its relation to the p -Laplace equation. In fact, when $1 < p < N$ and under suitable smoothness of the boundary of K , a unique p -capacitary function $u \in L^{\frac{pN}{N-p}}(\mathbb{R}^N)$ with $\nabla u \in L^p(\mathbb{R}^N; \mathbb{R}^N)$ exists and it satisfies the Euler–Lagrange equation

$$\begin{cases} -\Delta_p u = -\text{div}(\|\nabla u\|^{p-2} \nabla u) = 0 & \text{in } \mathbb{R}^N \setminus K, \\ u = 1 & \text{in } K, \\ u \rightarrow 0 & \text{as } \|x\| \rightarrow \infty. \end{cases}$$

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The purpose of this paper is to obtain existence of p -capacitary functions within anisotropic media, with regularity assumptions so mild as to include the case of a crystalline anisotropy. It is crucial to move out of classical Euclidean isotropic settings to allow the possibility of environments where properties differ with the direction, since this extra flexibility will be key for applications to crystal growth [3, 39], noise removal or image restoration [19, 36].

Accordingly, replacing the Euclidean norm in the definition of the p -capacity functional by a generic norm F in $\mathbb{R}^N$, leads to the *anisotropic p -capacity functional*:

$$\text{Cap}_p^F(K) := \inf \left\{ \int_{\mathbb{R}^N} F^p(\nabla u) \, dx : u \in C_c^\infty(\mathbb{R}^N), u \geq 1 \text{ in } K \right\}.$$

In the range $1 < p < N$, existence and uniqueness of minimizers (called anisotropic p -capacitary functions) have been shown under different conditions on the anisotropy F (most of the results deal with $C^\infty(\mathbb{R}^N \setminus \{0\})$ uniformly elliptic norms) and the domain K . Moreover, in these cases the minimizer is the unique solution to the following PDE:

$$\begin{cases} -\operatorname{div}(F^{p-1}(\nabla u) \nabla F(\nabla u)) = 0 & \text{in } \mathbb{R}^N \setminus K, \\ u = 1 & \text{in } K, \\ u \rightarrow 0 & \text{as } \|x\| \rightarrow \infty. \end{cases} \quad (1.1)$$

Our main goal is to find a similar characterization of minimizers of the anisotropic p -capacity by relaxing the assumptions on regularity of F and K as much as possible, and allowing a non-necessarily constant boundary condition. To state our main result in this direction, we need to introduce some definitions and notation.

1.2 Characterization of minimizers by means of the subdifferential

Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz-continuous boundary and $\varphi \in W^{1-\frac{1}{p},p}(\partial\Omega)$. Set

$$W_\varphi^{1,p}(\Omega) := \{u \in W^{1,p}(\Omega) : u = \varphi \text{ on } \partial\Omega\}.$$

For a norm F in $\mathbb{R}^N$, we consider the energy functional

$$\mathcal{F}_{\Omega,\varphi} : L^p(\Omega) \rightarrow [0, +\infty]$$

defined by

$$\mathcal{F}_{\Omega,\varphi}(u) := \begin{cases} \int_{\Omega} F^p(\nabla u) & \text{if } u \in W_\varphi^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases} \quad (1.2)$$

Notice that, since Ω is bounded and F^p is convex and coercive, [20, Theorem 5 in Section 8.2.4] ensures the existence of a minimizer. However, since F (and therefore F^p) is not required to be strictly convex, uniqueness is not anymore guaranteed.

Our objective is to characterize the minimizers of this energy functional by means of the corresponding Euler–Lagrange equation (notice that here we are not allowed to write ∇F as in (1.1)). Since $\mathcal{F}_{\Omega,\varphi}$ is a proper and convex functional, we have that

$$u \in \arg \min \{\mathcal{F}_{\Omega,\varphi}\} \quad \text{if, and only if,} \quad 0 \in \partial \mathcal{F}_{\Omega,\varphi}(u), \quad (1.3)$$

where the latter stands for the subdifferential of the energy functional, whose exact expression (and accordingly the concrete Euler–Lagrange equation for this problem) is unknown and does not follow from standard arguments, due to the non-smoothness of the integrand. Therefore, our first goal is to characterize the subdifferential; more precisely, we get:

Theorem 1.1 (Characterization of the subdifferential in Ω bounded). *Let $1 < p < \infty$, $\Omega \subset \mathbb{R}^N$ a bounded domain with Lipschitz-continuous boundary and $\varphi \in W^{1-\frac{1}{p},p}(\partial\Omega)$. If $u \in W_\varphi^{1,p}(\Omega)$ and $v \in L^{p'}(\Omega)$, with $p' = \frac{p}{p-1}$, then the*

following are equivalent:

- (a) $v \in \partial \mathcal{F}_{\Omega, \varphi}(u)$.
- (b) There exists $z \in L^\infty(\Omega; \mathbb{R}^N)$, with $F^\circ(z) \leq 1$ and $z \cdot \nabla u = F(\nabla u)$, such that $v = -\operatorname{div}(pF^{p-1}(\nabla u)z)$ in the weak sense, that is,

$$\int_{\Omega} vw = \int_{\Omega} pF^{p-1}(\nabla u)z \cdot \nabla w \quad \text{for every } w \in W_0^{1,p}(\Omega).$$

Here F° denotes the dual of the norm F .

The strategy and methods carried out to prove Theorem 1.1, suitably adapted to work with homogeneous Sobolev spaces, permit to obtain the corresponding characterization for unbounded exterior domains of the form $\mathbb{R}^N \setminus \overline{\Omega}$ (see Theorem 4.1), where $\Omega \subset \mathbb{R}^N$ is again a bounded domain with Lipschitz boundary. As a byproduct of this result, we get a characterization for all minimizers of $\operatorname{Cap}_p^F(\overline{\Omega})$ for bounded Lipschitz domains Ω .

Corollary 1.2. *Let $1 < p < \infty$, $\Omega \subset \mathbb{R}^N$ a bounded domain with Lipschitz boundary and $\varphi \in W^{1-\frac{1}{p}, p}(\partial\Omega)$. Consider either $D = \Omega$ or $D = \mathbb{R}^N \setminus \overline{\Omega}$. Then any minimizer u of the energy functional $\mathcal{F}_{D, \varphi}$ is a weak solution of*

$$\begin{cases} \operatorname{div}(F^{p-1}(\nabla u)z) = 0 & \text{in } D, \\ u = \varphi & \text{on } \partial D, \\ u \rightarrow 0 & \text{as } \|x\| \rightarrow \infty, \end{cases} \quad \text{if } D = \mathbb{R}^N \setminus \overline{\Omega}, \quad p < N, \quad (1.4)$$

for some $z \in L^\infty(D; \mathbb{R}^N)$, with $F^\circ(z) \leq 1$ and $z \cdot \nabla u = F(\nabla u)$. Here the equality on the boundary is understood in the sense of traces. In particular, if $\varphi \equiv 1$, then

$$\operatorname{Cap}_p^F(\overline{\Omega}) = \inf \{ \mathcal{F}_{\mathbb{R}^N \setminus \overline{\Omega}, 1}(u) : u \in L^{\frac{pN}{N-p}}(\mathbb{R}^N) \},$$

and hence anisotropic p -capacitary functions are solutions to (1.4).

Similar statements in the anisotropic framework were previously proven under hypotheses that are far from optimal. Indeed, up to our knowledge, the most general result in this spirit can be found in [10] (see also [2]), where the authors show existence, uniqueness and regularity of a p -capacitary function $u \in C^{2,\alpha}(\mathbb{R}^N \setminus K)$ in the case that K is convex and with boundary of class $C^{2,\alpha}$ for norms $F \in C^{2,\alpha}(\mathbb{R}^N \setminus \{0\})$ such that F^p is twice continuously differentiable in $\mathbb{R}^N \setminus \{0\}$ with a strictly positive definite Hessian matrix.

Here we generalize the results in the previous literature in several directions. Indeed, regarding the anisotropy, we only require it to be a norm, without any extra assumptions on smoothness or uniform ellipticity. In particular, we allow norms whose dual unit balls have corners and/or straight segments, including crystalline cases as the ℓ_∞ or ℓ_1 norms in $\mathbb{R}^N$.

Beyond these classical examples, all our results apply to any norm F whose unit sphere is a centrally symmetric and convex polytope (hence not strictly convex) or, more generally, unit balls which are in part uniformly convex and in part polyhedral while keeping the central symmetry, as well as more sophisticated cases where $F \in C^1(\mathbb{R}^N \setminus \{0\})$, while the dual norm has corners (see [14, Example A.1.19] or [16, Appendix A]).

Secondly, we study the Dirichlet problem both in bounded domains with Lipschitz boundary and in exterior domains, as well as for a generic Dirichlet boundary constraint. Moreover, we do not require that the set K is convex and, about its regularity, we just ask for Lipschitz continuity of the boundary, instead of the traditional $C^{2,\alpha}$ much stronger constraint. Lipschitz cannot be further weakened because it is the milder condition that guarantees, e.g., that the trace operator is surjective on the fractional Sobolev space $W^{1-\frac{1}{p}, p}$.

In short, we will be working in an unfriendly setting in the sense that we cannot perform any argument that involves second derivatives of the functions u nor principal curvatures (even in weak sense) of ∂K . To overcome these additional technical difficulties, the proof of Theorem 1.1 strongly relies on the theory of maximal monotone operators in Banach spaces as in the case of $p = 1$, previously studied in [31].

First, we associate a possibly multivalued operator $\mathcal{A}_\varphi : L^p(\Omega) \rightarrow L^{p'}(\Omega)$ to item (b) in Theorem 1.1. In order to ensure that this operator coincides with $\partial \mathcal{F}_{\Omega, \varphi}$, we show that $\mathcal{A}_\varphi \subseteq \partial \mathcal{F}_{\Omega, \varphi}$ and that both are maximal monotone (see Theorem 3.10). The maximal monotonicity of $\mathcal{A}_\varphi$ will be proved by verifying that the range condition $L^{p'}(\Omega) = R(\mathcal{J}_{L^p} + \mathcal{A}_\varphi)$ holds (cf. Proposition 3.8), with $\mathcal{J}_{L^p}$ being the duality mapping.

The latter, in turn, needs an approximation process with a sequence of coercive, monotone and weakly continuous operators defined on $W_{\phi}^{1,p}(\Omega)$. For the continuity, we will approximate the norm F by its Moreau–Yosida approximation while for the coercivity, we will add a p -Laplacian term. Suitable a-priori estimates and the use of the Minty–Browder technique (see Lemma 3.7) will permit to pass to the limit, first in the Yosida regularization and then in the p -Laplacian to finally achieve that the range condition holds.

1.3 Construction of minimizers with extra properties

We also show that there exists one p -capacitary function which is trapped between two explicit solutions of the Euler–Lagrange equations. By translation invariance, we can suppose that $0 \in \Omega$ and, since Ω is bounded, we can find $0 < r_1 < r_2$ such that $\mathcal{W}_{r_1} \subset \Omega \subset \mathcal{W}_{r_2}$, with $\mathcal{W}_r$ being the Wulff shape with radius r ; i.e., the ball of radius r with respect to the dual norm of F . Then, there are three p -capacitary functions u , u_{r_1} , u_{r_2} , minimizers of $\text{Cap}_p^F(\overline{\Omega})$, $\text{Cap}_p^F(\overline{\mathcal{W}_{r_1}})$ and $\text{Cap}_p^F(\overline{\mathcal{W}_{r_2}})$, respectively, such that $u_{r_1} \leq u \leq u_{r_2}$. Moreover, u_{r_1} and u_{r_2} are explicitly given as follows:

Theorem 1.3. *If $1 < p < N$ and $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary, there exists a minimizer u of $\text{Cap}_p^F(\overline{\Omega})$ such that:*

- (a) $u \in L^{p^*}(\mathbb{R}^N \setminus \overline{\Omega})$ is a weak solution of (1.4) with $D = \mathbb{R}^N \setminus \overline{\Omega}$, $p^* := \frac{Np}{N-p}$ and $\nabla u \in L^p(\mathbb{R}^N \setminus \overline{\Omega}; \mathbb{R}^N)$.
 (b) We can find constants $0 < r_1 < r_2$ with $\mathcal{W}_{r_1} \subset \Omega \subset \mathcal{W}_{r_2}$ so that

$$u_{r_1} \leq u \leq u_{r_2} \quad \text{with } u_{r_i} = (F^\circ)^{\frac{p-N}{p-1}}\left(\frac{\cdot}{r_i}\right), \quad i = 1, 2.$$

- (c) If F is required to be strictly convex, then the minimizer is unique.

In order to prove the above result, we need to approximate p -capacitary functions with minimizers of the relative p -capacity with respect to a ball. Recall that, in the anisotropic framework, Wulff shapes play the role of balls. Therefore, for $\overline{\Omega} \subset \mathcal{W}_R$, we consider

$$\text{Cap}_p^F(\overline{\Omega}; \mathcal{W}_R) := \inf \left\{ \int_{\mathcal{W}_R} F^p(\nabla u) : u \in C_0^\infty(\mathcal{W}_R), u \geq 1 \text{ in } \overline{\Omega} \right\},$$

which is also known as the anisotropic p -capacity of the condenser $(\overline{\Omega}; \mathcal{W}_R)$ or the condenser anisotropic p -capacity of the obstacle $\overline{\Omega}$ in the bounded domain $\mathcal{W}_R$. Notice that a minimizer clearly satisfies that $u|_{\overline{\Omega}} \equiv 1$, so we actually work within the annular domain $\Omega_R := \mathcal{W}_R \setminus \overline{\Omega}$, and try to let $R \rightarrow \infty$, as in the classical isotropic setting [25].

If we assume that F is strictly convex, a comparison argument directly yields two barriers $v_{r_1,R}$ (lower) and $v_{r_2,R}$ (upper) into which the minimizer u_R to $\text{Cap}_p^F(\overline{\Omega}; \mathcal{W}_R)$ is trapped. Since u_R are shown to be increasing with respect to R , we can pass to the limit when $R \rightarrow \infty$ and we get the result.

In the case that F is not a strictly convex norm, we need to approximate it with a sequence of strictly convex norms in a uniform way; to obtain uniform bounds on the barriers and then finish the argument as in the strictly convex case. Notice that, under this generality, it would be trickier to get uniqueness, as this does not happen in the extremal case $p = 1$ (see Appendix A for an example with infinitely many minimizers for $p = 1$).

1.4 Regularity of minimizers

We complete the paper with the study of the regularity of minimizers. We show that all minimizers both of the relative p -capacity with respect to a Wulff shape and of the anisotropic p -capacity are Lipschitz continuous, provided that the domain is regular enough in the following sense:

Definition 1.4 (Uniform interior ball condition). *Let $r > 0$. We say that Ω satisfies the $\mathcal{W}_r$ -condition if, for any $x \in \partial\Omega$, there exists $y \in \mathbb{R}^N$ such that*

$$\mathcal{W}_r + y \subseteq \overline{\Omega} \quad \text{and} \quad x \in \partial(\mathcal{W}_r + y).$$

This condition is milder than F -regularity for non-convex domains (see, e.g., the discussion in [15, Lemma 2.8 and Remark 2.9]). In this setting, we conclude:

Theorem 1.5. *Let $r > 0$ and suppose that Ω satisfies the $\mathcal{W}_r$ -condition. Then any minimizer of $\text{Cap}_p^F(\overline{\Omega}; \mathcal{W}_R)$ is Lipschitz continuous. Moreover, any minimizer of the energy functional $\mathcal{F}_{\mathbb{R}^N \setminus \overline{\Omega}, 1}$ is also Lipschitz continuous.*

We point out that this is the best expected regularity since the explicit solutions u_r in Theorem 1.3 are only Lipschitz continuous in the case that F does not have any extra regularity assumption. The proof requires an application of our comparison arguments, which is trickier than for the previous theorem, as we need that the upper and lower barrier coincide on the boundary, in order to exploit a regularity result from [27].

1.5 Geometric and physical meaning of p -capacity

Physically speaking, $\text{Cap}_2(K)$ measures the total electric charge (heat or fluid through a porous medium) flowing into $\mathbb{R}^N \setminus K$ across the boundary ∂K , driven by the field $J = -c\nabla u$, where u is the p -capacitary potential, and c the conductivity. However, the physical laws are empirical and linearity is just a simplifying assumption, hence the next level of complexity should consider flows driven by $J = -c\|\nabla u\|^{p-2}\nabla u$ for $p > 1$, which have already been applied to turbulent flows and deformation plasticity [6, 32]. Then (1.1) can be interpreted as the steady state of such flows.

The thermal analogy drives our geometric intuition [33]: take a cat, whose skin is at a constant temperature (say 1), within a uniform infinite medium with temperature vanishing at infinity. To protect from the cold, cats tend to curl up in a ball to minimize the heat loss, i.e., their capacity. This can be formalized by means of *isocapacitary inequalities*: $\text{Cap}_p(K) \geq \text{Cap}_p(B_r)$, where r is such that the volume of B_r coincides with that of K (see [28]). There are further characterizations of balls as the equality case of Minkowski type inequalities (cf. [1]). Anisotropic generalizations [26] come naturally by considering bodies embedded in non-uniform media.

1.6 Structure of the paper

The paper is organized as follows. We first introduce in Section 2 the basic background material about anisotropies, maximal monotone theory in Banach spaces and Yosida regularization in Hilbert ambients, while section 3 gathers all the approximation arguments needed to prove Theorem 1.1. These include estimates for the Moreau–Yosida approximation of F^p and the Yosida approximation of ∂F^p (Lemma 3.5), as well as a result in the spirit of Minty–Browder (Lemma 3.7) giving a sufficient condition for elements to belong to ∂F^p . Then in Section 4 we introduce the technical machinery of homogeneous Sobolev spaces to extend the characterization of the subdifferential to unbounded exterior domains (Theorem 4.1). Section 5 includes a comparison result (Lemma 5.1) for strictly convex norms; the obtaining of minimizers u_R for the relative p -capacity $\text{Cap}_p^F(\overline{\Omega}; \mathcal{W}_R)$ as limits of minimizers of energy functionals, which involve strictly convex anisotropies approximating a generic norm F (Lemma 5.5); and the proof that u_R are trapped between explicit solutions of the corresponding Euler–Lagrange problem within annular domains (Proposition 5.6). Then the next natural step is to carry over these barrier arguments in bigger and bigger rings to reach Theorem 1.3 as the outer boundary tends to infinity, which stands for the content of Section 6; in turn, Theorem 1.5 is proven in Section 7. Finally, we include an appendix with an explicit example to justify the lack of uniqueness in the case $p = 1$.

2 Notation and background material

2.1 Anisotropies and Wulff shape

A function $F : \mathbb{R}^N \rightarrow [0, \infty)$ is said to be an *anisotropy* if it is convex, positively 1-homogeneous (i.e., $F(\lambda x) = \lambda F(x)$ for all $\lambda > 0$ and all $x \in \mathbb{R}^N$) and coercive. We will always consider additionally that F is even, that is, a norm.

In particular, as all norms in $\mathbb{R}^N$ are equivalent, there exist constants $0 < c \leq C < \infty$ such that

$$c\|\xi\| \leq F(\xi) \leq C\|\xi\|, \quad (2.1)$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^N$.

We define the dual or polar function $F^\circ : \mathbb{R}^N \rightarrow [0, \infty)$ of F by

$$F^\circ(\xi) := \sup\{\xi \cdot \xi^* : \xi^* \in \mathbb{R}^N, F(\xi^*) \leq 1\} = \sup\left\{\frac{\xi \cdot \xi^*}{F(\xi^*)} : \xi^* \in \mathbb{R}^N \setminus \{0\}\right\}$$

for every $\xi \in \mathbb{R}^N$. It can be verified that F° is convex, lower semicontinuous and 1-positively homogeneous. Moreover, (2.1) leads to

$$\frac{1}{C}\|\xi\| \leq F^\circ(\xi) \leq \frac{1}{c}\|\xi\|. \quad (2.2)$$

From the definition of F° one gets a Cauchy–Schwarz-type inequality of the form

$$x \cdot \xi \leq F^\circ(x)F(\xi) \quad \text{for all } x, \xi \in \mathbb{R}^N. \quad (2.3)$$

The Wulff shape $\mathcal{W}_R$ of F is defined by

$$\mathcal{W}_R := B_{F^\circ}(R) := \{\xi^* \in \mathbb{R}^N : F^\circ(\xi^*) < R\}.$$

As we are dealing with even anisotropies, $\mathcal{W}_R$ is a centrally symmetric convex body. We say that F is *crystalline* if, furthermore, $\mathcal{W}_R$ is a convex polytope.

2.2 Maximal monotone operators on Banach spaces

Let X be a reflexive Banach space with dual X' , and denote by $\langle \cdot, \cdot \rangle$ the pairing between X' and X . Let $\mathcal{A} : X \rightarrow 2^{X'}$ be a multivalued operator on X (equivalently, we write $\mathcal{A} \subset X \times X'$ for its graph). Hereafter, $D(\mathcal{A})$ and $R(\mathcal{A})$ mean the domain and range of $\mathcal{A}$, respectively.

Definition 2.1. *The operator $\mathcal{A}$ is said to be monotone if*

$$\langle \tilde{\xi} - \tilde{\eta}, \xi - \eta \rangle \geq 0 \quad \text{for every } (\xi, \tilde{\xi}), (\eta, \tilde{\eta}) \in \mathcal{A}.$$

Moreover, $\mathcal{A}$ is called maximal monotone if there exists no other monotone multivalued map whose graph strictly contains the graph of $\mathcal{A}$.

Let $\Phi : X \rightarrow \mathbb{R} \cup \{\infty\}$ be lower semicontinuous, proper and convex; its subdifferential $\partial\Phi : X \rightarrow 2^{X'}$ is a multivalued operator given as follows:

$$\delta \in \partial\Phi(\zeta) \iff \Phi(\eta) - \Phi(\zeta) \geq \langle \delta, \eta - \zeta \rangle \quad \text{for all } \eta \in X. \quad (2.4)$$

This is a well-known example of a maximal monotone operator from X to X' (cf. [34, Theorem A]).

In turn, the duality mapping $\mathcal{J}_X : X \rightarrow 2^{X'}$ is defined as

$$\mathcal{J}_X(u) = \{v \in X' : \langle u, v \rangle = \|u\|_X^2 = \|v\|_{X'}^2\} \quad \text{for all } u \in X. \quad (2.5)$$

In particular, the duality mapping $\mathcal{J}_{L^p} : L^p(\Omega) \rightarrow L^{p'}(\Omega)$ is single-valued and has the following explicit expression:

$$\mathcal{J}_{L^p}(u)(x) = \|u(x)\|^{p-2}u(x)\|u\|_{L^p}^{2-p} \quad \text{for a.e. } x \in \Omega, \quad (2.6)$$

and for all $u \in L^p(\Omega)$.

This will be a crucial characterization (see [8, Theorem 2.2]):

Theorem 2.2 (Minty [29], Browder [13]). *Let $\mathcal{A} : L^p(\Omega) \rightarrow L^{p'}(\Omega)$ be a monotone operator and $\mathcal{J}_{L^p} : L^p(\Omega) \rightarrow L^{p'}(\Omega)$ the duality mapping of L^p . Then $\mathcal{A}$ is maximal monotone if, and only if, $R(\mathcal{A} + \mathcal{J}_{L^p}) = L^{p'}(\Omega)$.*

Therefore, we need results that ensure that the above range condition holds in order to deduce that an operator is maximal monotone. In particular, we will apply the following statement (see [22, Corollary 1.8]).

Theorem 2.3 (Hartman and Stampacchia [21]). *Let $\mathfrak{K} \neq \emptyset$ be a closed convex subset of a reflexive Banach space X , and let $\mathcal{B} : \mathfrak{K} \rightarrow X'$ be monotone, coercive and weakly continuous on finite-dimensional subspaces. Then $\mathcal{B}$ is surjective.*

Typically the literature refers to the hypotheses in the above theorem as the classical Leray–Lions assumptions (cf. [24]).

2.3 Yosida approximation of maps in Hilbert spaces

Consider I the identity map on a Hilbert space H . Then the *resolvent* of $\mathcal{A} : H \rightarrow 2^H$ is given by

$$J_\lambda = J_{\mathcal{A},\lambda} := (I + \lambda \mathcal{A})^{-1} \quad \text{for } \lambda > 0.$$

It holds that the resolvent of a monotone operator $\mathcal{A}$ is a single-valued non-expansive map from $R(I + \lambda \mathcal{A})$ to H (cf. [7, Proposition 3.5.3]).

The *Yosida approximation* of a maximal monotone map $\mathcal{A}$ is defined as

$$\mathcal{A}_\lambda := \frac{I - J_\lambda}{\lambda},$$

and hence it satisfies

$$\mathcal{A}_\lambda(\xi) \in \mathcal{A}(J_\lambda(\xi)) \quad \text{for all } \xi \in H. \quad (2.7)$$

In addition, we have the following properties (see [7, Theorem 3.5.9]):

Lemma 2.4. *The following statements hold:*

- (a) $\mathcal{A}_\lambda$ is a single-valued maximal monotone map which is Lipschitz with constant $\frac{1}{\lambda}$.
- (b) $\mathcal{A}_\lambda(\xi)$ converges to $\mathcal{A}^0(\xi)$ as $\lambda \rightarrow 0$, where $\mathcal{A}^0$ denotes the minimal section of $\mathcal{A}$ given by

$$\mathcal{A}^0(\xi) := \{\eta \in \mathcal{A}(\xi) : \eta \text{ has minimal norm in } \mathcal{A}(\xi)\}.$$

Recall that given a proper, convex and lower semicontinuous function $g : \mathbb{R}^N \rightarrow [0, \infty)$, for every $\lambda > 0$ the *Moreau–Yosida approximation* of g is defined by

$$g_\lambda(\xi) := \min_{\eta \in \mathbb{R}^N} \left\{ \frac{1}{2\lambda} \|\eta - \xi\|^2 + g(\eta) \right\} \leq g(\xi) \quad (2.8)$$

for all $\xi \in \mathbb{R}^N$. The minimum in (2.8) is attained at $J_{\partial g, \lambda}(\xi)$, where ∂g denotes the subdifferential of g . Furthermore, g_λ is convex and Fréchet differentiable with gradient

$$\nabla g_\lambda = (\partial g)_\lambda. \quad (2.9)$$

Thus the Yosida approximation of the subdifferential is equal to the gradient of the Moreau–Yosida approximation. We also have that

$$g_\lambda(\xi) \xrightarrow{\lambda \searrow 0} g(\xi) \quad \text{for every } \xi \in \mathbb{R}^N$$

(see [12], [37, Proposition 1.8] or [7, Theorem 6.5.7]).

In case that $g_\lambda(0) = 0$, the convexity of g_λ implies (cf. [37, Proposition 7.4])

$$\nabla g_\lambda(\xi) \cdot \xi \geq g_\lambda(\xi) \quad \text{for all } \xi \in D(g_\lambda). \quad (2.10)$$

3 Characterization of the subdifferential on bounded domains

3.1 An auxiliary operator containing the subdifferential

Hereafter Ω will denote an open bounded subset of $\mathbb{R}^N$ with Lipschitz-continuous boundary $\partial\Omega$. In order to study the minimizers of the functional $\mathcal{F}_{\Omega, \varphi}$ defined by (1.2), the well-known equivalence (1.3) tells that one needs to characterize the subdifferential of $\mathcal{F}_{\Omega, \varphi}$ in order to understand the meaning of $0 \in \partial \mathcal{F}_{\Omega, \varphi}(u)$.

Recall that the subdifferential $\partial\mathcal{F}_{\Omega,\varphi}$ of $\mathcal{F}_{\Omega,\varphi}$ is the multivalued operator from $L^p(\Omega)$ to $L^{p'}(\Omega)$ given by

$$v \in \partial\mathcal{F}_{\Omega,\varphi}(u) \iff \mathcal{F}_{\Omega,\varphi}(w) - \mathcal{F}_{\Omega,\varphi}(u) \geq \int_{\Omega} v(w - u) \text{ for all } w \in L^p(\Omega).$$

Remark 3.1. It is a well-known result (see, e.g., [8, Proposition 1.6]) that the effective domain $\mathcal{D}(\partial\mathcal{F}_{\Omega,\varphi})$ is dense in $\mathcal{D}(\mathcal{F}_{\Omega,\varphi}) = W_{\varphi}^{1,p}(\Omega)$, and consequently in $L^p(\Omega)$. Recall that u belongs to the effective domain of $\mathcal{A}$ if $\|\mathcal{A}(u)\|_{X'} < \infty$.

To get the sought characterization of $\partial\mathcal{F}_{\Omega,\varphi}$, we first introduce an auxiliary operator $\mathcal{A}_{\varphi}$.

Definition 3.2. We define $\mathcal{A}_{\varphi} \subset L^p(\Omega) \times L^{p'}(\Omega)$ as follows: $(u, v) \in \mathcal{A}_{\varphi}$ if

(A1) $u \in W_{\varphi}^{1,p}(\Omega)$,

(A2) $v \in L^{p'}(\Omega)$,

(A3) there exists $z \in \partial F^p(\nabla u)$ such that $v = -\operatorname{div}(z)$ weakly, meaning that

$$\int_{\Omega} vw = \int_{\Omega} z \cdot \nabla w \text{ for every } w \in W_0^{1,p}(\Omega).$$

The goal is to show that both operators $\mathcal{A}_{\varphi}$ and $\partial\mathcal{F}_{\Omega,\varphi}$ coincide (cf. Theorem 3.10). Once we have established this result, we can conclude that $0 \in \partial\mathcal{F}_{\Omega,\varphi}(u)$ is equivalent to u satisfying the following:

$$\begin{cases} \operatorname{div}(F^{p-1}(\nabla u)z) = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \quad (3.1)$$

in a weak sense, where $z \in \partial F(\nabla u)$ and we have used the chainrule for subdifferentials (see [9, Corollary 16.72] or [17, Theorem 2.3.9 (ii)]).

The first inclusion is quite straightforward:

Lemma 3.3. We have $\mathcal{A}_{\varphi} \subseteq \partial\mathcal{F}_{\Omega,\varphi}$. In particular, $\mathcal{A}_{\varphi}$ is monotone.

Proof. Let $(u, v) \in \mathcal{A}_{\varphi}$ and z as in (A3) above. In particular, by definition of subdifferential

$$z \in \partial F^p(\nabla u) \implies \int_{\Omega} z \cdot \nabla(w - u) \leq \int_{\Omega} F^p(\nabla w) - \int_{\Omega} F^p(\nabla u)$$

for every $w \in W_{\varphi}^{1,p}(\Omega)$. Consequently,

$$\int_{\Omega} v(w - u) = \int_{\Omega} z \cdot \nabla(w - u) \leq \mathcal{F}_{\Omega,\varphi}(w) - \mathcal{F}_{\Omega,\varphi}(u)$$

for every $v \in L^{p'}(\Omega)$. Hence the statement follows by definition of $\partial\mathcal{F}_{\Omega,\varphi}$. $\square$

3.2 Properties of the subdifferential of F^p

To establish the remaining inclusion, we need some properties of ∂F^p that will be used repeatedly. We start by writing the elements of ∂F^p in terms of those in ∂F because, as F is positively 1-homogeneous, its subdifferential has a useful well-known characterization (obtained by choosing $\eta = 0$ and $\eta = 2\zeta$ in (2.4)):

$$\delta \in \partial F(\zeta) \iff \begin{cases} \delta \cdot \zeta = F(\zeta), \\ \delta \cdot \eta \leq F(\eta) \end{cases} \text{ for every } \eta \in \mathbb{R}^N. \quad (3.2)$$

Notice that the above inequality is equivalent to $F^{\circ}(\delta) \leq 1$. In particular, for $\eta = \frac{\delta}{\|\delta\|}$ it follows by means of (2.1) that

$$\|\delta\| \leq C \text{ for all } \delta \in \partial F(\zeta). \quad (3.3)$$

The latter bound, combined with the chainrule for subdifferentials and again (2.1), leads directly to the following lemma.

Lemma 3.4. Every $z \in \partial F^p(\xi)$ can be written as

$$z = pF^{p-1}(\xi)\tilde{z} \quad \text{with } \tilde{z} \in \partial F(\xi)$$

and we can estimate

$$\|z\| \leq pC^p \|\xi\|^{p-1}.$$

Next we see that the same estimate holds for the corresponding Yosida approximation.

Lemma 3.5. The Moreau–Yosida approximation of F^p and the Yosida approximation of ∂F^p satisfy:

(a) If $1 < p < 2$, there exists a positive constant $K_\lambda = K_\lambda(c, p, \lambda)$ with $\lim_{\lambda \searrow 0} K_\lambda = 0$ so that

$$(F^p)_\lambda(\xi) \geq \frac{c^p}{2} \|\xi\|^p \quad \text{if } \|\xi\| \geq K_\lambda.$$

(b) $\|(\partial F^p)_\lambda(\xi)\| = \|(\nabla(F^p)_\lambda)(\xi)\| \leq pC^p \|\xi\|^{p-1}$.

Here c and C are the constants coming from (2.1).

Proof. (a) Note that, using (2.1) in the definition given by (2.8), one has

$$(F^p)_\lambda(\xi) \geq c^p \min_{\eta \in \mathbb{R}^N} \left\{ \frac{1}{2c^p\lambda} \|\eta - \xi\|^2 + \|\eta\|^p \right\}.$$

Hence we are done in the case that $\|\eta\|^p \geq \frac{\|\xi\|^p}{2}$. Otherwise, if $\|\eta\| < \frac{\|\xi\|}{2^{1/p}}$, by the reverse triangle inequality we get

$$\frac{1}{2c^p\lambda} \|\eta - \xi\|^2 \geq \frac{1}{2c^p\lambda} (\|\xi\| - \|\eta\|)^2 > \frac{\|\xi\|^2}{2c^p\lambda} (1 - 2^{-1/p})^2 > \frac{\|\xi\|^p}{2},$$

where the last inequality holds, if and only if,

$$\|\xi\| > \left(\frac{\lambda c^p}{(1 - 2^{-1/p})^2} \right)^{\frac{1}{2-p}} =: K_\lambda.$$

(b) By (2.7) with $\mathcal{A} = \partial F^p$ and the chainrule for subdifferentials, we can write

$$(\partial F^p)_\lambda(\xi) \in \partial F^p(J_{p,\lambda}(\xi)) = pF^{p-1}(J_{p,\lambda}(\xi))\partial F(J_{p,\lambda}(\xi)), \quad (3.4)$$

where $J_{p,\lambda}(\xi) = J_{\partial F^p, \lambda}(\xi)$ satisfies

$$\frac{1}{2\lambda} \|J_{p,\lambda}(\xi) - \xi\|^2 + F^p(J_{p,\lambda}(\xi)) \leq \frac{1}{2\lambda} \|\eta - \xi\|^2 + F^p(\eta) \quad \text{for all } \eta \in \mathbb{R}^N.$$

In particular, for $\eta = \xi$, we get

$$F^p(J_{p,\lambda}(\xi)) \leq F^p(\xi).$$

Raising the inequality to $\frac{1}{p}$ and using (2.1) leads to

$$F(J_{p,\lambda}(\xi)) \leq C\|\xi\|.$$

This, combined with (2.9) (with $g = F^p$) and (3.4), implies

$$\|(\partial F^p)_\lambda(\xi)\| = \|(\nabla(F^p)_\lambda)(\xi)\| \leq pF^{p-1}(J_{p,\lambda}(\xi))\|\delta\| \leq pC^p \|\xi\|^{p-1}$$

for every $\delta \in \partial F(J_{p,\lambda}(\xi))$. Note that we used (2.1) and (3.3) for the last inequality. $\square$

The existence of minimizers follows from the above properties.

Proposition 3.6. There exists at least one function u which is a minimizer of the functional $\mathcal{F}_{\Omega, \varphi}$ defined in (1.2) for any domain $\Omega \subset \mathbb{R}^N$. Moreover, $\mathcal{F}_{\Omega, \varphi}$ is lower semicontinuous with respect to the weak convergence in $W^{1,p}(\Omega)$.

Proof. As by (2.1) the functional $\mathcal{F}_{\Omega, \varphi}$ is bounded from below, there exists a minimizing sequence u_n . By the definition of the functional and the Poincaré inequality, $\{u_n\}_{n \in \mathbb{N}}$ is uniformly bounded in $W^{1,p}(\Omega)$ and hence it converges subsequentially and weakly in $W^{1,p}(\Omega)$ to $u \in W^{1,p}(\Omega)$.

Now let $z \in \partial F^p(\nabla u)$. Then $z \in L^{p'}(\Omega)$ and $z = pF^{p-1}(\nabla u)\delta$ with $\delta \in \partial F(\nabla u)$ by Lemma 3.4. By (3.2) and the weak convergence, we have

$$\begin{aligned} \int_{\Omega} F^p(\nabla u) &= \frac{1}{p} \int_{\Omega} z \cdot \nabla u = \frac{1}{p} \lim_{n \rightarrow \infty} \int_{\Omega} z \cdot \nabla u_n \leq \lim_{n \rightarrow \infty} \int_{\Omega} F^{p-1}(\nabla u) F(\nabla u_n) \\ &\leq \left(\int_{\Omega} F^p(\nabla u) \right)^{\frac{1}{p'}} \liminf_{n \rightarrow \infty} \left(\int_{\Omega} F^p(\nabla u_n) \right)^{\frac{1}{p}}, \end{aligned}$$

where the latter follows by application of Hölder's inequality. Thus

$$\int_{\Omega} F^p(\nabla u) \leq \liminf_{n \rightarrow \infty} \int_{\Omega} F^p(\nabla u_n),$$

as desired. $\square$

It remains to show that such a minimizer is a weak solution of (3.1), which will be a direct consequence of the characterization of $\partial \mathcal{F}_{\Omega, \varphi}$.

Throughout this paper, we will perform several approximation arguments whose outcome is a suitable weak limit $z \in L^{p'}(\Omega)$. The following result, in the spirit of the *Minty–Browder technique* (see [30]), gives a practical criterion to guarantee that z is indeed in the appropriate subdifferential. For brevity, here and in the sequel we will use the notation

$$\Psi(\zeta) = \|\zeta\|^{p-2}\zeta. \quad (3.5)$$

Lemma 3.7. *Given $u \in W^{1,p}(\Omega)$, $z \in L^{p'}(\Omega)$ and $\alpha \in \mathbb{R}$, suppose that the following inequality holds for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^N$:*

$$(z(x) - z' - \alpha\Psi(\xi)) \cdot (\nabla u(x) - \xi) \geq 0 \quad \text{for every } z' \in \partial F^p(\xi). \quad (3.6)$$

Then one can conclude that $z \in \partial F^p(\nabla u) + \alpha\Psi(\nabla u)$. The same conclusion holds if we only require that (3.6) holds for $z' = (\partial F^p)^0(\xi)$.

Proof. Notice that it is enough to prove the second claim, but we keep the stronger hypothesis (3.6) because of the applications afterwards. Therefore, there exists a null set $N_1 \subset \mathbb{R}^N$ such that

$$(z(x) - (\partial F^p)^0(\xi) - \alpha\Psi(\xi)) \cdot (\nabla u(x) - \xi) \geq 0$$

for every $x \in \Omega \setminus N_1$ and $\xi \in \mathbb{R}^N \setminus N_x$, where $N_x \subset \mathbb{R}^N$ is null. Let $x \in \Omega \setminus N_1$. Then, for a.e. $0 \neq \eta \in \mathbb{R}^N$, the set $\{t : \nabla u(x) + t\eta \in N_x\}$ is null and we may take a sequence $\{\varepsilon_n\}_{n \in \mathbb{N}}$ with $\varepsilon_n \searrow 0$ and $\{\nabla u(x) + \varepsilon_n\eta\}_{n \in \mathbb{N}} \subset \mathbb{R}^N \setminus N_x$. If we are in this case, we can write

$$\begin{aligned} &(z(x) - (\partial F^p)^0(\nabla u(x) + \varepsilon_n\eta) - \alpha\Psi(\nabla u(x) + \varepsilon_n\eta)) \cdot \eta \\ &= \frac{1}{\varepsilon_n} (z(x) - (\partial F^p)^0(\nabla u(x) + \varepsilon_n\eta) - \alpha\Psi(\nabla u(x) + \varepsilon_n\eta)) \cdot (\varepsilon_n\eta) \leq 0 \end{aligned}$$

for every $n \in \mathbb{N}$.

Since $(\partial F^p)^0(\nabla u(x) + \varepsilon_n\eta)$ is bounded we may suppose that, up to a subsequence, $(\partial F^p)^0(\nabla u(x) + \varepsilon_n\eta) \rightarrow \tilde{z} \in \partial F^p(\nabla u(x))$ as $n \rightarrow \infty$ (recall that ∂F^p is closed since it is maximal monotone). Hence, by the continuity of Ψ , taking limits in the previous equation we get that

$$(z(x) - \tilde{z} - \alpha\Psi(\nabla u(x))) \cdot \eta \leq 0.$$

Since this holds for a.e. $\eta \in \mathbb{R}^N$, we conclude that $z(x) - \alpha\Psi(\nabla u(x)) = \tilde{z} \in \partial F^p(\nabla u(x))$. $\square$

3.3 A range condition for the auxiliary operator

To prove Theorem 3.10, we need to show that the operator $\mathcal{A}_{\varphi}$ introduced in Definition 3.2 satisfies a range condition, which involves the duality mapping of $L^p(\Omega)$ given by (2.6).

Proposition 3.8. *If $\varphi \in W^{1-\frac{1}{p},p}(\partial\Omega)$, then $L^{p'}(\Omega) = R(\mathcal{J}_{L^p} + \mathcal{A}_\varphi)$.*

Proof. Given $f \in L^{p'}(\Omega)$, we aim to prove that there exists $u \in W_\varphi^{1,p}(\Omega)$ such that $(u, f - \mathcal{J}_{L^p} u) \in \mathcal{A}_\varphi$. With this purpose, let us consider

$$\mathcal{A}_\varphi^{n,m} : W_\varphi^{1,p}(\Omega) \rightarrow (W^{1,p}(\Omega))',$$

which is a sequence of operators approximating $\mathcal{A}_\varphi$ and defined as follows:

$$v = \mathcal{A}_\varphi^{n,m}(u) = -\operatorname{div}\left((\nabla(F^p)_{\frac{1}{n}})(\nabla u) + \frac{1}{m}\|\nabla u\|^{p-2}\nabla u\right) \quad (3.7)$$

in the weak sense. In particular, with the notation from (3.5) it holds

$$\int_{\Omega} vw = \int_{\Omega} (\nabla(F^p)_{\frac{1}{n}})(\nabla u) \cdot \nabla w + \frac{1}{m} \int_{\Omega} \Psi(\nabla u) \cdot \nabla w \quad \text{for every } w \in W_0^{1,p}(\Omega).$$

Claim 1. *The operators $\mathcal{A}_\varphi^{n,m} + \mathcal{J}_{L^p}$ satisfy the hypotheses of Theorem 2.3.*

Proof of Claim 1. First, $\mathcal{J}_{L^p}$ is uniformly continuous on every bounded subset of $L^p(\Omega)$ (cf. [8, Theorem 1.2]) and it is monotone because it coincides with the subdifferential of $\frac{1}{2}\|\cdot\|_{L^p}^2$ (cf. [8, p.7]).

Since $\mathcal{A}_\varphi^{n,m}(u) = -\operatorname{div}((\nabla(F^p)_{\frac{1}{n}})(\nabla u)) - \frac{1}{m}\Delta_p u$, monotonicity and continuity for the first term follow from (2.9) and Lemma 2.4 (a). In turn, the p -Laplacian is known (cf. [38, Lemma 2.2]) to be Hölder continuous on $W^{1,p}(\Omega)$, and the monotonicity follows from the inequalities (see [38, (2.10)])

$$(\Psi(\xi) - \Psi(\eta)) \cdot (\xi - \eta) \geq \begin{cases} c_p \|\xi - \eta\|^p & \text{if } p \geq 2, \\ c_p \frac{\|\xi - \eta\|^2}{(\|\xi\| + \|\eta\|)^{2-p}} & \text{if } 1 < p < 2, \end{cases}$$

which hold for every $\xi, \eta \in \mathbb{R}^N$. This, combined with Poincaré inequality, easily leads to coercivity when $p \geq 2$; in fact, for any $u, w \in W_\varphi^{1,p}(\Omega)$ we get

$$\frac{\langle \Psi(\nabla u) - \Psi(\nabla w), \nabla u - \nabla w \rangle}{\|u - w\|_{W^{1,p}}} \geq C_{p,\Omega} \|\nabla u - \nabla w\|_{L^p}^{p-1} \rightarrow \infty \quad \text{as } \|\nabla u\|_{L^p} \rightarrow \infty.$$

To establish the coercivity of $\mathcal{A}_\varphi^{n,m} + \mathcal{J}_{L^p}$ in the remaining case $1 < p < 2$, again by Poincaré inequality, it is enough to show that, given a fixed $w \in W_\varphi^{1,p}(\Omega)$, it holds

$$\frac{\langle \Psi(\nabla u) - \Psi(\nabla w), \nabla u - \nabla w \rangle}{\|\nabla u - \nabla w\|_{L^p}} \rightarrow \infty \quad \text{as } \|\nabla u\|_{L^p} \rightarrow \infty$$

for any $u \in W_\varphi^{1,p}(\Omega)$. With this aim, the triangle inequality leads to

$$\frac{\langle \Psi(\nabla u), \nabla u \rangle + \langle \Psi(\nabla w), \nabla w \rangle}{\|\nabla u - \nabla w\|_{L^p}} \geq \frac{\|\nabla u\|_{L^p}^p + \|\nabla w\|_{L^p}^p}{\|\nabla u\|_{L^p} + \|\nabla w\|_{L^p}},$$

which tends to ∞ as $\|\nabla u\|_{L^p} \rightarrow \infty$. It remains to check that the crossed terms keep bounded in the limit. Indeed, we can assume that $\|\nabla u\|_{L^p} > \|\nabla w\|_{L^p}$, and then

$$\begin{aligned} \left| \frac{\langle \Psi(\nabla u), \nabla w \rangle + \langle \Psi(\nabla w), \nabla u \rangle}{\|\nabla u - \nabla w\|_{L^p}} \right| &\leq \frac{\int_{\Omega} \|\nabla u\|^{p-1} \|\nabla w\|}{\|\nabla u - \nabla w\|_{L^p}} + \frac{\int_{\Omega} \|\nabla w\|^{p-1} \|\nabla u\|}{\|\nabla u - \nabla w\|_{L^p}} \\ &\leq \frac{\|\nabla u\|_{L^p}^{p-1} \|\nabla w\|_{L^p}}{\|\nabla u\|_{L^p} - \|\nabla w\|_{L^p}} + \frac{\|\nabla w\|_{L^p}^{p-1} \|\nabla u\|_{L^p}}{\|\nabla u\|_{L^p} - \|\nabla w\|_{L^p}}, \end{aligned}$$

where we applied Hölder and the reverse triangle inequality. Notice that, as $p \in (1, 2)$, the right-hand side tends to 0 as $\|\nabla w\|_{L^p}^{p-1}$ as $\|\nabla u\|_{L^p} \rightarrow \infty$, which is a bounded limit, as desired. $\square$

Now, since $W_\varphi^{1,p}(\Omega)$ is closed and convex in $W^{1,p}(\Omega)$, by Theorem 2.3 we deduce that

$$L^{p'}(\Omega) \subset (W^{1,p}(\Omega))' \subseteq R(\mathcal{J}_{L^p} + \mathcal{A}_\varphi^{n,m}).$$

Thus there exist $u_{n,m} \in W_\varphi^{1,p}(\Omega)$ such that

$$\mathcal{A}_\varphi^{n,m}(u_{n,m}) = f - \mathcal{J}_{L^p}(u_{n,m}) \in L^{p'}(\Omega). \quad (3.8)$$

Therefore,

$$\int_{\Omega} (f - \partial_{L^p}(u_{n,m}))w = \int_{\Omega} (\nabla(F^p)^{\frac{1}{n}})(\nabla u_{n,m}) \cdot \nabla w + \frac{1}{m} \int_{\Omega} \Psi(\nabla u_{n,m}) \cdot \nabla w \quad (3.9)$$

for every $w \in W_0^{1,p}(\Omega)$.

Claim 2. *The sequence $\{u_{n,m}\}_{n \in \mathbb{N}}$ is uniformly bounded in $W^{1,p}(\Omega)$.*

Proof of Claim 2. Given $w \in W_0^{1,p}(\Omega)$, letting $w - u_{n,m}$ as a test function in (3.9) yields

$$\int_{\Omega} (f - \partial_{L^p}(u_{n,m}))(w - u_{n,m}) = \int_{\Omega} (\nabla(F^p)^{\frac{1}{n}})(\nabla u_{n,m}) \cdot (\nabla w - \nabla u_{n,m}) + \frac{1}{m} \int_{\Omega} \Psi(\nabla u_{n,m}) \cdot \nabla w - \frac{1}{m} \|\nabla u_{n,m}\|_{L^p}^p.$$

First, the integral on the left-hand side, by means of the definition of the duality mapping in (2.5), can be written as

$$\int_{\Omega} (f - \partial_{L^p}(u_{n,m}))(w - u_{n,m}) = \int_{\Omega} f(w - u_{n,m}) - \int_{\Omega} \partial_{L^p}(u_{n,m})w + \|u_{n,m}\|_{L^p}^2.$$

Rearranging terms, we obtain

$$\begin{aligned} & \|u_{n,m}\|_{L^p}^2 + \frac{1}{m} \|\nabla u_{n,m}\|_{L^p}^p + \int_{\Omega} (\nabla(F^p)^{\frac{1}{n}})(\nabla u_{n,m}) \cdot \nabla u_{n,m} \\ &= \int_{\Omega} f(u_{n,m} - w) + \int_{\Omega} \partial_{L^p}(u_{n,m})w + \int_{\Omega} (\nabla(F^p)^{\frac{1}{n}})(\nabla u_{n,m}) \cdot \nabla w + \frac{1}{m} \int_{\Omega} \Psi(\nabla u_{n,m}) \cdot \nabla w. \end{aligned}$$

Now, by (2.10) for $g = F^p$ and by Hölder's inequality, we can estimate

$$\begin{aligned} \|u_{n,m}\|_{L^p}^2 + \frac{\|\nabla u_{n,m}\|_{L^p}^p}{m} &\leq \|f\|_{L^{p'}}(\|w\|_{L^p} + \|u_{n,m}\|_{L^p}) + \|\partial_{L^p}(u_{n,m})\|_{L^{p'}}\|w\|_{L^p} \\ &\quad + \int_{\Omega} (\nabla(F^p)^{\frac{1}{n}})(\nabla u_{n,m}) \cdot \nabla w + \frac{\|\nabla u_{n,m}\|_{L^p}^{p-1}}{m} \|\nabla w\|_{L^p}. \end{aligned}$$

On the other hand, by Cauchy–Schwarz for the inner product in $\mathbb{R}^N$, Lemma 3.5 (b) and Young's inequality $ab \leq \varepsilon a^{p'} + C(\varepsilon)b^p$, where $C(\varepsilon) = \frac{(\varepsilon p')^{1-p}}{p}$ with $\varepsilon = \frac{1}{2mpC^p}$,

$$\begin{aligned} \int_{\Omega} (\nabla(F^p)^{\frac{1}{n}})(\nabla u_{n,m}) \cdot \nabla w &\leq pC^p \int_{\Omega} \|\nabla u_{n,m}\|^{p-1} \|\nabla w\| \\ &\leq \frac{1}{2m} \int_{\Omega} \|\nabla u_{n,m}\|^p + m^{p-1} \mathcal{C} \int_{\Omega} \|\nabla w\|^p, \end{aligned}$$

where $\mathcal{C}$ denotes hereafter any constant which depends only on p and the constants from (2.1), whose concrete meaning may change from line to line (and we use $\tilde{\mathcal{C}}$ whenever two of such constants appear on the same line).

Bringing these bounds together, using $\|\partial_{L^p}(u_{n,m})\|_{L^{p'}} = \|u_{n,m}\|_{L^p}$, Cauchy's inequality $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$ and Young's inequality with $\varepsilon = \frac{1}{4}$, we get

$$\begin{aligned} \|u_{n,m}\|_{L^p}^2 + \frac{1}{2m} \|\nabla u_{n,m}\|_{L^p}^p &\leq \|f\|_{L^{p'}}\|w\|_{L^p} + \frac{1}{2} \|u_{n,m}\|_{L^p}^2 + \frac{1}{2} (\|f\|_{L^{p'}} + \|w\|_{L^p})^2 \\ &\quad + m^{p-1} \mathcal{C} \|\nabla w\|_{L^p}^p + \frac{1}{4m} \|\nabla u_{n,m}\|_{L^p}^p + \left(\frac{4}{p'}\right)^{p-1} \frac{\|\nabla w\|_{L^p}^p}{pm}. \end{aligned}$$

In short, we reach

$$\frac{1}{2} \|u_{n,m}\|_{L^p}^2 + \frac{1}{4m} \|\nabla u_{n,m}\|_{L^p}^p \leq \frac{3}{2} (\|f\|_{L^{p'}}^2 + \|w\|_{L^p}^2) + (m^{p-1} + 1) \mathcal{C} \|\nabla w\|_{L^p}^p,$$

from which the claim follows. $\square$

Accordingly, up to a subsequence, we may assume that $u_{n,m}$ converges in $L^p(\Omega)$ and a.e. to some $u_m \in L^p(\Omega)$ as $n \rightarrow \infty$; and $\nabla u_{n,m} \rightharpoonup \nabla u_m$ weakly in $L^p(\Omega)$ as $n \rightarrow \infty$. The continuity of the trace operator with respect to the weak convergence in $W^{1,p}(\Omega)$ (see [23, Corollary 18.4]) guarantees that $u_m \in W_0^{1,p}(\Omega)$.

Claim 3. *There exists $z_m \in \partial F^p(\nabla u_m)$ such that*

$$\int_{\Omega} (f - \mathcal{J}_{L^p}(u_m))\psi = \int_{\Omega} z_m \cdot \nabla \psi + \frac{1}{m} \int_{\Omega} \Psi(\nabla u_m) \cdot \nabla \psi \quad (3.10)$$

for every $\psi \in W_0^{1,p}(\Omega)$.

Proof of Claim 3. First, Lemma 3.5 (b) ensures that

$$\|(\nabla(F^p)_{\frac{1}{n}})(\nabla u_{n,m})\| \leq pC^p \|\nabla u_{n,m}\|^{p-1},$$

thus

$$\{(\nabla(F^p)_{\frac{1}{n}})(\nabla u_{n,m})\}_{n \in \mathbb{N}} \text{ is bounded in } L^{p'}(\Omega).$$

Consequently, we may assume that

$$(\nabla(F^p)_{\frac{1}{n}})(\nabla u_{n,m}) \rightharpoonup \bar{z}_m \text{ weakly in } L^{p'}(\Omega) \text{ as } n \rightarrow \infty. \quad (3.11)$$

In turn, we also have that $\|\Psi(\nabla u_{n,m})\| = \|\nabla u_{n,m}\|^{p-1}$, and hence

$$\Psi(\nabla u_{n,m}) \rightharpoonup g_m \text{ weakly in } L^{p'}(\Omega) \text{ as } n \rightarrow \infty. \quad (3.12)$$

Therefore, taking $\psi \in W_0^{1,p}(\Omega)$ as test function in (3.9) and letting $n \rightarrow \infty$, we get

$$\int_{\Omega} (f - \mathcal{J}_{L^p}(u_m))\psi = \int_{\Omega} \bar{z}_m \cdot \nabla \psi + \frac{1}{m} \int_{\Omega} g_m \cdot \nabla \psi; \quad (3.13)$$

in particular,

$$-\operatorname{div}\left(\bar{z}_m + \frac{1}{m}g_m\right) = f - \mathcal{J}_{L^p}(u_m) \text{ in the weak sense}$$

and

$$\mathcal{A}_{\phi}^{n,m}(u_{n,m}) \rightharpoonup -\operatorname{div}\left(\bar{z}_m + \frac{1}{m}g_m\right) \text{ weakly in } L^{p'}(\Omega) \text{ as } n \rightarrow \infty, \quad (3.14)$$

which comes from (3.8).

Moreover, since for a fixed $\xi \in \mathbb{R}^N$, $(\nabla(F^p)_{\frac{1}{n}})(\xi) = (\partial F^p)_{\frac{1}{n}}(\xi)$ tends to $(\partial F^p)^0(\xi)$ as $n \rightarrow \infty$ (by Lemma 2.4 (b)) and $\|(\nabla(F^p)_{\frac{1}{n}})(\xi)\| \leq pCF^{p-1}(\xi)$ (by Lemma 3.5 (b)), by the dominated convergence theorem, we have that

$$(\nabla(F^p)_{\frac{1}{n}})(\nabla g) \rightarrow (\partial F^p)^0(\nabla g) \text{ in } L^{p'}(\Omega) \text{ as } n \rightarrow \infty \quad (3.15)$$

for every $g \in W^{1,p}(\Omega)$.

Next we aim to show that $\bar{z}_m + \frac{1}{m}g_m \in \partial F^p(\nabla u_m) + \frac{1}{m}\Psi(\nabla u_m)$ by means of Lemma 3.7. With this aim, let $\xi \in \mathbb{R}^N$ and take $\omega : \Omega \rightarrow \mathbb{R}$ defined by $\omega(\eta) := \xi \cdot \eta$ so that $\nabla \omega = \xi$. Then, letting $\phi \in C_0^\infty(\Omega)$ such that $\phi \geq 0$, by the monotonicity of $\nabla(F^p)_{\frac{1}{n}} = (\partial F^p)_{\frac{1}{n}}$ and Ψ we have, for every $n \in \mathbb{N}$,

$$\begin{aligned} 0 &\leq \langle (\nabla(F^p)_{\frac{1}{n}})(\nabla u_{n,m}) - (\nabla(F^p)_{\frac{1}{n}})(\nabla \omega), \nabla(u_{n,m} - \omega)\phi \rangle + \left\langle \frac{\Psi(\nabla u_{n,m}) - \Psi(\nabla \omega)}{m}, \nabla(u_{n,m} - \omega)\phi \right\rangle \\ &= \int_{\Omega} \mathcal{A}_{\phi}^{n,m}(u_{n,m})(u_{n,m} - \omega)\phi - \int_{\Omega} \left((\nabla(F^p)_{\frac{1}{n}})(\nabla u_{n,m}) + \frac{1}{m}\Psi(\nabla u_{n,m}) \right) \cdot \nabla \phi(u_{n,m} - \omega) \\ &\quad - \int_{\Omega} \left((\nabla(F^p)_{\frac{1}{n}})(\nabla \omega) + \frac{1}{m}\Psi(\nabla \omega) \right) \cdot (\nabla u_{n,m} - \nabla \omega)\phi, \end{aligned}$$

thus, letting $n \rightarrow \infty$, by (3.11), (3.12), (3.14) and (3.15), we get

$$\begin{aligned} 0 &\leq - \int_{\Omega} \operatorname{div}\left(\bar{z}_m + \frac{1}{m}g_m\right)(u_m - \omega)\phi - \int_{\Omega} \left(\bar{z}_m + \frac{1}{m}g_m\right) \cdot \nabla \phi(u_m - \omega) \\ &\quad - \int_{\Omega} \left((\partial F^p)^0(\nabla \omega) + \frac{1}{m}\Psi(\nabla \omega) \right) \cdot (\nabla u_m - \nabla \omega)\phi \\ &= \int_{\Omega} \left(\bar{z}_m + \frac{1}{m}g_m - (\partial F^p)^0(\nabla \omega) - \frac{1}{m}\Psi(\nabla \omega) \right) \cdot (\nabla u_m - \nabla \omega)\phi. \end{aligned}$$

Hence, since $\phi \in C_0^\infty(\Omega)$ is arbitrary and $\nabla\omega = \xi$, we conclude (3.6) for $z' = (\partial F^p)^0(\xi)$ and $\alpha = \frac{1}{m}$. Then

$$\bar{z}_m + \frac{1}{m}g_m \in \partial F^p(\nabla u_m) + \frac{1}{m}\Psi(\nabla u_m)$$

by application of Lemma 3.7. By substitution in (3.13), the claim follows. $\square$

Claim 4. *The sequence $\{u_m\}_{m \in \mathbb{N}}$ is uniformly bounded in $W^{1,p}(\Omega)$.*

Proof of Claim 4. We proceed as before, but now taking $w - u_m$ as test function in (3.10) with $w \in W_\phi^{1,p}(\Omega)$. Then, as $z_m = pF^{p-1}(\nabla u_m)\bar{z}_m$ with $\bar{z}_m \in \partial F(\nabla u_m)$, by means of (3.2) we can estimate

$$\int_{\Omega} (f - \mathcal{J}_{L^p}(u_m))(w - u_m) \leq p \int_{\Omega} F^{p-1}(\nabla u_m)F(\nabla w) - p \int_{\Omega} F^p(\nabla u_m) + \frac{1}{m} \int_{\Omega} \Psi(\nabla u_m) \cdot \nabla w - \frac{1}{m} \|\nabla u_m\|_{L^p}^p.$$

Rearranging terms, using (2.5) and applying Hölder's inequality, we have

$$\|u_m\|_{L^p}^2 + \left(pC^p + \frac{1}{m}\right) \|\nabla u_m\|_{L^p}^p \leq \|f\|_{L^{p'}}(\|w\|_{L^p} + \|u_m\|_{L^p}) + \|\mathcal{J}_{L^p}(u_m)\|_{L^{p'}}\|w\|_{L^p} + \left(\frac{1}{m} + pC^p\right) \|\nabla u_m\|_{L^p}^{p-1} \|\nabla w\|_{L^p},$$

where c, C are the constants coming from (3.2). By definition of the duality map, Cauchy's inequality and Young's inequality with $\varepsilon = \frac{1}{2} \frac{pC^p + \frac{1}{m}}{pC^p + \frac{1}{m}}$,

$$\frac{1}{2} \|u_m\|_{L^p}^2 + \frac{1}{2} \left(pC^p + \frac{1}{m}\right) \|\nabla u_m\|_{L^p}^p \leq \frac{3}{2} (\|f\|_{L^{p'}}^2 + \|w\|_{L^p}^2) + C(\varepsilon) \|\nabla w\|_{L^p}^p,$$

where

$$C(\varepsilon) = \frac{1}{p} \left(\frac{2(pC^p + \frac{1}{m})}{p'(pC^p + \frac{1}{m})} \right)^{p-1} \leq \frac{1}{p} \left(\frac{2(pC^p + 1)}{p'pC^p} \right)^{p-1}.$$

Accordingly, we reach

$$\frac{1}{2} \|u_m\|_{L^p}^2 + \mathcal{C} \|\nabla u_m\|_{L^p}^p \leq \tilde{\mathcal{C}} (\|f\|_{L^{p'}}^2 + \|w\|_{L^p}^2 + \|\nabla w\|_{L^p}^p),$$

as desired. $\square$

Therefore, u_m converges subsequentially in $L^p(\Omega)$ and a.e. to some $u \in L^p(\Omega)$ as $m \rightarrow \infty$; and $\nabla u_m \rightharpoonup \nabla u$ weakly in $L^p(\Omega)$ as $m \rightarrow \infty$. The continuity of the trace operator with respect to the weak convergence in $W^{1,p}(\Omega)$ ensures that $u \in W_\phi^{1,p}(\Omega)$.

Next, by Lemma 3.4, as $z_m \in \partial F^p(\nabla u_m)$, we have

$$\|z_m\| \leq pC^{p-1} \|\nabla u_m\|^{p-1}.$$

This implies that $\{z_m\}$ is uniformly bounded in $L^{p'}(\Omega)$, and hence $z_m \rightharpoonup z$ weakly in $L^{p'}(\Omega)$ as $m \rightarrow \infty$. On the other hand, by Claim 4, $\{\Psi(\nabla u_m)\}_{m \in \mathbb{N}}$ is bounded in $L^{p'}(\Omega)$ so

$$\frac{1}{m} \Psi(\nabla u_m) \rightharpoonup 0 \quad \text{weakly in } L^{p'}(\Omega) \text{ as } m \rightarrow \infty. \quad (3.16)$$

Thus, taking limits as $m \rightarrow \infty$ in (3.10), we get that

$$\int_{\Omega} (f - \mathcal{J}_{L^p}(u))\psi = \int_{\Omega} z \cdot \nabla \psi \quad \text{for every } \psi \in W_0^{1,p}(\Omega),$$

that is,

$$-\operatorname{div}(z) = f - \mathcal{J}_{L^p}(u) \quad \text{in the weak sense.} \quad (3.17)$$

It remains to see that $z \in \partial F^p(\nabla u)$, which can be obtained after establishing (3.6). Therefore, let $\xi \in \mathbb{R}^N$, $z' \in \partial F^p(\xi)$ and take $\omega : \Omega \rightarrow \mathbb{R}$ defined by $\omega(\eta) := \xi \cdot \eta$ so that $\nabla\omega = \xi$. Then, letting $\phi \in C_0^\infty(\Omega)$ such that $\phi \geq 0$, by the monotonicity of $\partial F^p + \frac{1}{m}\Psi$ and Claim 3 we have, for every $m \in \mathbb{N}$,

$$\begin{aligned} 0 &\leq \int_{\Omega} \left(z_m + \frac{1}{m} \Psi(\nabla u_m) - z' - \frac{1}{m} \Psi(\nabla \omega) \right) \cdot (\nabla u_m - \nabla \omega) \phi \\ &= \int_{\Omega} (f - \mathcal{J}_{L^p}(u_m))(u_m - \omega) \phi - \int_{\Omega} \left(z_m + \frac{1}{m} \Psi(\nabla u_m) \right) \cdot \nabla \phi (u_m - \omega) - \int_{\Omega} \left(z' + \frac{1}{m} \Psi(\nabla \omega) \right) \cdot (\nabla u_m - \nabla \omega) \phi, \end{aligned}$$

thus, since by Fatou's Lemma

$$\liminf_{n \rightarrow \infty} \int_{\Omega} \mathcal{J}_{L^p}(u_m) \cdot u_m \geq \int_{\Omega} \mathcal{J}_{L^p}(u) \cdot u,$$

taking the infimum limit as $m \rightarrow \infty$, by means of (3.16) we get

$$0 \leq \int_{\Omega} (f - \mathcal{J}_{L^p}(u))(u - \omega)\phi - \int_{\Omega} z \cdot \nabla \phi(u - \omega) - \int_{\Omega} z' \cdot (\nabla u - \nabla \omega)\phi = \int_{\Omega} (z - z') \cdot (\nabla u - \nabla \omega)\phi,$$

where the last equality comes from (3.17). Consequently, since $\phi \in C_0^\infty(\Omega)$ is arbitrary and $\nabla \omega = \xi$, we conclude (3.6) with $\alpha = 0$, as desired. Finally, Lemma 3.7 yields $z \in \partial F^p(\nabla u)$, which leads to $(u, f - \mathcal{J}_{L^p}(u)) \in \mathcal{A}_\phi$, which finishes the proof. $\square$

Remark 3.9. Let us point out that in the singular case, that is, when $1 < p < 2$, there is a significant shortcut of the above proof, as one can exploit Lemma 3.5 (a) to prove that the operators $\mathcal{A}_\phi^n(u) := -\operatorname{div}((\nabla(F^p)_\frac{1}{n})(\nabla u))$ are coercive, and then the result follows by performing only one limiting argument. The drawback of this shorter proof is that some bounds depend explicitly on the measure of Ω , and in the next section we plan to extend the results (with minor changes) to unbounded domains.

3.4 Proof of the characterization

After all the machinery developed in the previous subsections, the following result is almost straightforward.

Theorem 3.10. *We have $\mathcal{A}_\phi = \partial \mathcal{F}_{\Omega, \phi}$ for any $\phi \in W^{1-\frac{1}{p}, p}(\partial\Omega)$.*

Proof. Recall that, by Lemma 3.3, $\mathcal{A}_\phi \subseteq \partial \mathcal{F}_{\Omega, \phi}$. Next by Proposition 3.8 we have that

$$L^{p'}(\Omega) = R(\mathcal{J}_{L^p} + \mathcal{A}_\phi),$$

thus, by Theorem 2.2, we have that $\mathcal{A}_\phi$ is a maximal monotone operator, and hence the claim follows. $\square$

By Definition 3.2, Lemma 3.4 and (3.2), this finishes the proof of Theorem 1.1.

4 Characterization of the subdifferential in unbounded exterior domains

After suitable adaptations that we will sketch here, we can reproduce the proof of the previous section and make it work for unbounded domains of the form $\mathbb{R}^N \setminus \overline{\Omega}$, where Ω is a bounded domain with Lipschitz boundary. The main difference is that we work with the homogeneous Sobolev space $\dot{W}^{1,p}(\mathbb{R}^N)$, defined as the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{\dot{W}^{1,p}} := \|\nabla u\|_{L^p}, \quad (4.1)$$

see, e.g., [11]. Setting $p^* = \frac{Np}{N-p}$, $\dot{W}^{1,p}(\mathbb{R}^N)$ can be identified with the following space of functions:

$$\{u \in L^{p^*}(\mathbb{R}^N) : \nabla u \in L^p(\mathbb{R}^N; \mathbb{R}^N)\}.$$

By means of the Gagliardo–Nirenberg–Sobolev inequality, this ensures that

$$\|u\|_{L^{p^*}} \leq C_{p,N} \|\nabla u\|_{L^p} \quad \text{for every } u \in \dot{W}^{1,p}(\mathbb{R}^N),$$

and hence on these spaces we have a Poincaré-type inequality, which is exactly what we need for our estimates to work also in unbounded settings. This also tells us that the norm (4.1) is equivalent to

$$\|u\|_{L^{p^*}} + \|\nabla u\|_{L^p}.$$

With these considerations, $\dot{W}^{1,p}(\mathbb{R}^N)$ is a reflexive Banach space (see [18, Theorem 12.2.3]) and, therefore, Theorem 2.3 can be applied in this setting.

Given $\varphi \in W^{1-\frac{1}{p},p}(\partial\Omega)$, we note that [23, Theorem 18.40] ensures that the trace operator is surjective on sets with Lipschitz continuous boundary, thus we can find $\Phi \in W^{1,p}(\Omega)$ whose trace on $\partial\Omega$ is φ . Then, given $u \in L^{p^*}(\mathbb{R}^N \setminus \overline{\Omega})$ with $\nabla u \in L^p(\mathbb{R}^N \setminus \overline{\Omega})$ such that $u = \varphi$ on $\partial\Omega$, we will consider the extension

$$\tilde{u} := u \chi_{\mathbb{R}^N \setminus \Omega} + \Phi \chi_{\Omega} \in \dot{W}^{1,p}(\mathbb{R}^N).$$

Hence we work with functions on the space

$$\dot{W}_{\Phi}^{1,p}(\mathbb{R}^N) = \{v \in \dot{W}^{1,p}(\mathbb{R}^N) : v = \Phi \text{ in } \Omega\}.$$

Now we consider the energy functional

$$\mathcal{F}_{\mathbb{R}^N \setminus \overline{\Omega}, \Phi} : L^{p^*}(\mathbb{R}^N) \rightarrow [0, +\infty]$$

defined by

$$\mathcal{F}_{\mathbb{R}^N \setminus \overline{\Omega}, \Phi}(u) := \begin{cases} \int_{\mathbb{R}^N} F^p(\nabla u) & \text{if } u \in \dot{W}_{\Phi}^{1,p}(\mathbb{R}^N), \\ +\infty & \text{otherwise,} \end{cases}$$

and observe that it is a proper and convex functional. To characterize its subdifferential, we proceed exactly as before, that is, we introduce an auxiliary operator $\mathcal{A}_{\Phi} \subset L^{p^*}(\mathbb{R}^N) \times L^{(p^*)'}(\mathbb{R}^N)$ by doing the obvious modifications in Definition 3.2, and repeat the arguments from the previous section for a sequence of approximating operators $\mathcal{A}_{\Phi}^{n,m} : \dot{W}_{\Phi}^{1,p}(\mathbb{R}^N) \rightarrow (\dot{W}_{\Phi}^{1,p}(\mathbb{R}^N))'$ given again by the expression (3.7). The aforementioned properties of homogeneous Sobolev spaces allow us to repeat almost verbatim the steps in the previous section to conclude the following result:

Theorem 4.1 (Characterization of the subdifferential on $\mathbb{R}^N \setminus \overline{\Omega}$). *Let $1 < p < N$, $\Omega \subset \mathbb{R}^N$ an open bounded set with Lipschitz boundary and $\Phi \in W^{1,p}(\Omega)$. If $u \in \dot{W}_{\Phi}^{1,p}(\mathbb{R}^N)$ and $v \in L^{(p^*)'}(\mathbb{R}^N)$, then the following are equivalent:*

- $v \in \partial \mathcal{F}_{\mathbb{R}^N \setminus \overline{\Omega}, \Phi}(u)$.
- *There exists $z \in L^{\infty}(\mathbb{R}^N \setminus \overline{\Omega}; \mathbb{R}^N)$, with $F^{\circ}(z) \leq 1$ and $z \cdot \nabla u = F(\nabla u)$, such that $v = -\operatorname{div}(pF^{p-1}(\nabla u)z)$ in the weak sense, that is,*

$$\int_{\mathbb{R}^N \setminus \overline{\Omega}} v w = \int_{\mathbb{R}^N \setminus \overline{\Omega}} pF^{p-1}(\nabla u)z \cdot \nabla w \quad \text{for every } w \in \dot{W}_0^{1,p}(\mathbb{R}^N),$$

where $\dot{W}_0^{1,p}(\mathbb{R}^N) = \{w \in \dot{W}^{1,p}(\mathbb{R}^N) : w = 0 \text{ in } \Omega\}$.

We notice that Theorem 4.1 gives a characterization for all minimizers of $\operatorname{Cap}_p^F(\overline{\Omega})$ for bounded Lipschitz domains Ω since

$$\operatorname{Cap}_p^F(\overline{\Omega}) = \inf \{ \mathcal{F}_{\mathbb{R}^N \setminus \overline{\Omega}, 1}(u) : u \in L^{p^*}(\mathbb{R}^N) \}.$$

5 A comparison principle for strictly convex norms and a barrier argument

5.1 Results for strictly convex anisotropies

We start by obtaining a comparison principle for problem (3.1) with a strictly convex even anisotropy F . It is well-known (cf. [35, Corollary 1.7.3]) that strict convexity of F is closely related to differentiability of F° . More precisely,

$$F^{\circ} \in C^1(\mathbb{R}^N \setminus \{0\}) \quad \text{if, and only if,} \quad B_F(1) = \{\xi : F(\xi) < 1\} \text{ is strictly convex.}$$

In the above conditions, it may happen that $F \notin C^1(\mathbb{R}^N \setminus \{0\})$, as shown by the examples in [14, Example A.1.19] or [16, Section A]. In this spirit, the following result generalizes [10, Lemma 2.3] by relaxing the regularity requirements for F and for the functions u_i involved.

Lemma 5.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz continuous boundary and F a strictly convex norm in $\mathbb{R}^N$. If $u_1, u_2 \in W^{1,p}(\Omega)$ satisfy*

$$\begin{cases} \operatorname{div}(F^{p-1}(\nabla u_1)z_1) \geq \operatorname{div}(F^{p-1}(\nabla u_2)z_2) & \text{in } \Omega, \\ u_1 \leq u_2 & \text{on } \partial\Omega, \end{cases}$$

in the weak sense for some $z_i \in \partial F(\nabla u_i)$, $i = 1, 2$, then $u_1 \leq u_2$ in Ω .

Proof. Setting $\Omega^+ = \Omega \cap \{u_1 > u_2\}$ and using $(u_1 - u_2)^+$ as a test function, we have

$$\int_{\Omega^+} (F^{p-1}(\nabla u_1)z_1 - F^{p-1}(\nabla u_2)z_2) \cdot \nabla(u_1 - u_2) \leq 0.$$

Notice that no boundary terms arise since $(u_1 - u_2)^+ = 0$ on $\partial\Omega^+$.

On the other hand, as $pF^{p-1}(\nabla u_i)z_i \in \partial F^p(\nabla u_i)$ and the latter is monotone, we reach

$$(F^{p-1}(\nabla u_1)z_1 - F^{p-1}(\nabla u_2)z_2) \cdot (\nabla u_1 - \nabla u_2) \geq 0.$$

Therefore, the integrand on left-hand side vanishes a.e. in Ω^+ . Accordingly, by means of (3.2), we get

$$\begin{aligned} 0 &= F^p(\nabla u_1) + F^p(\nabla u_2) - F^{p-1}(\nabla u_1)z_1 \cdot \nabla u_2 - F^{p-1}(\nabla u_2)z_2 \cdot \nabla u_1 \\ &\geq (F^{p-1}(\nabla u_1) - F^{p-1}(\nabla u_2))(F(\nabla u_1) - F(\nabla u_2)), \end{aligned}$$

meaning that $F(\nabla u_1) = F(\nabla u_2)$ a.e. in Ω^+ . Hence the strict convexity of F ensures that $\nabla u_1 = \nabla u_2$ a.e. in Ω^+ . As $u_1 = u_2$ on $\partial\Omega^+$, we conclude that Ω^+ has null measure and thus the claim follows. $\square$

Remark 5.2. We observe that this result gives uniqueness of minimizers of $\mathcal{F}_{\Omega,\varphi}$ (and solutions to (3.1)) if F is a strictly convex norm.

We also need the following auxiliary result:

Lemma 5.3. *Let F be a strictly convex norm in $\mathbb{R}^N$. Then for every $x \neq 0$,*

$$F(\nabla F^\circ(x)) = 1 \quad \text{and} \quad \frac{x}{F^\circ(x)} \in \partial F(\nabla F^\circ(x)).$$

Proof. The proof of the first equality is exactly as in [40, Proposition 1.3], since it only uses the differentiability of F° , which follows from the strict convexity of F . For the second claim, from the 1-homogeneity of F° (recall (3.2)), we can write

$$x \cdot \nabla F^\circ(x) = F^\circ(x) = F(\nabla F^\circ(x))F^\circ(x).$$

By (2.3) this implies that $\nabla F^\circ(x)$ minimizes $g(\xi) := F^\circ(x)F(\xi) - x \cdot \xi$. Accordingly,

$$0 \in \partial g(\nabla F^\circ(x)) = F^\circ(x)\partial F(\nabla F^\circ(x)) - x,$$

meaning that $x \in F^\circ(x)\partial F(\nabla F^\circ(x))$, as desired. $\square$

5.2 Existence of minimizers within annular domains

Notation 5.4. For any $R > 0$ and Ω a bounded domain with Lipschitz-continuous boundary in $\mathbb{R}^N$ such that $\overline{\Omega} \subset \mathcal{W}_R$, we consider the annular region $\Omega_R := \mathcal{W}_R \setminus \overline{\Omega}$ and the function ψ defined on $\partial\Omega_R$ by $\psi = 1$ on $\partial\Omega$ and $\psi = 0$ on $\partial\mathcal{W}_R$. Now it makes sense to work with the functional $\mathcal{F}_R := \mathcal{F}_{\Omega_R,\psi}$ given by (1.2).

In this setting, we aim to obtain a minimizer of $\mathcal{F}_R$ and then apply suitable comparison results to take limits as $R \rightarrow \infty$. However, in order to exploit Lemma 5.1 in general, that is, for non-necessarily strictly convex norms, we will approximate the anisotropy F by a sequence of strictly convex anisotropies $\{F_\varepsilon = F + \varepsilon\|\cdot\|\}_{\varepsilon>0}$, and consider the subsequent energy functional

$$\mathcal{F}_R^\varepsilon : L^p(\Omega_R) \rightarrow [0, +\infty]$$

defined by

$$\mathcal{F}_R^\varepsilon(u) := \begin{cases} \int_{\Omega_R} F_\varepsilon^p(\nabla u) & \text{if } u \in W_\psi^{1,p}(\Omega_R), \\ +\infty & \text{otherwise,} \end{cases}$$

where, by simplicity, we denote $(F_\varepsilon)^p$ by F_ε^p .

Lemma 5.5. *Let $\{u_\varepsilon\}_{\varepsilon>0}$ be a sequence of minimizers of $\mathcal{F}_R^\varepsilon$. Then there exists a subsequence that converges weakly in $W^{1,p}(\Omega_R)$ to $u_R \in W_\psi^{1,p}(\Omega_R)$. Moreover, u_R is a minimizer of $\mathcal{F}_R$.*

Proof. Let u_ε be a minimizer of $\mathcal{F}_R^\varepsilon$ and let $w \in W_\psi^{1,p}(\Omega_R)$. By the uniform convergence of $\{F_\varepsilon\}_{\varepsilon>0}$ to F as $\varepsilon \searrow 0$ and (2.1), there exist $k > 0$ not depending on ε such that

$$(C+1)^p \|\nabla w\|_{L^p}^p \geq \mathcal{F}_R^\varepsilon(w) \geq \mathcal{F}_R^\varepsilon(u_\varepsilon) \geq k \|\nabla u_\varepsilon\|_{L^p}^p \quad \text{for all } \varepsilon \text{ small enough.}$$

Accordingly, by Poincaré inequality and up to a subsequence, we may suppose that $u_\varepsilon \rightharpoonup u_R \in W^{1,p}(\Omega_R)$ weakly in $W^{1,p}(\Omega_R)$ and strongly in $L^p(\Omega_R)$.

Now, since $\mathcal{F}_R$ is lower semicontinuous with respect to the weak convergence in $W^{1,p}(\Omega_R)$ (recall Proposition 3.6), we can write

$$\begin{aligned} \int_{\Omega_R} F^p(\nabla u_R) &\leq \liminf_{\varepsilon \rightarrow 0} \int_{\Omega_R} F_\varepsilon^p(\nabla u_\varepsilon) \\ &\leq \limsup_{\varepsilon \rightarrow 0} \int_{\Omega_R} (F(\nabla u_\varepsilon) + \varepsilon \|\nabla u_\varepsilon\|)^p \\ &\leq \limsup_{\varepsilon \rightarrow 0} \int_{\Omega_R} (F(\nabla w) + \varepsilon \|\nabla w\|)^p = \int_{\Omega_R} F^p(\nabla w) \end{aligned}$$

for every $w \in W^{1,p}(\Omega_R)$. Therefore, we get that u_R is a minimizer of $\mathcal{F}_R$. By the continuity of traces, $u_R = 1$ on $\partial\Omega$ and equals zero on $\partial\mathcal{W}_R$. $\square$

5.3 Construction of explicit barriers trapping a minimizer

Proposition 5.6. *Let $0 < r_1 < r_2 < R$ be such that $\mathcal{W}_{r_1} \subseteq \Omega \subseteq \mathcal{W}_{r_2}$, and let u_R be a minimizer of $\mathcal{F}_R$ which arises as a subsequential limit of minimizers of $\{\mathcal{F}_R^\varepsilon\}_{\varepsilon>0}$ as $\varepsilon \searrow 0$. Then*

$$\frac{(F^\circ)^{\frac{p-N}{p-1}}(x) - R^{\frac{p-N}{p-1}}}{r_1^{\frac{p-N}{p-1}} - R^{\frac{p-N}{p-1}}} \leq u_R(x) \leq \frac{(F^\circ)^{\frac{p-N}{p-1}}(x) - R^{\frac{p-N}{p-1}}}{r_2^{\frac{p-N}{p-1}} - R^{\frac{p-N}{p-1}}} \quad (5.1)$$

for a.e. $x \in \Omega_R$.

Proof. Given $r < R$, it is a routine computation to check that

$$v_{r,R}^\varepsilon(x) = \frac{(F_\varepsilon^\circ)^{\frac{p-N}{p-1}}(x) - R^{\frac{p-N}{p-1}}}{r^{\frac{p-N}{p-1}} - R^{\frac{p-N}{p-1}}}$$

satisfies

$$\begin{cases} \operatorname{div} \left(F_\varepsilon^{p-1}(\nabla v_{r,R}^\varepsilon) \frac{x}{F_\varepsilon^\circ(x)} \right) = 0 & \text{in } \mathcal{W}_R^\varepsilon \setminus \overline{\mathcal{W}_r^\varepsilon}, \\ v_{r,R}^\varepsilon = 1 & \text{on } \partial\mathcal{W}_r^\varepsilon, \\ v_{r,R}^\varepsilon = 0 & \text{on } \partial\mathcal{W}_R^\varepsilon. \end{cases}$$

Here $\mathcal{W}_r^\varepsilon$ is the Wulff shape of radius r associated to the anisotropy F_ε . Notice that the first equality follows after checking that $\partial F_\varepsilon(\nabla v_{r,R}^\varepsilon) = \partial F_\varepsilon(\nabla F_\varepsilon^\circ)$ and using Lemma 5.3. It can be easily checked, as in [15, Lemma 5.3], that

$$\mathcal{W}_r^\varepsilon = \mathcal{W}_r + B_{\|\cdot\|}(\varepsilon r). \quad (5.2)$$

On the other hand, as $u_\varepsilon \in W_\psi^{1,p}(\Omega_R)$ is a minimizer for $\mathcal{F}_R^\varepsilon$, then it holds

$$\begin{cases} \operatorname{div}(F_\varepsilon^{p-1}(\nabla u_\varepsilon)z) = 0 & \text{in } \Omega_R, \\ u_\varepsilon = 1 & \text{on } \partial\Omega, \\ u_\varepsilon = 0 & \text{on } \partial\mathcal{W}_R, \end{cases} \quad \text{with } z \in \partial F_\varepsilon(\nabla u_\varepsilon).$$

Let $r_2 < R_1 < R$ be such that $\mathcal{W}_{r_1}^\varepsilon \subseteq \Omega \subseteq \mathcal{W}_{r_2}^\varepsilon \subseteq \mathcal{W}_{R_1}^\varepsilon \subseteq \mathcal{W}_R$. Notice that $v_{r_1,R_1}^\varepsilon \leq u_\varepsilon$ on $\partial\Omega_R$, where v_{r_1,R_1}^ε is extended by zero in $\mathcal{W}_R \setminus \mathcal{W}_{R_1}^\varepsilon$. Therefore Lemma 5.1 ensures that

$$u_\varepsilon(x) \geq v_{r_1,R_1}^\varepsilon(x) = \frac{(F_\varepsilon^\circ)^{\frac{p-N}{p-1}}(x) - R_1^{\frac{p-N}{p-1}}}{r_1^{\frac{p-N}{p-1}} - R_1^{\frac{p-N}{p-1}}} \quad \text{for a.e. } x \in \Omega_R. \quad (5.3)$$

Similarly, for $R_2 > R$ with $\mathcal{W}_R \subseteq \mathcal{W}_{R_2}^\varepsilon$, if we extend v_{r_2,R_2}^ε by 1 in $\mathcal{W}_{R_2}^\varepsilon \setminus \overline{\Omega}$, we have that $u_\varepsilon \leq v_{r_2,R_2}^\varepsilon$ on $\partial\Omega_R$, and hence

$$u_\varepsilon(x) \leq v_{r_2,R_2}^\varepsilon(x) = \frac{(F_\varepsilon^\circ)^{\frac{p-N}{p-1}}(x) - R_2^{\frac{p-N}{p-1}}}{r_2^{\frac{p-N}{p-1}} - R_2^{\frac{p-N}{p-1}}} \quad \text{for a.e. } x \in \Omega_R. \quad (5.4)$$

Finally, given $\varepsilon > 0$, we set

$$R_1(\varepsilon) = \max\{\rho > 0 : \mathcal{W}_\rho^\varepsilon \subseteq \mathcal{W}_R\} \quad \text{and} \quad R_2(\varepsilon) = \min\{\rho > 0 : \mathcal{W}_R \subseteq \mathcal{W}_\rho^\varepsilon\}.$$

Then, by (5.2), we have

$$R \leq \lim_{\varepsilon \searrow 0} R_1(\varepsilon) \leq \lim_{\varepsilon \searrow 0} R_2(\varepsilon) \leq R.$$

So both R_1 and R_2 converge to R and, taking limits in (5.3) and (5.4), we conclude (5.1), as desired. $\square$

6 Existence of minimizers with double side bounds

Proof of Theorem 1.3. Recall that for any $R > 0$ one can actually construct a minimizer u_R , by means of Lemma 5.5, for which we have upper and lower barriers as stated in Proposition 5.6. Now the aim is to show that these estimates pass to the limit as $R \rightarrow \infty$.

Let $0 < r_1 < r_2$ such that $\mathcal{W}_{r_1} \subseteq \Omega \subseteq \mathcal{W}_{r_2}$. For each big enough R , let u_R be a minimizer of $\mathcal{F}_R$ which arises as a limit of minimizers of $\{\mathcal{F}_R^\varepsilon\}_{\varepsilon>0}$ and extend u_R by zero to $\mathbb{R}^N \setminus \overline{\Omega}$ (without renaming). Then by Lemma 5.1 we have that $\{u_R\}$ is increasing in R and, by Proposition 5.6, we get

$$\frac{(F^\circ)^{\frac{p-N}{p-1}}(x) - R^{\frac{p-N}{p-1}}}{r_1^{\frac{p-N}{p-1}} - R^{\frac{p-N}{p-1}}} \leq u_R(x) \leq \frac{(F^\circ)^{\frac{p-N}{p-1}}(x) - R^{\frac{p-N}{p-1}}}{r_2^{\frac{p-N}{p-1}} - R^{\frac{p-N}{p-1}}} \quad \text{a.e. in } \mathbb{R}^N \setminus \overline{\Omega}.$$

Consequently, there exists a measurable function $u : \mathbb{R}^N \setminus \overline{\Omega} \rightarrow \mathbb{R}$ such that, up to a subsequence, $u_R(x) \nearrow u(x)$ as $R \rightarrow \infty$ for a.e. $x \in \mathbb{R}^N \setminus \overline{\Omega}$ and

$$(F^\circ)^{\frac{p-N}{p-1}}\left(\frac{x}{r_1}\right) \leq u(x) \leq (F^\circ)^{\frac{p-N}{p-1}}\left(\frac{x}{r_2}\right) \quad \text{a.e. in } \mathbb{R}^N \setminus \overline{\Omega}. \quad (6.1)$$

In particular, $u \rightarrow 0$ as $\|x\| \rightarrow \infty$. On the other hand, notice that the bounding functions in (6.1) belong to $L^q(\mathbb{R}^N \setminus \overline{\Omega})$ if $q > \frac{N(p-1)}{N-p}$, and this ensures by dominated convergence that $u_R \rightarrow u$ in $L^{p^*}(\mathbb{R}^N \setminus \overline{\Omega})$.

Now, let $\varphi \in C_c^\infty(\mathbb{R}^N)$ with $\varphi = 1$ on $\partial\Omega$, and consider R large enough so that $\operatorname{supp} \varphi \subset \mathcal{W}_R$. Since u_R is a minimizer of $\mathcal{F}_R$, we have that

$$c^p \|\nabla u_R\|_{L^p(\mathbb{R}^N \setminus \overline{\Omega})}^p \leq \mathcal{F}_R(u_R) \leq \mathcal{F}_R(\varphi) \leq C^p \|\nabla \varphi\|_{L^p(\mathbb{R}^N \setminus \overline{\Omega})}^p.$$

Therefore, $\{\|\nabla u_R\|_p\}_R$ is bounded thus $u \in W^{1,p}(\mathbb{R}^N \setminus \overline{\Omega})$ and $\nabla u_R \rightharpoonup \nabla u$ weakly in $L^p(\mathbb{R}^N \setminus \overline{\Omega})$; in particular, $u = 1$ on $\partial\Omega$. Moreover, since $0 \in \partial\mathcal{F}_R(u_R)$, by Theorem 1.1 we have that there exists $z_R \in \partial F(\nabla u_R)$ (which we extend by zero in $\mathbb{R}^N \setminus \mathcal{W}_R$) such that

$$\int_{\mathbb{R}^N \setminus \overline{\Omega}} p F^{p-1}(\nabla u_R) z_R \cdot \nabla w = 0 \quad (6.2)$$

for every $w \in W_0^{1,p}(\mathcal{W}_R \setminus \overline{\Omega})$. Now, by (3.3), we have $\|z_R\|_\infty \leq C$ so, up to a subsequence,

$$pF^{p-1}(\nabla u_R)z_R \rightharpoonup \tilde{z} \quad \text{in the weak topology of } L^{p'}(\mathbb{R}^N \setminus \overline{\Omega}). \quad (6.3)$$

Taking limits as $R \rightarrow \infty$ in (6.2), we get

$$\int_{\mathbb{R}^N \setminus \overline{\Omega}} \tilde{z} \cdot \nabla w = 0$$

for every $w \in C_c^\infty(\mathbb{R}^N \setminus \overline{\Omega})$ (thus, by approximation, for any $w \in W_0^{1,p}(\mathbb{R}^N \setminus \overline{\Omega})$ that is,

$$-\operatorname{div}(\tilde{z}) = 0 \quad \text{in the weak sense.} \quad (6.4)$$

It remains to show that $\tilde{z} \in \partial F^p(\nabla u)$, which will be deduced as usual from Lemma 3.7. Therefore, it suffices to get (3.6) for $z = \tilde{z}$ and $\alpha = 0$. With this goal, let $\xi \in \mathbb{R}^N$ and take $\omega : \mathbb{R}^N \setminus \overline{\Omega} \rightarrow \mathbb{R}$ defined by $\omega(\eta) := \xi \cdot \eta$ so that $\nabla \omega = \xi$. Then, letting $\phi \in C_c^\infty(\mathbb{R}^N \setminus \overline{\Omega})$ such that $\phi \geq 0$, by the monotonicity of ∂F^p and (6.2) we have, for every $R > 0$ and any $z' \in \partial F^p(\xi)$,

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}^N \setminus \overline{\Omega}} (pF^{p-1}(\nabla u_R)z_R - z') \cdot (\nabla u_R - \nabla \omega) \phi \\ &= - \int_{\operatorname{supp} \nabla \phi} pF^{p-1}(\nabla u_R)z_R \cdot \nabla \phi (u_R - \omega) - \int_{\mathbb{R}^N \setminus \overline{\Omega}} z' \cdot (\nabla u_R - \nabla \omega) \phi. \end{aligned}$$

Since by (6.1) we have that $u_R \rightarrow u$ strongly in $L^p(K)$ for every compact set $K \subset \mathbb{R}^N \setminus \overline{\Omega}$, letting $R \rightarrow \infty$, by (6.3) and (6.4), we get

$$0 \leq \int_{\mathbb{R}^N \setminus \overline{\Omega}} -\tilde{z} \cdot \nabla \phi (u - \omega) - \int_{\mathbb{R}^N \setminus \overline{\Omega}} z' \cdot (\nabla u - \nabla \omega) \phi = \int_{\mathbb{R}^N \setminus \overline{\Omega}} (\tilde{z} - z') \cdot (\nabla u - \nabla \omega) \phi.$$

Finally, since $0 \leq \phi \in C_c^\infty(\mathbb{R}^N \setminus \overline{\Omega})$ is arbitrary and $\nabla \omega = \xi$ we reach (3.6) as desired. $\square$

7 Lipschitz regularity of minimizers for domains satisfying a uniform interior ball condition

If we further assume that Ω satisfies the condition stated in Definition 1.4, by a careful application of our comparison results, we will show that a minimizer constructed as in Lemma 5.5 is bounded above and below by two Lipschitz solutions which coincide on the boundary, and accordingly we will deduce the Lipschitz regularity of the minimizer by application of the following result (see [27, Corollary 4.2] for a more general version):

Theorem 7.1. *Let $f : \mathbb{R}^N \rightarrow \mathbb{R}$ be a convex function which is bounded below by an affine function and let*

$$W_{u_1, u_2}^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega) : u_1 \leq u \leq u_2 \text{ a.e. and } u = u_1 = u_2 \text{ on } \partial\Omega\},$$

where u_1 and u_2 are Lipschitz functions on Ω coinciding on $\partial\Omega$. Then the problem of minimizing $I(u) = \int_\Omega f(\nabla u)$ in $W_{u_1, u_2}^{1,p}(\Omega)$ admits at least one solution; moreover, at least one of the minimizers is Lipschitz with Lipschitz constant $L \leq \max\{\operatorname{Lip}(u_1), \operatorname{Lip}(u_2)\}$.

With the conventions from Notation 5.4, we are now in a position to prove that any minimizer w_R of the energy functional $\mathcal{F}_R$ is Lipschitz continuous. Moreover, any minimizer u of $\mathcal{F}_{\mathbb{R}^N \setminus \overline{\Omega}, 1}$ is Lipschitz continuous.

Proof of Theorem 1.5. Let $R_0 := r + \frac{1}{2} \operatorname{dist}_{F^*}(\partial\Omega, \partial\mathcal{W}_R)$. For each $z \in \partial\Omega$ let $y_z \in \mathbb{R}^N$ such that $\mathcal{W}_r + y_z \subseteq \overline{\Omega}$ and $z \in \partial(\mathcal{W}_r + y_z)$. Given $z \in \partial\Omega$, define $\eta_z : \mathbb{R}^N \setminus \{y_z\} \rightarrow \mathbb{R}$ by

$$\eta_z(x) := \frac{(F^\circ)^{\frac{p-N}{p-1}}(x - y_z) - R_0^{\frac{p-N}{p-1}}}{r^{\frac{p-N}{p-1}} - R_0^{\frac{p-N}{p-1}}}.$$

Note that $\eta_z(z) = 1$, $\eta_z \leq 1$ on $\partial\Omega$ and $\eta_z \leq 0$ on $\partial\mathcal{W}_R$. Moreover, as in the proof of Proposition 5.6, one shows that $\eta_z(x) \leq u_R(x)$ for a.e. $x \in \Omega_R = \mathcal{W}_R \setminus \overline{\Omega}$ and every $z \in \partial\Omega$. In addition, since $F^\circ(x - y_z) \geq r$ for every $x \in \Omega_R$ and

$$\partial\eta_z(x) := \frac{\frac{p-N}{p-1}(F^\circ)^{\frac{1-N}{p-1}}(x - y_z)\partial F^\circ(x - y_z)}{r^{\frac{p-N}{p-1}} - R_0^{\frac{p-N}{p-1}}},$$

we have, for every $x \in \Omega_R$ and $w \in \partial\eta_z(x)$, that

$$\|w\| \leq \frac{\frac{|N-p|}{p-1}r^{\frac{1-N}{p-1}}}{r^{\frac{p-N}{p-1}} - R_0^{\frac{p-N}{p-1}}} \|\tilde{w}\|, \quad \text{where } \tilde{w} \in \partial F^\circ(x - y_z).$$

Hence by (2.2) and (3.3) we get the bound

$$\|w\| \leq \frac{\frac{|N-p|}{p-1}r^{\frac{1-N}{p-1}}}{r^{\frac{p-N}{p-1}} - R_0^{\frac{p-N}{p-1}}} \frac{1}{c} =: L_1$$

Therefore, η_z is L_1 -Lipschitz for every $z \in \partial\Omega$ (with L_1 independent of R).

Let I be a countable dense subset of $\partial\Omega$ and

$$\eta(x) := \sup_{z \in I} \{\eta_z(x) \vee 0\}.$$

Since the η_z are L_1 -Lipschitz, we get that η is also L_1 -Lipschitz. Moreover, $\eta = 1$ on $\partial\Omega$, $\eta = 0$ on $\partial\mathcal{W}_R$ and $\eta \leq u_R$ a.e. in Ω_R .

Therefore, if $\mathcal{W}_{r_1} \subseteq \Omega \subseteq \mathcal{W}_{r_2} \subseteq \mathcal{W}_R$, Proposition 5.6 leads to

$$\eta(x) \leq u_R(x) \leq \frac{(F^\circ)^{\frac{p-N}{p-1}}(x) - R^{\frac{p-N}{p-1}}}{r_2^{\frac{p-N}{p-1}} - R^{\frac{p-N}{p-1}}} \wedge 1 =: \eta'(x) \quad \text{for a.e. } x \in \Omega_R.$$

Working as above we get that η' is L_2 -Lipschitz for some constant L_2 which decreases with R . In short, we have shown that $u_R \in W_{\eta, \eta'}^{1,p}(\Omega_R)$.

Now, by Theorem 7.1, there exists at least one minimizer of $\mathcal{F}_R$ within $W_{\eta, \eta'}^{1,p}(\Omega_R)$ which is Lipschitz continuous with constant $L \leq \max\{L_1, L_2\}$. If we denote this minimizer by v_R , notice that

$$\mathcal{F}_R(u_R) = \min_{W^{1,p}(\Omega_R)} \mathcal{F}_R = \min_{W_{\eta, \eta'}^{1,p}(\Omega_R)} \mathcal{F}_R = \mathcal{F}_R(v_R).$$

Next consider any minimizer w_R of $\mathcal{F}_R$ (not necessarily constructed by a limiting procedure or trapped between η and η'). Therefore,

$$\begin{aligned} \mathcal{F}_R\left(\frac{w_R + v_R}{2}\right) &= \int_{\Omega_R} F^p\left(\frac{\nabla(w_R + v_R)}{2}\right) \\ &\leq \int_{\Omega_R} \frac{F^p(\nabla w_R) + F^p(\nabla v_R)}{2} = \int_{\Omega_R} F^p(\nabla w_R) = \mathcal{F}_R(w_R). \end{aligned}$$

As the reverse inequality holds because $w_R \in \arg \min \mathcal{F}_R$, we conclude

$$F^p\left(\frac{\nabla(w_R + v_R)}{2}\right) = \frac{F^p(\nabla w_R) + F^p(\nabla v_R)}{2} \quad \text{a.e. in } \Omega_R.$$

Now, if $F(\nabla w_R) \neq F(\nabla v_R)$, by the strict convexity of $t \rightarrow |t|^p$ and the convexity of F , we have that (a.e. in Ω_R)

$$\begin{aligned} \left(\frac{F(\nabla w_R) + F(\nabla v_R)}{2}\right)^p &< \frac{F^p(\nabla w_R) + F^p(\nabla v_R)}{2} = F^p\left(\frac{\nabla(w_R + v_R)}{2}\right) \\ &\leq \left(\frac{F(\nabla w_R) + F(\nabla v_R)}{2}\right)^p, \end{aligned}$$

which is a contradiction. Therefore, $F(\nabla w_R) = F(\nabla v_R)$ a.e. in Ω_R , thus w_R is $\frac{c}{C}L$ -Lipschitz (as v_R is L -Lipschitz), where c and C come from (2.1).

Now, since the u_R are uniformly Lipschitz and $u_R \nearrow u$ by the proof of Theorem 1.3, we get that u is Lipschitz continuous (with Lipschitz constant $\frac{C}{c}L$). Now take w an arbitrary minimizer of $\mathcal{F}_{\mathbb{R}^N \setminus \overline{\Omega}, 1}$ and redo the previous argument with u and w playing the role of v_R and w_R , respectively. Then we conclude that w is also Lipschitz continuous with constant $\frac{C^2}{c^2}L$. $\square$

A Non-uniqueness of minimizers in the case $p = 1$

In this appendix, we show an example of non-uniqueness for relative 1-capacitary functions; i.e., minimizers of the relative 1-capacity with respect to a ball $\mathcal{W}_R$. We take $N = 2$ and $F(\xi) = \|\xi\|_1 = |\xi_1| + |\xi_2|$ as the anisotropy. Let $\Omega := B_1$ be the (Euclidean) unit ball in $\mathbb{R}^2$ centered at the origin and set $R = 2$, thus $\mathcal{W}_2 = \{\xi \in \mathbb{R}^2 : \|\xi\|_\infty = \max\{|\xi_1|, |\xi_2|\} \leq 2\}$. Consider

$$\text{Cap}_1^{\|\cdot\|_1}(\overline{B_1}; \mathcal{W}_2) = \inf \left\{ \int_{\mathcal{W}_2} \|\nabla u\|_1 dx : u \in C_0^\infty(\mathcal{W}_2), u \geq 1 \text{ in } B_1 \right\}.$$

It turns out that, in general, due to the lack of compactness of minimizing sequences in $W^{1,1}(\mathcal{W}_R)$, minimizers of linear growth functionals with respect to the gradient, as that in $\text{Cap}_1^{\|\cdot\|_1}$, can be functions that do not belong to the Sobolev space $W^{1,1}(\mathcal{W}_R)$. Therefore, one needs to consider the relaxed functional to $G : L^1(\mathcal{W}_2 \setminus \overline{B_1}) \rightarrow [0, +\infty]$,

$$G(u) := \begin{cases} \int_{\mathcal{W}_2 \setminus \overline{B_1}} \|\nabla u\|_1 & \text{if } u \in W^{1,1}(\mathcal{W}_2 \setminus \overline{B_1}), \\ +\infty & \text{otherwise,} \end{cases} \quad \begin{cases} u = 1 & \text{in } \partial B_1, \\ u = 0 & \text{in } \partial \mathcal{W}_2 \end{cases}$$

which is (see [31, Theorem 4])

$$\mathcal{G}(u) := |Du|_1(\mathcal{W}_2 \setminus \overline{B_1}) + \int_{\partial B_1} \|v^{B_1}\|_1 |u - 1| d\mathcal{H}^1 + \int_{\partial \mathcal{W}_2} \|v^{\mathcal{W}_2}\|_1 |u| d\mathcal{H}^1,$$

where $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure in $\mathbb{R}^2$. Moreover, v^U represents the unit exterior normal to the open set $U \subset \mathbb{R}^2$; and the term $|Du|_1(U)$ means the anisotropic total variation measure of the open set $U \subset \mathbb{R}^2$, defined as (see [4, Definition 3.1])

$$|Du|_1(U) := \sup \left\{ \int_U u \operatorname{div} z : z \in X_1(U) \right\},$$

with

$$X_1(U) := \{z \in L^\infty(U; \mathbb{R}^2) : \operatorname{supp} z \subset U, \operatorname{div} z \in L^2(U), \|z\|_\infty \leq 1 \text{ a.e. in } U\}.$$

We note that $|Du|_1(U)$ is finite if, and only if, u is a bounded variation function in U . Similarly, $E \subset \mathbb{R}^2$ is a set of finite perimeter if, and only if,

$$\operatorname{Per}_1(E) := |D\chi_E|_1(\mathbb{R}^2) < +\infty,$$

where χ_E is the characteristic function of the measurable set E ; i.e.,

$$\chi_E(x) := \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

For a comprehensive treatment of bounded variation functions and sets of finite perimeter, we refer to [5].

The characterization of the subdifferential of the energy functional $\mathcal{G}$ is included in [31, Theorem 9]. We will show that $0 \in \partial \mathcal{G}(u)$ in the case that $u = \chi_E$ with $E \subset \mathcal{W}_1 \setminus \overline{B_1}$ of finite perimeter such that $\partial B_1 \subset \partial E$.

In this setting, the characterization of the subdifferential is much simpler than for a generic bounded variation function and we obtain that $0 \in \partial \mathcal{G}(\chi_E)$ if, and only if, there exists $z \in L^\infty(\mathcal{W}_2 \setminus \overline{B_1})$ such that $\|z\|_\infty \leq 1$, a.e. in $\mathcal{W}_2 \setminus \overline{B_1}$, $\operatorname{div} z = 0$ in the distributional sense and

$$\mathcal{G}(\chi_E) = |D\chi_E|_1(\mathcal{W}_2 \setminus \overline{B_1}) = \operatorname{Per}_1(E) - \operatorname{Per}_1(B_1) = \operatorname{Per}_1(B_1). \quad (\text{A.1})$$

We observe that the first two equalities hold because of the particular assumption on E .

We define

$$z(x, y) := \begin{cases} -\left(\frac{x}{|x|}, \frac{y}{|y|}\right) & \text{if } (x, y) \in \mathcal{W}_1 \setminus \overline{B_1}, \\ -\frac{(x, y)}{\|(x, y)\|_\infty^2} & \text{if } (x, y) \in \mathcal{W}_2 \setminus \mathcal{W}_1. \end{cases}$$

It is easy to show that the vector field z satisfies $\|z\|_\infty \leq 1$ in $\mathcal{W}_2 \setminus \overline{B_1}$ and $\operatorname{div} z = 0$ in the distributional sense.

On the other hand, a routine computation ensures that for (A.1) to hold, we just need that

$$v^E(x, y) \cdot z(x, y) = -1 \quad \mathcal{H}^1\text{-a.e. in } \partial E \setminus \partial B_1. \quad (\text{A.2})$$

Since there are infinite sets of finite perimeter $E \subset \mathcal{W}_1 \setminus \overline{B_1}$ satisfying (A.2), we conclude that there are infinitely many different 1-capacitary functions. For instance, $E_1 := \mathcal{W}_1 \cap (B_{\|\cdot\|_1}(\sqrt{2}) \setminus \overline{B_1})$ and $E_2 := \mathcal{W}_1 \setminus \overline{B_1}$ are two different examples. We finally note that it can be also proved, though the proof requires the general characterization of the subdifferential, that $u = \chi_\emptyset = 0$ is a minimizer.

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References

- [1] V. Agostiniani, M. Fogagnolo and L. Mazziere, Minkowski inequalities via nonlinear potential theory, *Arch. Ration. Mech. Anal.* **244** (2022), no. 1, 51–85.
- [2] M. Akman, J. Gong, J. Hineman, J. Lewis and A. Vogel, The Brunn–Minkowski inequality and a Minkowski problem for nonlinear capacity, *Mem. Amer. Math. Soc.* **275** (2022), no. 1348, 1–115.
- [3] F. Almgren, J. E. Taylor and L. Wang, Curvature-driven flows: A variational approach, *SIAM J. Control Optim.* **31** (1993), no. 2, 387–438.
- [4] M. Amar and G. Belletini, A notion of total variation depending on a metric with discontinuous coefficients, *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **11** (1994), no. 1, 91–133.
- [5] L. Ambrosio, N. Fusco and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford Math. Monogr., Clarendon Press, New York, 2000.
- [6] C. Atkinson and C. R. Champion, Some boundary-value problems for the equation $\nabla \cdot (|\nabla \varphi|^N \nabla \varphi) = 0$, *Quart. J. Mech. Appl. Math.* **37** (1984), no. 3, 401–419.
- [7] J.-P. Aubin and H. Frankowska, *Set-Valued Analysis*, Mod. Birkhäuser Class., Birkhäuser, Boston, 2009.
- [8] V. Barbu, *Nonlinear Differential Equations of Monotone Types in Banach Spaces*, Springer Monogr. Math., Springer, New York, 2010.
- [9] H. H. Bauschke and P. L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, CMS Books Math./Ouvrages Math. SMC, Springer, New York, 2011.
- [10] C. Bianchini and G. Ciraolo, Wulff shape characterizations in overdetermined anisotropic elliptic problems, *Comm. Partial Differential Equations* **43** (2018), no. 5, 790–820.
- [11] L. Brasco, D. Gómez-Castro and J. L. Vázquez, Characterisation of homogeneous fractional Sobolev spaces, *Calc. Var. Partial Differential Equations* **60** (2021), no. 2, Paper No. 60.
- [12] H. Brezis, *Opérateurs Maximaux Monotones*, American Elsevier, New York, 1973.
- [13] F. E. Browder, *Problèmes nonlinéaires*, Sémin. Math. Supér. 15, Les Presses de l'Université de Montréal, Montréal, 1966.
- [14] P. Cannarsa and C. Sinestrari, *Semiconcave Functions, Hamilton–Jacobi equations, and Optimal Control*, Progr. Nonlinear Differential Equations Appl. 58, Birkhäuser, Boston, 2004.
- [15] V. Caselles, A. Chambolle, S. Moll and M. Novaga, A characterization of convex calibrable sets in $\mathbb{R}^N$ with respect to anisotropic norms, *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **25** (2008), no. 4, 803–832.

- [16] G. Ciraolo and A. Greco, An overdetermined problem associated to the Finsler Laplacian, *Commun. Pure Appl. Anal.* **20** (2021), no. 3, 1025–1038.
- [17] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Class. Appl. Math. 5, Society for Industrial and Applied Mathematics, Philadelphia, 1990.
- [18] L. Diening, P. Harjulehto, P. Hästö and M. Růžička, *Lebesgue and Sobolev Spaces with Variable Exponents*, Lecture Notes in Math. 2017, Springer, Heidelberg, 2011.
- [19] S. Esedoğlu and S. J. Osher, Decomposition of images by the anisotropic Rudin–Osher–Fatemi model, *Comm. Pure Appl. Math.* **57** (2004), no. 12, 1609–1626.
- [20] L. C. Evans, *Partial Differential Equations*, 2nd ed., Grad. Stud. Math. 19, American Mathematical Society, Providence, 2010.
- [21] P. Hartman and G. Stampacchia, On some non-linear elliptic differential-functional equations, *Acta Math.* **115** (1966), 271–310.
- [22] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, Class. Appl. Math. 31, Society for Industrial and Applied Mathematics, Philadelphia, 2000.
- [23] G. Leoni, *A First Course in Sobolev Spaces*, 2nd ed., Grad. Stud. Math. 181, American Mathematical Society, Providence, 2017.
- [24] J. Leray and J.-L. Lions, Quelques résultats de Višik sur les problèmes elliptiques nonlinéaires par les méthodes de Minty–Browder, *Bull. Soc. Math. France* **93** (1965), 97–107.
- [25] J. L. Lewis, Capacitary functions in convex rings, *Arch. Ration. Mech. Anal.* **66** (1977), no. 3, 201–224.
- [26] R. Li and C. Xiong, Sharp bounds for the anisotropic p -capacity of Euclidean compact sets, *J. Differential Equations* **317** (2022), 196–224.
- [27] C. Mariconda and G. Treu, Lipschitz regularity for minima without strict convexity of the Lagrangian, *J. Differential Equations* **243** (2007), no. 2, 388–413.
- [28] V. Maz'ya, Lectures on isoperimetric and isocapacitary inequalities in the theory of Sobolev spaces, in: *Heat Kernels and Analysis on Manifolds, Graphs, and Metric Spaces* (Paris 2002), Contemp. Math. 338, American Mathematical Society, Providence (2003), 307–340.
- [29] G. J. Minty, Monotone (nonlinear) operators in Hilbert space, *Duke Math. J.* **29** (1962), 341–346.
- [30] G. J. Minty, On a “monotonicity” method for the solution of non-linear equations in Banach spaces, *Proc. Natl. Acad. Sci. USA* **50** (1963), 1038–1041.
- [31] J. S. Moll, The anisotropic total variation flow, *Math. Ann.* **332** (2005), no. 1, 177–218.
- [32] J. R. Philip, n -diffusion, *Austral. J. Phys.* **14** (1961), 1–13.
- [33] G. Pólya, Estimating electrostatic capacity, *Amer. Math. Monthly* **54** (1947), 201–206.
- [34] R. T. Rockafellar, On the maximal monotonicity of subdifferential mappings, *Pacific J. Math.* **33** (1970), 209–216.
- [35] R. Schneider, *Convex Bodies: The Brunn–Minkowski Theory*, 2nd ed., Encyclopedia Math. Appl. 151, Cambridge University, Cambridge, 2013.
- [36] S. Setzer, G. Steidl and T. Teuber, Restoration of images with rotated shapes, *Numer. Algorithms* **48** (2008), no. 1–3, 49–66.
- [37] R. E. Showalter, *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, Math. Surveys Monogr. 49, American Mathematical Society, Providence, 1997.
- [38] J. Simon, Régularité de la solution d'un problème aux limites non linéaires, *Ann. Fac. Sci. Toulouse Math. (5)* **3** (1981), no. 3–4, 247–274.
- [39] J. E. Taylor, Crystalline variational problems, *Bull. Amer. Math. Soc.* **84** (1978), no. 4, 568–588.
- [40] C. Xia, *On a class of anisotropic problem*, Ph.D. Thesis, Albert-Ludwigs University Freiburg, 2012.