ON ISOMETRIC EMBEDDINGS INTO THE SET OF STRONGLY NORM-ATTAINING LIPSCHITZ FUNCTIONS

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Abstract. In this paper, we provide an infinite metric space M such that the set $\mathrm{SNA}(M)$ of strongly norm-attaining Lipschitz functions on M does not contain a subspace which is linearly isometric to c_0. This answers a question posed by Antonio Avilés, Gonzalo Martínez-Cervantes, Abraham Rueda Zoca, and Pedro Tradacete. On the other hand, we prove that $\mathrm{SNA}(M)$ contains an isometric copy of c_0 whenever M is an infinite metric space which is not uniformly discrete. In particular, the latter holds true for all infinite compact metric spaces while it does not hold true for all proper metric spaces. We also provide some positive results in the non-separable setting.

Contents

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1. Introduction

Let M be a pointed metric space. We consider the subset $\mathrm{SNA}(M)$ of $\mathrm{Lip}_0(M)$ of all strongly norm-attaining Lipschitz functions on M. In 2016, Marek Cúth, Michal Doucha, and Przemysław Wojtaszczyk [8] proved that ℓ_∞ (and hence c_0) embeds isomorphically in $\mathrm{Lip}_0(M)$ for any infinite metric space M. One year later, this result was improved by Marek Cúth and Michal Johanis by showing that ℓ_∞ (and hence c_0) embeds isometrically in $\mathrm{Lip}_0(M)$ [9]. Motivated by the papers [1, 13, 19], we turn our attention to the study of the analogous problems for the subset $\mathrm{SNA}(M)$. In big contrast to what happens in the classical norm-attainment theory, where Martin Rmoutil [21] proved that the set of all norm-attaining functionals needs not contain 2-dimensional spaces, Antonio Avilés, Gonzalo Martínez Cervantes, Abraham Rueda Zoca, and Pedro Tradacete [1] provided a beautiful and interesting construction, and showed that $\mathrm{SNA}(M)$ contains an isomorphic copy of c_0 for every infinite complete metric space M (answering [19, Question 2] in the positive). At the very ending of that paper, the authors wondered whether an isometric version of this result holds true (see [1, Remark 3.6]).

In this paper, we answer the latter question by providing an example of an infinite uniformly discrete metric space M such that $\mathrm{SNA}(M)$ does *not* contain any subspace which is linearly isometric to c_0 (see Theorem 4.1). On the other hand, we prove that $\mathrm{SNA}(M)$ does contain a linearly isometric copy of c_0 whenever M is infinite but not uniformly discrete (see Theorem 4.2). It turns out that this is no longer true even for proper metric spaces (see Theorem 4.4). We conclude the paper by tackling the problem in the non-separable setting; we prove that whenever $\mathrm{dens}(M') = \Gamma > \aleph_0$, then $\mathrm{SNA}(M)$ contains a linearly isometric copy of $c_0(\Gamma)$ (see Theorem 5.2).

2. Preliminaries and notation

Throughout the text, all the vector spaces will be considered to be *real*. Let (M, d) be a pointed metric space (that is, a metric space with a distinguished point 0). We denote by $\mathrm{Lip}_0(M)$ the Banach space of all Lipschitz functions $f : M \to \mathbb{R}$ such that $f(0) = 0$, endowed with the Lipschitz norm

$$\|f\|_{\mathrm{Lip}} := \sup\left\{\frac{|f(y) - f(x)|}{d(x,y)} : x, y \in M,\, x \neq y\right\}.$$

We say that a Lipschitz function $f \in \mathrm{Lip}_0(M)$ *strongly attains its norm*, or that it is *strongly norm-attaining*, if there exist two different points $p, q \in M$ such that

$$\|f\|_{\mathrm{Lip}} = \frac{|f(p) - f(q)|}{d(p,q)}.$$

The set of strongly norm-attaining Lipschitz functions on M will be denoted by $\mathrm{SNA}(M)$. In the last few years, this topic has been intensively studied. We send the reader to [1, 2, 3, 4, 5, 6, 7, 11, 12, 14, 15, 17, 18, 19] and the references therein.

In this paper, we will discuss the possibility of finding a linear space isometrically isomorphic to c_0 inside the set $\mathrm{SNA}(M)$. Clearly, this situation is reduced only to infinite metric spaces. Note also that, for our purposes, the choice of the distinguished point 0 in the pointed metric space M is irrelevant in our context. Indeed, if 0 and $0'$ are two distinguished points in M, then the mapping from $\mathrm{Lip}_0(M)$ to $\mathrm{Lip}_{0'}(M)$ defined as $f \mapsto f - f(0')$ is a linear isometry that completely preserves the strong norm-attainment behaviour of the mappings, so we do not need to worry about the choice of the distinguished point.

In this document, the expression *linear subspaces of* $\mathrm{SNA}(M)$ should be understood as linear subspaces of $\mathrm{Lip}_0(M)$ consisting of strongly norm-attaining Lipschitz functions. Also, if Y is a Banach space, then we say that Y isometrically embeds in $\mathrm{SNA}(M)$ (or, equivalently, $\mathrm{SNA}(M)$ contains a linearly isometric copy of Y), whenever there exists a linear isometric embedding $U : Y \to \mathrm{Lip}_0(M)$ such that $U(Y) \subseteq \mathrm{SNA}(M)$.

The *separation radius* of a point $x \in M$ is defined by

$$R(x) := \inf \Big\{ d(x, y) \colon y \in M \setminus \{x\} \Big\},$$

and it will play an important role in some of the upcoming results. We will say that a point x from a metric space M *attains its separation radius* whenever there is some $y \in M$ such that $R(x) = d(x, y)$. The symbol M' stands for the set of all cluster points of M. Recall that a metric space M is said to be *discrete* if $M' = \emptyset$, *uniformly discrete* if $\inf\{R(x) \colon x \in M\} > 0$, and *proper* if every closed and bounded subset of M is compact. The notation $B(x, R)$ stands for the closed ball of center $x \in M$ and radius $R > 0$.

Throughout the entire note, we will denote by $c_0(\Gamma)$ the space of all real valued functions over a set Γ satisfying that for every element $x \in c_0(\Gamma)$, the set $\{\gamma \in \Gamma \colon |x(\gamma)| \geqslant \varepsilon\}$ is finite for every $\varepsilon > 0$. If Γ is countable, we will refer to $c_0(\Gamma)$ simply as c_0.

Let X be a separable Banach space with a Schauder basis denoted by $\{x_n\}_{n=1}^\infty$. We say that a sequence $\{y_n\}_{n=1}^\infty$ in a Banach space Y is *(isometrically) equivalent* to the basis $\{x_n\}_{n=1}^\infty$ if there exists a linear (isometric) isomorphism $T \colon \overline{\mathrm{span}}\{y_n \colon n \in \mathbb{N}\} \to X$ such that $T(y_n) = x_n$ for all $n \in \mathbb{N}$. The following straightforward facts will be used throughout the text without any explicit reference.

- (i) A sequence $\{x_n\}_{n=1}^\infty$ is isometrically equivalent to the canonical basis of c_0 if and only if the equality $\big\| \sum_{n=1}^\infty \lambda_n x_n \big\| = \max_n |\lambda_n|$ holds for every sequence $\{\lambda_n\}_{n=1}^\infty \in c_0$.
- (ii) If a sequence $\{x_n\}_{n=1}^\infty$ is isometrically equivalent to the canonical basis of c_0, then so is the sequence $\{\varepsilon_n x_n\}_{n=1}^\infty$, where $\varepsilon_n \in \{-1, 1\}$ for every $n \in \mathbb{N}$.
- (iii) Any subsequence of a sequence which is isometrically equivalent to the canonical basis of c_0 is once again isometrically equivalent to the same basis.

Given any set A and a natural number $k \in \mathbb{N}$, we denote by $A^{[k]}$ the set of all subsets of A with exactly k elements. We will use Ramsey's Theorem intensively throughout the text, which ensures that given any infinite set A and any finite partition of the set $A^{[k]}$, $\{B_1, \dots, B_n\}$ for some $n \in \mathbb{N}$, there exists an infinite subset S of A and a number $i \in \{1, \dots n\}$ such that $S^{[k]}$ is contained in B_i (see, for instance, [10, Proposition 6.4]).

Finally, let us note the following. If M is any metric space and M_c is its completion, then, as Lipschitz mappings extend uniquely by uniform continuity, one clearly has $\mathrm{Lip}_0(M) = \mathrm{Lip}_0(M_c)$. However, it can happen that a Lipschitz mapping strongly attains its norm on M_c but not on M, so $\mathrm{SNA}(M)$ can be a strict subset of $\mathrm{SNA}(M_c)$ sometimes. For this reason, we will state our main positive results for general metric spaces (complete or not) whenever possible.

3. Some useful tools

In this section, we state and prove some auxiliary results that will be crucial for the rest of the note. The following are three essential yet straightforward lemmas that hold in any metric space. We provide their proofs for the sake of completeness.

Lemma 3.1. *Let M be a metric space. Suppose that $\{f_n\}_{n=1}^\infty \subseteq \mathrm{Lip}_0(M)$ is a sequence isometrically equivalent to the canonical basis of c_0. If for every $n \in \mathbb{N}$, the function f_n strongly attains its Lipschitz norm at a pair of points $x_n, y_n \in M$, then $f_m(x_n) = f_m(y_n)$ for every $m \in \mathbb{N} \setminus \{n\}$.*

Proof. Let $n \in \mathbb{N}$ be fixed. Suppose that the function f_n strongly attains its Lipschitz norm at a pair of points $x_n, y_n \in M$. Without loss of generality, we may (and we do) assume that $|f_n(x_n) - f_n(y_n)| = f_n(x_n) - f_n(y_n) = d(x_n, y_n)$. Let us suppose by contradiction that there exist natural numbers $m \neq n$ such that $f_m(x_n) \neq f_m(y_n)$. We may again suppose without loss of generality that $f_m(x_n) > f_m(y_n)$ (otherwise we may consider the sequence $\{g_k\}_{k=1}^\infty$ defined as $g_m = -f_m$ and $g_k = f_k$ for $k \neq m$, which is still equivalent to the c_0 basis). Set $f := f_n + f_m$. Then, we have that

$$\begin{aligned} |f(x_n) - f(y_n)| &\geqslant (f_n + f_m)(x_n) - (f_n + f_m)(y_n) \\ &= f_n(x_n) - f_n(y_n) + f_m(x_n) - f_m(y_n) \\ &> d(x_n, y_n), \end{aligned}$$

which yields a contradiction with the fact that f is 1-Lipschitz. ∎

Lemma 3.2. *Let M be a metric space. Suppose that $\{f_n\}_{n=1}^\infty \subseteq \mathrm{Lip}_0(M)$ is a sequence equivalent to the canonical basis of c_0. Then, for all $p \in M$, $\lim_{n\to\infty} |f_n(p)| = 0$.*

Proof. Let $T\colon c_0 \to \overline{\mathrm{span}}\{f_n\colon n \in \mathbb{N}\}$ be a linear isomorphism with $T(e_n) = f_n$ for all $n \in \mathbb{N}$ and set $C = \|T\|$. Suppose that for some $p \in M$, the sequence $\{f_n(p)\}_{n=1}^\infty$ does not converge to 0. Then, there exists $N \in \mathbb{N}$ such that $\sum_{n=1}^N |f_n(p)| > C \cdot d(p, 0)$. However, this implies that there exist $\{\varepsilon_n\}_{n=1}^N \subseteq \{-1, 1\}^N$ such that the function $\sum_{k=1}^N \varepsilon_n f_n$ is not C-Lipschitz, contradicting the fact that the operator norm of T is C. ∎

Finally, for the upcoming positive results of the paper, we need the following generalization of [1, Lemma 3.1].

Lemma 3.3. *Let Γ be a nonempty index set. Let M be a pointed metric space such that there exist $\{(x_\gamma, y_\gamma)\}_{\gamma\in\Gamma} \subseteq M \times M$ with $x_\gamma \neq y_\gamma$ for every $\gamma \in \Gamma$ satisfying that*

$$d(x_\alpha, x_\beta) \geqslant d(x_\alpha, y_\alpha) + d(x_\beta, y_\beta) \tag{3.1}$$

for every $\alpha \neq \beta \in \Gamma$. Then there is a linear subspace of $\mathrm{SNA}(M)$ linearly isometric to $c_0(\Gamma)$.

Proof. Since the choice of the point 0 is not relevant as noted before, we may assume that $0 \in \{y_\gamma\}_{\gamma\in\Gamma}$. Pick $\gamma' \in \Gamma$ such that $y_{\gamma'} := 0$. For each $\gamma \in \Gamma$, define $f_\gamma : M \to \mathbb{R}$ by

$$f_\gamma(x) := \max\{0, d(x_\gamma, y_\gamma) - d(x, x_\gamma)\} \quad (x \in M).$$

Clearly, $f_{\gamma'}(0) = 0$. Also, if $\gamma \neq \gamma'$ we have by means of the triangle inequality,

$$d(x_\gamma, y_\gamma) \overset{(3.1)}{\leqslant} d(x_\gamma, x_{\gamma'}) - d(x_{\gamma'}, y_{\gamma'}) \leq d(x_\gamma, y_{\gamma'}).$$

Therefore, $f_\gamma(0) = \max\{0, d(x_\gamma, y_\gamma) - d(x_\gamma, y_{\gamma'})\} = 0$ for every $\gamma \in \Gamma$.

Notice that for every $\gamma \in \Gamma$, $f_\gamma(x) \neq 0$ if and only if x is such that $d(x_\gamma, x) < d(x_\gamma, y_\gamma)$. Also, notice that by (3.1), for every $\alpha \neq \beta$ in Γ, $B(x_\alpha, d(x_\alpha, y_\alpha)) \cap B(x_\beta, d(x_\beta, y_\beta))$ has empty interior.

Let $\lambda := (\lambda_\gamma)_{\gamma\in\Gamma} \in c_0(\Gamma)$ and let $\gamma_0 \in \Gamma$ be such that $|\lambda_{\gamma_0}| = \|\lambda\|_\infty$. Set $f = \sum_{\gamma\in\Gamma} \lambda_\gamma f_\gamma$. Clearly,

$$|f(x_{\gamma_0}) - f(y_{\gamma_0})| = |\lambda_{\gamma_0}| d(x_{\gamma_0}, y_{\gamma_0}).$$

Therefore, we will be done if we show that $\|f\|_{\mathrm{Lip}} \leqslant |\lambda_{\gamma_0}|$:

Let $x \neq y$ be two points in M. We will distinguish several cases.

(a) If both x and y lie outside of $\bigcup_{\gamma\in\Gamma} B(x_\gamma, d(x_\gamma, y_\gamma))$, then clearly $|f(x) - f(y)| = 0$.

(b) Assume that $x \notin \bigcup_{\gamma\in\Gamma} B(x_\gamma, d(x_\gamma, y_\gamma))$ and that there exists some $\alpha \in \Gamma$ such that $y \in B(x_\alpha, d(x_\alpha, y_\alpha))$. Since $0 \leqslant d(x_\alpha, y_\alpha) - d(x_\alpha, y) \leqslant d(x_\alpha, x) - d(x_\alpha, y) \leqslant d(x, y)$, we have

$$|f(x) - f(y)| = |\lambda_\alpha|(d(x_\alpha, y_\alpha) - d(x_\alpha, y)) \leqslant |\lambda_{\gamma_0}| d(x, y).$$

(c) Assume now that there is some $\gamma \in \Gamma$ such that $x, y \in B(x_\gamma, d(x_\gamma, y_\gamma))$. Then, by the triangle inequality,

$$\begin{aligned} |f(x) - f(y)| &= |\lambda_\gamma||(d(x_\gamma, y_\gamma) - d(x_\gamma, x)) - (d(x_\gamma, y_\gamma) - d(x_\gamma, y))| \\ &= |\lambda_{\gamma_0}|\, |d(x_\gamma, x) - d(x_\gamma, y)| \\ &\leq |\lambda_{\gamma_0}| d(x, y). \end{aligned}$$

(d) Finally, if there are different $\alpha, \beta \in \Gamma$ such that $x \in B(x_\alpha, d(x_\alpha, y_\alpha))$ and $y \in B(x_\beta, d(x_\beta, y_\beta))$, thanks to (3.1), we know that $d(x_\alpha, y_\alpha) \leqslant d(x_\alpha, x_\beta) - d(x_\beta, y_\beta)$. Hence, by means of this last inequality and the triangle inequality, we have that

$$\begin{aligned} d(x_\alpha, y_\alpha) - d(x_\alpha, x) + d(x_\beta, y_\beta) - d(x_\beta, y) &\leqslant d(x_\alpha, x_\beta) - d(x_\alpha, x) - d(x_\beta, y) \\ &\leqslant d(x, y). \end{aligned} \tag{3.2}$$

Then,

$$\begin{aligned} |f(x) - f(y)| &= |\lambda_\alpha f_\alpha(x) - \lambda_\beta f_\beta(y)| \\ &\leqslant |\lambda_{\gamma_0}|(f_\alpha(x) + f_\beta(y)) \\ &= |\lambda_{\gamma_0}|(d(x_\alpha, y_\alpha) - d(x_\alpha, x) + d(x_\beta, y_\beta) - d(x_\beta, y)) \\ &\overset{(3.2)}{\leqslant} |\lambda_{\gamma_0}| d(x, y). \end{aligned}$$

■

4. The isometric containment of c_0 in SNA(M)

In this section we turn our attention to the main results of the paper.

4.1. **A bounded and uniformly discrete counterexample.**

In this subsection, we construct an infinite complete metric space M such that the set SNA(M) of strongly norm-attaining Lipschitz functions does not contain a linearly isometric copy of c_0, answering a question posed in [1, Remark 3.6]. It is worth mentioning that no point of the constructed metric space attains its separation radius.

Theorem 4.1. *There exists an infinite bounded uniformly discrete metric space M such that c_0 is not isometrically contained in* SNA(M) *and for which no point in M attains its separation radius.*

Proof. Let $M = \{p_n\}_{n\in\mathbb{N}}$ be any countable set endowed with the metric d given by

$$d(p_n, p_m) = \begin{cases} 1 + \frac{1}{\max\{m,n\}} & \text{if } m \neq n, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the diameter of M is $3/2$.

For the sake of contradiction, let us suppose that there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ of strongly norm-attaining functions which is isometrically equivalent to the canonical basis of c_0. For each $n \in \mathbb{N}$, let $x_n, y_n \in M$ be such that $x_n \neq y_n$ and $\big|f_n(x_n) - f_n(y_n)\big| = d(x_n, y_n)$. Our goal is to find two natural numbers $n_0 \neq m_0$ and $\delta \in \{-1, 1\}$ such that the Lipschitz function $f_{n_0} + \delta f_{m_0}$ has Lipschitz norm strictly greater than 1. This will lead to a contradiction.

Let us consider the sets

$$\begin{aligned} A &:= \left\{\{n,m\} \in \mathbb{N}^{[2]} : \{x_n, y_n\} \cap \{x_m, y_m\} = \emptyset\right\}, \\ B_1 &:= \left\{\{n,m\} \in \mathbb{N}^{[2]} : x_n = x_m\right\}, \\ B_2 &:= \left\{\{n,m\} \in \mathbb{N}^{[2]} : y_n = y_m\right\}, \text{ and} \\ B_3 &:= \left\{\{n,m\} \in \mathbb{N}^{[2]} : x_n = y_m \text{ or } x_m = y_n\right\}. \end{aligned}$$

Note that, as an immediate consequence of Lemma 3.1, those sets form a partition of $\mathbb{N}^{[2]}$. By Ramsey's theorem, there exists $C \in \{A, B_1, B_2, B_3\}$ and an infinite set $S \subseteq \mathbb{N}$ such that $S^{[2]} \subseteq C$.

Case 1: $C = A$.

By restricting ourselves to S, we have that $\{x_n, y_n\} \cap \{x_m, y_m\} = \emptyset$ for every $n, m \in S$ with $n \neq m$. For each $n \in S$, let us set

$$\varepsilon_n := \frac{1}{2k(n)}, \quad \text{where} \quad k(n) := \max\{k \in S : p_k = x_n \quad \text{or} \quad p_k = y_n\}.$$

Let us fix some $n_0 \in S$. Since $\{x_n, y_n\} \cap \{x_m, y_m\} = \emptyset$ for every $n, m \in S$ with $n \neq m$, by Lemma 3.2 and the definition of the metric d, there exists $m_0 \in S \setminus \{n_0\}$ such that

(*i*) $\max\{|f_{m_0}(x_{n_0})|, |f_{m_0}(y_{n_0})|\} \leqslant \dfrac{\varepsilon_{n_0}}{3}$ and

(*ii*) $\max\{d(x_{n_0}, x_{m_0}), d(x_{n_0}, y_{m_0}), d(y_{n_0}, x_{m_0}), d(y_{n_0}, y_{m_0})\} \leqslant 1 + \dfrac{\varepsilon_{n_0}}{3}$.

Now, by Lemma 3.1, there is a constant $C_{m_0} \in \mathbb{R}$ such that $f_{n_0}(x_{m_0}) = f_{n_0}(y_{m_0}) = C_{m_0}$. Since all the previous inequalities still hold if we relabel any of the pairs (x_{n_0}, y_{n_0}) or (x_{m_0}, y_{m_0}), we may assume that

$$|f_{n_0}(x_{n_0}) - C_{m_0}| \geqslant |f_{n_0}(y_{n_0}) - C_{m_0}| \qquad \text{and} \qquad |f_{m_0}(x_{m_0})| \geqslant |f_{m_0}(y_{m_0})|.$$

Note that the triangle inequality yields that

$$|f_{n_0}(x_{n_0}) - C_{m_0}| + |f_{n_0}(y_{n_0}) - C_{m_0}| \geqslant d(x_{n_0}, y_{n_0}) \text{ and } |f_{m_0}(x_{m_0})| + |f_{m_0}(y_{m_0})| \geqslant d(x_{m_0}, y_{m_0}),$$

and so, under our previous assumption we get that

$$|f_{m_0}(x_{m_0})| \geqslant \frac{1}{2} + \varepsilon_{m_0} \qquad \text{and} \qquad |f_{n_0}(x_{n_0}) - C_{m_0}| \geqslant \frac{1}{2} + \varepsilon_{n_0}. \tag{4.1}$$

In particular, $f_{m_0}(x_{m_0}) \neq 0$. Set now $\delta := \frac{|f_{m_0}(x_{m_0})|}{f_{m_0}(x_{m_0})} \in \{-1, 1\}$. To finish the proof of this case, we distinguish two possibilities depending on the sign of $f_{n_0}(x_{n_0}) - C_{m_0}$:

If $f_{n_0}(x_{n_0}) < C_{m_0}$, consider the function $f = f_{n_0} + \delta f_{m_0}$, which is 1-Lipschitz by assumption. However, using properties (i) and (ii) and equation (4.1) we obtain that

$$\begin{aligned}|f(x_{n_0}) - f(x_{m_0})| &\geqslant -f_{n_0}(x_{n_0}) - \delta f_{m_0}(x_{n_0}) + f_{n_0}(x_{m_0}) + \delta f_{m_0}(x_{m_0}) \\ &> \frac{1}{2} + \varepsilon_{n_0} + \frac{1}{2} - |f_{m_0}(x_{n_0})| > d(x_{n_0}, x_{m_0}),\end{aligned}$$

a contradiction.

On the other hand, if $f_{n_0}(x_{n_0}) \geqslant C_{m_0}$, an analogous procedure shows that the function $g = f_{n_0} - \delta f_{m_0}$ has a Lipschitz norm greater than 1 witnessed by the same pair (x_{n_0}, x_{m_0}). This again yields a contradiction.

Case 2: $C = B_1$.

Restricting ourselves to S once more, note that there exists $k^* \in S$ such that $x_n = p_{k^*}$ for every $n \in S$. By Lemma 3.1, we have that $y_n \neq y_m$ for every $n, m \in S$ with $n \neq m$. Using Lemma 3.2, we may find $n_0, m_0 \in S$ with $n_0 \neq m_0$ such that

$$|f_{n_0}(x_{n_0})| \leqslant \frac{1}{10} \qquad \text{and} \qquad |f_{m_0}(x_{m_0})| \leqslant \frac{1}{10}, \tag{4.2}$$

and so, by the triangle inequality, we have that

$$|f_{n_0}(y_{n_0})| \geqslant \frac{9}{10} \qquad \text{and} \qquad |f_{m_0}(y_{m_0})| \geqslant \frac{9}{10}. \tag{4.3}$$

Since $x_{n_0} = x_{m_0}$, another consequence of equation (4.2) together with Lemma 3.1 is that

$$|f_{n_0}(y_{m_0})| \leqslant \frac{1}{10} \qquad \text{and} \qquad |f_{m_0}(y_{n_0})| \leqslant \frac{1}{10}. \tag{4.4}$$

Changing signs of f_{n_0} and f_{m_0} if necessary, we may assume that $f_{n_0}(y_{n_0}) > 0$ and $f_{m_0}(y_{m_0}) > 0$. Finally, consider the function $f := f_{n_0} - f_{m_0}$, which is 1-Lipschitz by the assumption on the sequence $\{f_n\}_{n\in\mathbb{N}}$. However, applying (4.3) and (4.4), and recalling that the diameter of M is $3/2$ we obtain that

$$\begin{aligned}|f(y_{n_0}) - f(y_{m_0})| &\geqslant f_{n_0}(y_{n_0}) + f_{m_0}(y_{m_0}) - f_{m_0}(y_{n_0}) - f_{n_0}(y_{m_0}) \\ &\geqslant \frac{9}{5} - (|f_{m_0}(y_{n_0})| + |f_{n_0}(y_{m_0})|) > d(y_{n_0}, y_{m_0}).\end{aligned}$$

We have then a contradiction and the proof of this case is over.

Case 3: $C = B_2$.

If $C = B_2 = \big\{\{n,m\} \in \mathbb{N}^{[2]} \colon y_n = y_m\big\}$ and $S^{[2]} \subset B_2$, defining $x'_n = y_n$ and $y'_n = x_n$ for all $n \in \mathbb{N}$ we observe that $S^{[2]} \subset \big\{\{n,m\} \in \mathbb{N}^{[2]} \colon x'_n = x'_m\big\}$, so this case can be solved in an analogous way to the case $C = B_1$.

Case 4: $C = B_3$. We will show that this case cannot happen. Suppose for the sake of contradiction that $S^{[2]} \subset B_3 = \big\{\{n,m\} \in \mathbb{N}^{[2]} : x_n = y_m \text{ or } x_m = y_n\big\}$, and fix $n_0 \in S$. Defining $S_1 = \{m \in S \setminus \{n_0\} \colon x_m = y_{n_0}\}$ and $S_2 = \{m \in S \setminus \{n_0\} \colon y_m = x_{n_0}\}$ we obtain that

$$S_1^{[2]} \subset \big\{\{n,m\} \in \mathbb{N}^{[2]} \colon x_n = x_m = y_{n_0}\big\} \subset B_1,$$
$$S_2^{[2]} \subset \big\{\{n,m\} \in \mathbb{N}^{[2]} \colon y_n = y_m = x_{n_0}\big\} \subset B_2.$$

The sets S_1 and S_2 must be singletons. Indeed, suppose first that there existed $n_1 \neq n_2 \in S_1$. Then $x_{n_1} = x_{n_2}$, and since $\{n_1, n_2\} \in B_3$, one of $x_{n_1} = y_{n_2}$ or $y_{n_1} = x_{n_2}$ holds. This implies that either $x_{n_1} = y_{n_1}$ or $x_{n_2} = y_{n_2}$, which is a contradiction with the fact that the functions f_{n_1} and f_{n_2} strongly attain their Lipschitz norm at the corresponding pairs of points. Similarly, we obtain that S_2 is a singleton as well.

Finally, since for each $m \in S \setminus \{n_0\}$ we have that either $x_{n_0} = y_m$ or $x_m = y_{n_0}$, it follows that $S = S_1 \cup S_2 \cup \{n_0\}$, and thus the cardinality of S is at most 3. This is a contradiction with the choice of S. ■

It is worth mentioning that there exist countable bounded uniformly discrete metric spaces M with the condition that no point x in M attains its separation radius, but such that c_0 embeds isometrically in $\mathrm{SNA}(M)$. Indeed, it suffices to consider a countable collection $\{M_n\}_{n\in\mathbb{N}}$ of copies of the previous space in such a way that $d(M_n, M_m) = 3$ for all different $n, m \in \mathbb{N}$, and observe that, in this context, Lemma 3.3 applies. This means that the aforementioned property is not sufficient for c_0 not to be contained in $\mathrm{SNA}(M)$ isometrically.

Likewise, one could be tempted to assume that the condition of not attaining the separation radii is at least necessary in negative results like Theorem 4.1. However, this is far from being true as well. In fact, later in this section we will exhibit a proper unbounded uniformly discrete metric space M such that c_0 cannot be embedded in $\mathrm{SNA}(M)$ isometrically (see Theorem 4.4). In particular, every point of M attains its separation radius, since closed bounded sets in M are compact. On the other hand, we will see in the next subsection that the property of being uniformly discrete is indeed necessary in order to get such negative results (see Theorem 4.2).

4.2. **Non uniformly discrete metric spaces.** Let us move on to the main positive result of the paper. Going in the opposite direction of Theorem 4.1, here we show that we can *always* embed c_0 isometrically in $\mathrm{SNA}(M)$ whenever M is infinite but not uniformly discrete. Recall that M' is the set of cluster points of M. Let us remark that the complete metric spaces such that the set of cluster points is either empty or infinite were already covered in [1], and we do not address these cases at all in the proof.

Theorem 4.2. *Let M be an infinite non uniformly discrete metric space. Then, the set $\mathrm{SNA}(M)$ contains a linearly isometric copy of c_0.*

Proof. Assume first that M is complete. By [1, Theorems 3.2 and 3.4] it suffices to assume that M' is non-empty and finite. We can also assume without loss of generality that $0 \in M'$. Now we can find a sequence $\{x_n\}_{n\in\mathbb{N}}$ in M converging to 0 such that $R(x_n) > 0$ for all $n \in N$. It is clear that the sequence $\{R(x_n)\}_{n\in\mathbb{N}}$ converges to 0.

We are going to define a sequence $\{f_k\}_{k\in\mathbb{N}}$ of 1-Lipschitz functions in $\mathrm{SNA}(M)$ which will be isometrically equivalent to the canonical basis of c_0 and such that the subspace $\overline{\mathrm{span}}\{f_k : k \in \mathbb{N}\}$ (which is linearly isometric to c_0) is contained in $\mathrm{SNA}(M)$.

We define the following sets:

$$A := \big\{\{n, m\} \in \mathbb{N}^{[2]} : d(x_n, x_m) \geqslant R(x_n) + R(x_m)\big\},$$
$$B := \big\{\{n, m\} \in \mathbb{N}^{[2]} : d(x_n, x_m) < R(x_n) + R(x_m)\big\},$$

which form a partition of $\mathbb{N}^{[2]}$. By Ramsey's theorem, there is $C \in \{A, B\}$ and an infinite subset $S \subseteq \mathbb{N}$ such that $S^{[2]} \subseteq C$. These two possibilities give us two separate cases.

Case 1: $C = A$.

Consider the subset $\{x_n\}_{n\in S}$, which satisfies that $d(x_n, x_m) \geqslant R(x_n) + R(x_m)$ for all $n \neq m \in S$. Assume first that there is an infinite subset of S, which we denote by S_0, such that $R(x_n)$ is attained for every $n \in S_0$. Consider now for each $n \in S_0$ an element $y_n \in M$ such that $d(x_n, y_n) = R(x_n)$. It is straightforward to see that the sequences $\{x_n\}_{n\in S_0}, \{y_n\}_{n\in S_0}$ satisfy the assumptions of Lemma 3.3 and we are done.

Otherwise, by passing to a subsequence if necessary, we may assume that $R(x_n)$ is not attained for any $n \in S$. Let us then choose inductively a sequence $\{a_k\}_{k\in\mathbb{N}}$ among the elements of the sequence $\{x_n\}_{n\in S}$ satisfying that for every $k \in \mathbb{N}$,

$$d(a_k, 0) \leqslant \frac{d(a_j, 0) - R(a_j)}{3} \quad \forall j < k. \tag{4.5}$$

For the sake of clarity, let us denote $\Delta_k = \frac{d(a_k,0)-R(a_k)}{3}$ for every $k \in \mathbb{N}$. It is clear from (4.5) that $\{\Delta_k\}_{k=1}^{\infty}$ is a decreasing sequence. Now, by definition of $R(a_k)$, we deduce that for every $k \in \mathbb{N}$, there is some $b_k \in B(a_k, R(a_k) + \Delta_k)$. Finally, let us prove that the sequences $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$ are under the assumptions of Lemma 3.3. Pick $n, m \in \mathbb{N}$ with $n < m$. Clearly, the following expressions hold

$$\begin{aligned} d(a_n, 0) =& R(a_n) + 3\Delta_n, \quad d(a_m, 0) \leqslant \Delta_n, \quad d(a_n, b_n) \leqslant R(a_n) + \Delta_n \\ & d(a_m, b_m) \leqslant R(a_m) + \Delta_m \leqslant R(a_m) + 3\Delta_m = d(a_m, 0) \leqslant \Delta_n. \end{aligned} \tag{4.6}$$

Hence, by (4.6) we have that

$$d(a_n, a_m) \geqslant d(a_n, 0) - d(a_m, 0) \geqslant R(a_n) + 2\Delta_n \geq d(a_n, b_n) + d(a_m, b_m).$$

This finishes the first case.

Case 2: $C = B$.

Since the set S is infinite and the sequence $\{x_n\}_{n\in S}$ is convergent, we can inductively define a pair of sequences $\{a_k\}_{k\in\mathbb{N}} \subseteq \{x_n\}_{n\in S}$ and $\{b_k\}_{k\in\mathbb{N}} \subseteq \{x_n\}_{n\in S}$ satisfying the following properties:

(*i*) $R(a_k) < \varepsilon_j/2$, for $j,k\in\mathbb{N}$ with $j<k$, where $\varepsilon_j = R(a_j)+R(b_j)-d(a_j,b_j) > 0$.
(*ii*) $R(b_k) < R(a_k)/2$ for every $k\in\mathbb{N}$.

Fixed $k\in\mathbb{N}$, we define $f_k : M\to\mathbb{R}$ by

$$f_k(p) = \begin{cases} R(a_k) - \dfrac{\varepsilon_k}{2} & \text{if } p = a_k, \\ -R(b_k) + \dfrac{\varepsilon_k}{2} & \text{if } p = b_k, \\ 0 & \text{otherwise.} \end{cases}$$

Property (i) and the definition of ε_k ensure that $f_k(a_k) \geqslant 0$ and $f_k(b_k) \leqslant 0$ for every $k\in\mathbb{N}$. With this, we obtain that

$$|f_k(a_k)| = R(a_k) - \frac{\varepsilon_k}{2}, \qquad \text{and} \qquad |f_k(b_k)| = R(b_k) - \frac{\varepsilon_k}{2}, \qquad \text{for all } k\in\mathbb{N}. \tag{4.7}$$

Let $\{\lambda_k\}_{k\in\mathbb{N}} \in c_0$. Again we will show that $f := \sum_{k\in\mathbb{N}} \lambda_k f_k \in \mathrm{SNA}(M)$ and also that $\|f\|_{\mathrm{Lip}} = \max_{k\in\mathbb{N}}\{|\lambda_k|\}$. Choose $k_0\in\mathbb{N}$ such that $|\lambda_{k_0}| = \max_{k\in\mathbb{N}}\{|\lambda_k|\}$.

We again start by proving that f is $|\lambda_{k_0}|$-Lipschitz. Take $p,q\in M$ with $p\neq q$. We will show that $|f(p)-f(q)| \leqslant |\lambda_{k_0}| d(p,q)$. If both p and q form a pair $\{a_k,b_k\}$ for some $k\in\mathbb{N}$, the previous inequality is clear. We need to study now the two remaining possibilities:

(a) Suppose that there exist $k_1,k_2\in\mathbb{N}$ with $k_1<k_2$ such that $p\in\{a_{k_1},b_{k_1}\}$ and $q\in\{a_{k_2},b_{k_2}\}$. Then, by (4.7) we have in particular that $|f(p)| = |\lambda_{k_1}|\left(R(p)-\frac{\varepsilon_{k_1}}{2}\right)$ and $|f(q)| \leqslant |\lambda_{k_2}| R(a_{k_2})$. Hence, we obtain that
$$\begin{aligned} |f(p)-f(q)| \leqslant& |\lambda_{k_0}|\cdot\left(R(p) - \frac{\varepsilon_{k_1}}{2} + R(a_{k_2})\right) \\ \leqslant& |\lambda_{k_0}|\cdot R(p) \leqslant |\lambda_{k_0}|\cdot d(p,q). \end{aligned}$$
(b) If $p\in M\setminus\{x_k\}_{k\in\mathbb{N}}$, then $f(p)=0$ and, using (4.7) again, we have that
$$|f(p)-f(q)| = |f(q)| \leqslant |\lambda_{k_0}|\cdot R(q) \leqslant |\lambda_{k_0}|\cdot d(p,q).$$

We have proven then that the Lipschitz norm of f is smaller or equal than $|\lambda_{k_0}|$. Finally, considering the pair of points a_{k_0} and b_{k_0}, we quickly observe that $\|f\|_{\mathrm{Lip}} = |\lambda_{k_0}|$ and that f strongly attains its Lipschitz norm at this pair of points. This finishes the proof for complete metric spaces.

Finally, assume now that M is not complete and let M_c be its completion. Note that if M_c' is empty or infinite, we already have the result using again the constructions from [1, Theorems 3.2 and 3.4]. Otherwise, if M_c' is finite, it suffices to do the same procedure we described but on M_c instead of M, and also asking to our original sequence $\{x_n\}_{n\in\mathbb{N}}$ to be in M. Then, the constructed sequence of functions $\{f_k\}_{k\in\mathbb{N}}$ from $\mathrm{SNA}(M_c)$ will still be strongly norm-attaining when restricted to M. Finally, it suffices to consider the functions $f - f(0_M)$ if needed (for instance, if our original limit point 0 was in $M_c \setminus M$, which means that it cannot be the distinguished point from the original space M). ■

It is an immediate consequence of [1, Theorem 3.3] that if M is any infinite compact metric space, then c_0 is isomorphically embedded into $\mathrm{SNA}(M)$ (for countable compact metric spaces this was achieved non constructively using the little Lipschitz space). Note that our previous theorem provides a constructive proof that for any infinite compact metric space M with a finite amount of cluster points, $\mathrm{SNA}(M)$ actually contains c_0 isometrically.

Corollary 4.3. *Let M be an infinite compact metric space. Then, the subset $\mathrm{SNA}(M)$ contains a linearly isometric copy of c_0.*

4.3. **A proper and uniformly discrete counterexample.** To finish this section, we show that Corollary 4.3 cannot be improved to include all proper metric spaces. Indeed, we have the following result.

Theorem 4.4. *There exists an infinite proper uniformly discrete metric space M such that c_0 is not isometrically contained in $\mathrm{SNA}(M)$ and for which every point in M attains its separation radius.*

Proof. Let $M = \{p_k\}_{k=0}^{\infty}$, with distinguished point $p_0 = 0$, be a countable set endowed with the metric $d\colon M \times M \to \mathbb{R}$ given by:

$$d(p_k, p_j) = \begin{cases} k + j - \varepsilon_{\max\{k,j\}} & \text{if } k \neq j \in \mathbb{N} \setminus \{0\}, \\ k & \text{if } j = 0, \\ j & \text{if } k = 0, \\ 0 & \text{if } j = k, \end{cases}$$

where $\{\varepsilon_k\}_{k\in\mathbb{N}}$ is a sequence of positive numbers such that $\varepsilon_{k+1} > \varepsilon_k$ and $\varepsilon_k < 1/2$ for all $k \in \mathbb{N}$. For convenience, write $\delta_k = \varepsilon_{k+1} - \varepsilon_k > 0$ for all $k \in \mathbb{N}$. It is clear that M is proper since every bounded set is finite.

As in the proof of Theorem 4.1, we start by assuming that there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions in $\mathrm{SNA}(M)$ isometrically equivalent to the canonical basis of c_0, and we are going to find two natural numbers $n_0 \neq m_0$ such that $f_{n_0} - f_{m_0}$ is not 1-Lipschitz, which will yield a contradiction. For each $n \in \mathbb{N}$, since f_n is strongly norm-attaining, we may consider two points $x_n \neq y_n$ such that $|f_n(x_n) - f_n(y_n)| = d(x_n, y_n)$.

We write $k(n)$ and $j(n)$ to denote the natural numbers such that $x_n = p_{k(n)}$ and $y_n = p_{j(n)}$ for every $n \in \mathbb{N}$. By relabelling the pairs (x_n, y_n) if needed, we may assume that $k(n) < j(n)$ for all $n \in \mathbb{N}$.

We now define the sets A, B_1, B_2, and B_3 as in the proof of Theorem 4.1. By Ramsey's theorem, there exists $C \in \{A, B_1, B_2, B_3\}$ and an infinite set $S \subseteq \mathbb{N}$ such that $S^{[2]} \subseteq C$.

Case 1: $C = A$.

In this case, choose an arbitrary $n_0 \in S$ such that $k(n_0), j(n_0) \neq 0$. By Lemma 3.2, and using that $\{x_n, y_n\} \cap \{x_m, y_m\} = \emptyset$ for all $n \neq m \in S$, we can find $m_0 \in S$ with $k(m_0) > j(n_0)$ such that

$$|f_{m_0}(x_{n_0})| < \frac{1}{2}\delta_{j(n_0)} \qquad \text{and} \qquad |f_{m_0}(y_{n_0})| < \frac{1}{2}\delta_{j(n_0)}. \tag{4.8}$$

Using Lemma 3.1 we can define $C_{m_0} \in \mathbb{R}$ such that $C_{m_0} = f_{n_0}(x_{m_0}) = f_{n_0}(y_{m_0})$. Using the triangle inequality, we know that

$$|f_{n_0}(x_{n_0}) - C_{m_0}| + |f_{n_0}(y_{n_0}) - C_{m_0}| \geqslant d(x_{n_0}, y_{n_0}),$$

and so, we obtain that either

(a_0) $|f_{n_0}(x_{n_0}) - C_{m_0}| \geqslant k(n_0) - \frac{1}{2}\varepsilon_{j(n_0)}$, or
(a_1) $|f_{n_0}(y_{n_0}) - C_{m_0}| \geqslant j(n_0) - \frac{1}{2}\varepsilon_{j(n_0)}$.

Similarly, since $|f_{m_0}(x_{m_0})| + |f_{m_0}(y_{m_0})| \geqslant d(x_{m_0}, y_{m_0})$ by the triangle inequality, we get that either

(b_0) $|f_{m_0}(x_{m_0})| \geqslant k(m_0) - \big(\frac{1}{2}\varepsilon_{j(n_0)} + \frac{1}{2}\delta_{j(n_0)}\big)$, or
(b_1) $|f_{m_0}(y_{m_0})| \geqslant j(m_0) - \big(\varepsilon_{j(m_0)} - \frac{1}{2}\varepsilon_{j(n_0)} - \frac{1}{2}\delta_{j(n_0)}\big)$.

In total, there are now 4 different possibilities that must be checked for contradiction. We will only expand on the two possibilities where (a_0) holds, since the two remaining possibilities (where (a_1) holds) are proven similarly. Hence, suppose first that (a_0) and (b_0) hold. By changing the signs of f_{n_0} and f_{m_0} if necessary, we may suppose that $f_{n_0}(x_{n_0}) - C_{m_0} \geqslant k(n_0) - \frac{1}{2}\varepsilon_{j(n_0)}$ and $f_{m_0}(x_{m_0}) \geqslant k(m_0) - \big(\frac{1}{2}\varepsilon_{j(n_0)} + \frac{1}{2}\delta_{j(n_0)}\big)$. Consider the function $f = f_{n_0} - f_{m_0}$, which is 1-Lipschitz since we are assuming that $\{f_n\}_{n\in\mathbb{N}}$ is isometrically equivalent to the canonical basis of c_0. However, using (4.8), we have that

$$\begin{aligned} |f(x_{n_0}) - f(x_{m_0})| &\geqslant f_{n_0}(x_{n_0}) - f_{m_0}(x_{n_0}) - C_{m_0} + f_{m_0}(x_{m_0}) \\ &> k(n_0) - \frac{1}{2}\varepsilon_{j(n_0)} - \frac{1}{2}\delta_{j(n_0)} + k(m_0) - \frac{1}{2}\varepsilon_{j(n_0)} - \frac{1}{2}\delta_{j(n_0)} \\ &\geqslant k(n_0) + k(m_0) - \varepsilon_{k(m_0)} = d(x_{n_0}, x_{m_0}), \end{aligned}$$

which yields a contradiction. Suppose now that (a_0) and (b_1) hold. Again we may suppose that $f_{n_0}(x_{n_0}) - C_{m_0} \geqslant k(n_0) - \frac{1}{2}\varepsilon_{j(n_0)}$ and $f_{m_0}(y_{m_0}) \geqslant j(m_0) - \big(\varepsilon_{j(m_0)} - \frac{1}{2}\varepsilon_{j(n_0)} - \frac{1}{2}\delta_{j(n_0)}\big)$. Using (4.8) again, the 1-Lipschitz function $f = f_{n_0} - f_{m_0}$ now tested at the pair (x_{n_0}, y_{m_0})

yields

$$\begin{aligned}|f(x_{n_0}) - f(y_{m_0})| &> k(n_0) - \frac{1}{2}\varepsilon_{j(n_0)} - \frac{1}{2}\delta_{j(n_0)} + j(m_0) - \varepsilon_{j(m_0)} + \frac{1}{2}\varepsilon_{j(n_0)} + \frac{1}{2}\delta_{j(n_0)}\\ &= k(n_0) + j(m_0) - \varepsilon_{j(m_0)} = d(x_{n_0}, y_{m_0}),\end{aligned}$$

which is again a contradiction. This finishes the proof for Case 1.

Case 2: $C = B_1$.

Write k^* to denote the non-negative integer (including 0) such that $p_{k^*} = x_n$ for all $n \in S$. Suppose first that $k^* = 0$. Then, choose any two different numbers $n_0 \neq m_0 \in S$. Since both f_{n_0} and f_{m_0} strongly attain their norm, at the pairs $(0, y_{n_0})$ and $(0, y_{m_0})$ respectively, and both f_{n_0} and f_{m_0} vanish at 0, we have that $|f_{n_0}(y_{n_0})| = j(n_0)$ and $|f_{m_0}(y_{m_0})| = j(m_0)$. With Lemma 3.1 we obtain that $f_{n_0}(y_{m_0}) = f_{m_0}(y_{n_0}) = 0$. By changing the signs of both functions if needed, we may suppose that $f_{n_0}(y_{n_0}) = j(n_0)$ and $f_{m_0}(y_{m_0}) = j(m_0)$, producing a contradiction directly by considering the mapping $f = f_{n_0} - f_{m_0}$, which is not 1-Lipschitz as witnessed by the pair (y_{n_0}, y_{m_0}). Indeed,

$$|f(y_{n_0}) - f(y_{m_0})| = j(n_0) + j(m_0) > d(y_{n_0}, y_{m_0}).$$

Suppose now that $k^* \neq 0$. Using Lemma 3.2, choose two different natural numbers $n_0 \neq m_0 \in S$ with $j(m_0) > j(n_0) > k^*$ such that

$$|f_{n_0}(p_{k^*})| < \frac{1}{4} \qquad \text{and} \qquad |f_{m_0}(p_{k^*})| < \frac{1}{4}.$$

On the one hand, these inequalities imply, by the triangle inequality, that $|f_{n_0}(y_{n_0})| > k^* + j(n_0) - \varepsilon_{j(n_0)} - \frac{1}{4}$ and $|f_{m_0}(y_{m_0})| > k^* + j(m_0) - \varepsilon_{j(m_0)} - \frac{1}{4}$, while, on the other hand, they imply by Lemma 3.1 that

$$|f_{n_0}(y_{m_0})| < \frac{1}{4} \qquad \text{and} \qquad |f_{m_0}(y_{n_0})| < \frac{1}{4}.$$

Finally, we may again suppose without loss of generality that f_{n_0} and f_{m_0} are both positive at the points y_{n_0} and y_{m_0} respectively, and consider the function $f = f_{n_0} - f_{m_0}$, which is assumed to be 1-Lipschitz. However, we have that

$$\begin{aligned}|f(y_{n_0}) - f(y_{m_0})| &\geqslant f_{n_0}(y_{n_0}) - f_{m_0}(y_{n_0}) - f_{n_0}(y_{m_0}) + f_{m_0}(y_{m_0})\\ &\geqslant j(n_0) + j(m_0) + 2k^* - 1 - \varepsilon_{j(n_0)} - \varepsilon_{j(m_0)}\\ &> j(n_0) + j(m_0) > d(y_{n_0}, y_{m_0}),\end{aligned}$$

a contradiction. This finishes the proof of Case 2.

Case 3: $C = B_2$.

We will show that this case is impossible. Suppose, for the sake of contradiction, that $S^{[2]} \subseteq B_2$. Then for all $m, n \in S$, $y_n = y_m = p_{j^*}$ for some fixed $j^* \in \mathbb{N}$. Since $k(n) < j(n)$ for all $n \in \mathbb{N}$, we obtain that $k(n) < j^*$ for all $n \in S$. The set S is infinite, so there exist

$n \neq m \in S$ such that $x_n = x_m$ and $y_n = y_m$. It follows that the functions f_n and f_m from the basis strongly attain their norms at the same pair of points, contradicting Lemma 3.1.

Case 4: $C = B_3$.

Similarly to Case 4 of the proof of Theorem 4.1, if $S \subset \mathbb{N}$ satisfies $S^{[2]} \subset B_3$, the cardinality of S is at most 3. Hence, Ramsey's Theorem cannot apply to B_3 either and the proof is finished. ■

5. The non-separable case

In this section we tackle the problem of embedding $c_0(\Gamma)$ in $\mathrm{SNA}(M)$ isometrically, where Γ is an arbitrary set of large cardinality. Let us first introduce some basic concepts and results of set theory that will be heavily used in this section.

We denote by $\mathrm{dens}(M)$ the *density character* of a metric space M, defined as the smallest cardinal Γ such that there is a dense subset of M of cardinality Γ. The *cofinality* $\mathrm{cof}(\alpha)$ of an ordinal α is the smallest ordinal β such that $\alpha = \sup_{\gamma<\beta} \alpha_\gamma$, where $\{\alpha_\gamma\}_{\gamma<\beta}$ is an ordinal sequence with $\alpha_\gamma < \alpha$ for all $\gamma < \beta$. A cardinal Γ is *regular* if $\mathrm{cof}(\Gamma) = \Gamma$. Following the notation of [20], for an ordinal α, we denote by α^+ the least cardinal strictly bigger than α, which is always a regular cardinal (see [20, Lemma 10.37]). We again refer the reader to [20] for a comprehensive background on this topic. Finally, recall that a subset of a metric space $S \subseteq M$ is called *r-separated* for some $r > 0$ whenever $d(x, y) \geqslant r$ for all $x \neq y \in S$.

The next result is essentially based on the proof of [16, Proposition 3].

Proposition 5.1. *Let M be a metric space with $\mathrm{dens}(M) = \Gamma$ for some cardinal Γ. Then, there exists a discrete set $L \subseteq M$ with $\mathrm{card}(L) = \Gamma$. Moreover, if $\mathrm{cof}(\Gamma) > \aleph_0$, then L can be chosen to be uniformly discrete.*

Proof. Note that if Γ is finite, the result is trivial, so we will only prove the case where Γ is infinite. Note also that it is well known that if M is any infinite Hausdorff topological space (in particular if M is an infinite metric space), then it always contains an infinite discrete set, so the case where $\Gamma = \aleph_0$ is already finished. Hence, we will assume now that Γ is uncountable. For every $k \in \mathbb{N}$, let M_k be some maximal $\frac{1}{2^k}$-separated subset of M. Denote $\Gamma_k := |M_k|$ for all $k \in \mathbb{N}$, and note that $\overline{\bigcup_{k\in\mathbb{N}} M_k} = M$, and so, $\sup_{k\in\mathbb{N}} \Gamma_k = \Gamma$. If $\mathrm{cof}(\Gamma) > \aleph_0$, then, since $\overline{\bigcup_{k\in\mathbb{N}} M_k} = M$, we have that there is $k_0 \in \mathbb{N}$ such that $\Gamma_{k_0} = \Gamma$ and so we take $L := M_{k_0}$.

Now, let us assume that $\mathrm{cof}(\Gamma) = \aleph_0$. If there exists $k_0 \in \mathbb{N}$ such that $\Gamma_{k_0} = \Gamma$, we are done, since we can take once again $L := M_{k_0}$. On the other hand, if this is not the case, we have that $\Gamma_k < \Gamma$ for every $k \in \mathbb{N}$ and $\mathrm{cof}(\Gamma) = \aleph_0$. Since Γ is not regular, we know that $\Gamma_k^+ < \Gamma$, for every $k \in \mathbb{N}$. Using this, and the fact that $\sup_{k\in\mathbb{N}} \Gamma_k = \Gamma$, it is straightforward to inductively construct a subsequence $\{\Gamma_{k_n}\}_{n\in\mathbb{N}}$ of $\{\Gamma_k\}_{k\in\mathbb{N}}$ with Γ_{k_1} infinite and such that

$\Gamma_{k_n}^+ < \Gamma_{k_{n+1}}$ for all $n \in \mathbb{N}$. Notice that since $\{\Gamma_k\}_{k\in\mathbb{N}}$ is an increasing sequence, we have that $\sup_{n\in\mathbb{N}} \Gamma_{k_n} = \Gamma$.

Now, for each $n \in \mathbb{N}$, let us consider a sequence of sets $\{\widetilde{M}_n\}_{n\in\mathbb{N}}$ such that $\widetilde{M}_n$ is a subset of $M_{k_{n+1}}$ with $|\widetilde{M}_n| = \Gamma_{k_n}^+$ for all $n \in \mathbb{N}$. Let us write $\widetilde{M}_n = \{x_\alpha^n : \alpha \in \Gamma_{k_n}^+\}$.

For each $n \in \mathbb{N}$, each $j \leqslant n$, and each $\alpha \in \Gamma_{k_j}^+$, we define

$$A_{j,\alpha}^n := \widetilde{M}_{n+1} \cap B\left(x_\alpha^j, \frac{1}{2^{k_{j+1}+1}}\right).$$

We will inductively construct, for every $n \in \mathbb{N}$, a set $L_n \subseteq \widetilde{M}_n$ with $|L_n| = \Gamma_{k_n}^+$ and a finite subset $N_n \subseteq M$ such that whenever $j < n$,

$$d(L_n, L_j \setminus N_n) \geqslant \frac{1}{2^{k_{j+1}+2}}. \tag{5.1}$$

Set $L_1 := \widetilde{M}_1$ and $N_1 := \emptyset$. Now, assuming that for some $n \in \mathbb{N}$ we have constructed L_j and N_j for all $j \leqslant n$, we can do the inductive step towards $n+1$.

(a) Suppose that $|A_{j,\alpha}^n| < \Gamma_{k_{n+1}}^+$ for every $j \leqslant n$ and $\alpha \in \Gamma_{k_n}^+$. Since $\Gamma_{k_n}^+ < \Gamma_{k_{n+1}}^+$ and $\Gamma_{k_{n+1}}^+$ is regular, we have that

$$\left| \bigcup_{j\leqslant n,\, \alpha\in\Gamma_{k_j}^+} A_{j,\alpha}^n \right| < \Gamma_{k_{n+1}}^+ = \operatorname{card}(\widetilde{M}_{n+1}).$$

Therefore, the set

$$L_{n+1} := \widetilde{M}_{n+1} \setminus \bigcup_{j\leqslant n,\, \alpha\in\Gamma_{k_j}^+} A_{j,\alpha}^n$$

satisfies $|L_{n+1}| = \Gamma_{k_{n+1}}^+$ and (5.1) holds by setting $N_{n+1} = \emptyset$. Indeed, for any $j \in \{1, \dots, n\}$, every point in L_j is of the form x_α^j for some $\alpha \in \Gamma_{k_j}^+$. Hence, if there exists a point $p \in L_{n+1}$ such that $d(p, x_\alpha^j) < \frac{1}{2^{k_{j+1}+2}}$, then p belongs to the set $A_{j,\alpha}^n$, which leads to a contradiction with the definition of L_{n+1}.

(b) Suppose now that $|A_{j_0,\alpha_0}^n| = \Gamma_{k_{n+1}}^+$ for some $j_0 \leqslant n$ and some $\alpha_0 \in \Gamma_{k_{j_0}}^+$. Without loss of generality we consider $j_0 \in \{1, \dots, n\}$ to be such that $|A_{j,\alpha}^n| < \Gamma_{k_{n+1}}^+$ for all $j_0 < j \leqslant n$ and all $\alpha \in \Gamma_{k_j}^+$. Define

$$L_{n+1} := A_{j_0,\alpha_0}^n \setminus \bigcup_{j_0<j\leqslant n,\, \alpha\in\Gamma_{k_j}^+} A_{j,\alpha}^n.$$

Arguing as in case (a), we obtain that $|L_{n+1}| = \Gamma_{k_{n+1}}^+$. Finally, define

$$N_{n+1} := \left\{ x \in M \colon \exists i \in \{1, \dots, j_0\} \text{ such that } x \in L_i \text{ and } d(x, L_{n+1}) < \frac{1}{2^{k_{i+1}+2}} \right\}$$

which is finite since for each $i \in \{1, \dots, j_0\}$, there can only be at most a single point x_i in L_i such that $d(x_i, L_{n+1}) < \frac{1}{2^{k_{i+1}+2}}$. Indeed, if $i = j_0$, the only point in L_{j_0} that can satisfy that property is $x_{\alpha_0}^{j_0}$, since for every $\beta \in \Gamma_{k_{j_0}}^+ \setminus \{\alpha_0\}$, $d(x_\beta^{j_0}, A_{j_0,\alpha_0}^n) \geqslant \frac{1}{2^{k_{j_0+1}+1}}$. On the other hand, if $i < j_0$, if there were two points $x_i \neq y_i \in L_i$ with that property, we would have that

$$d(x_i, y_i) \leqslant d(x_i, A_{j_0,\alpha_0}^n) + d(y_i, A_{j_0,\alpha_0}^n) + \operatorname{diam}(A_{j_0,\alpha_0}^n) < \frac{1}{2^{k_{i+1}}},$$

a contradiction with the fact that L_i is $\frac{1}{2^{k_{i+1}}}$-separated.

Let us check that the sets L_{n+1} and N_{n+1} satisfy equation (5.1) for each $j \in \{1, \dots, n\}$. Fix $j \in \{1, \dots, n\}$. If $j \leqslant j_0$ then the inequality follows directly by definition of N_{n+1}. Otherwise, if $j > j_0$, then the inequality holds following the same argument as in case (a).

Having discussed both possibilities, the induction is finished. To finish the proof, set $L := \left(\bigcup_{n\in\mathbb{N}} L_n\right) \setminus \left(\bigcup_{n\in\mathbb{N}} N_n\right)$. It is clear that $|L| = \Gamma$, and using equation (5.1), it is straightforward to prove that all convergent sequences in L are eventually constant, and thus L is discrete. ∎

As an application of Lemma 3.3 and Proposition 5.1, we have the following isometric result.

Theorem 5.2. *Let M be a pointed metric space such that $\operatorname{dens}(M') = \Gamma$ for some infinite cardinal Γ. Then there is a linear subspace of $\mathrm{SNA}(M)$ that is isometrically isomorphic to $c_0(\Gamma)$.*

Proof. The case where $\Gamma = \aleph_0$ is already covered in [1, Theorem 3.2]. Assume now that $\Gamma > \aleph_0$. If we apply Proposition 5.1 to the set M', we find a discrete set $L \subseteq M'$ with $\operatorname{card}(L) = \operatorname{dens}(L) = \Gamma$ and such that all points of L are cluster points of M. Finally, Lemma 3.3 can be applied now if we consider $\{x_\gamma\}_{\gamma\in\Gamma}$ to be L itself and for each $\gamma \in \Gamma$, we set y_γ to be sufficiently close to x_γ. ∎

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