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Photorefractive four-wave mixing cannot be treated perturbatively. Revisiting the undepleted-pump approximation

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Abstract

We theoretically consider photorefractive degenerate four-wave mixing oscillators and show that, for this particular system, the linear stability analysis technique leads to incorrect results. The reason for this astonishing failure lies in the unphysical predictions of the undepleted-pump approximation, which appear naturally during the linearization process of the full model. As a consequence, photorefractive four-wave mixing does not seem to permit a perturbative treatment that allows the derivation of mean-field models, and one is forced to use the full model in order to make sensible predictions.

Supplementary material for this article is available [online](#)

Keywords: wave mixing, photorefractive oscillators, undepleted-pump approximation

(Some figures may appear in colour only in the online journal)

1. Introduction

In modeling nonlinear optical cavities, the derivation of mean-field models in which the axial coordinate z can be removed is often pursued. In this way, one gets systems of partial differential equations in $(x, y; t)$, which further simplify to ordinary differential equations for so-called single-mode configurations [1–3]. Moreover, these systems of equations often permit the application of standard perturbative techniques, which enables us to connect them with universal order parameter equations that are very useful for describing dynamical systems close to bifurcation points [4, 5].

Photorefractive oscillators, in both two-wave and four-wave mixing (degenerate as well as non-degenerate) configurations, have been heavily used as nonlinear optical cavities for the study of nonlinear dynamics, both theoretically and experimentally [3, 6, 7]. However, in these studies, there is a striking difference between two-wave mixing and four-wave mixing; mean-field models have been derived only for the former [8–11], never for the latter. In fact, the theory used for interpreting the experimental results obtained with degenerate four-wave mixing photorefractive resonators has always been heuristic [12–15].

In order to fill this gap, we have tried to derive a mean-field model for photorefractive four-wave mixing through a perturbative treatment of the full model equations that assumes small variations of the fields along a single cavity round-trip. This is the usual procedure and, in particular, the one successfully applied to two-wave mixing resonators. However, the predictions of the derived mean-field model contradict basic experimental facts of four-wave mixing resonators as detailed

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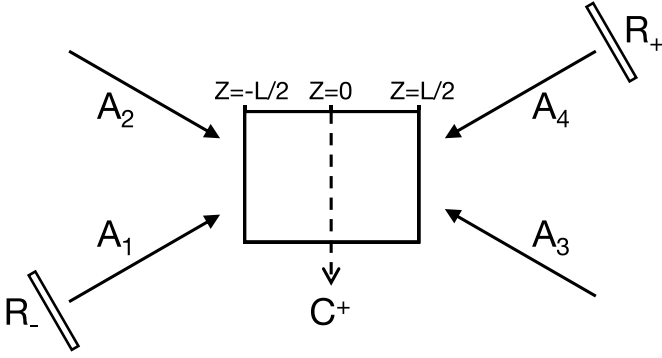


Figure 1. Scheme of a photorefractive degenerate four-wave mixing oscillator. A_2 and A_3 denote the slowly varying amplitudes of the pumping fields, and A_1 and A_4 denote those of the intracavity fields.

below. In this article, we reflect on the rationale for this surprising failure, and show that it lies in the fact that the oscillation threshold of photorefractive degenerate four-wave mixing does not permit a perturbative treatment since the linear stability analysis of its trivial state leads to incorrect results.

In essence, this occurs because the perturbative treatment breaks energy conservation, which has catastrophic consequences for this particular system, since it introduces non-physical solutions, i.e. solutions not supported by the original full model or by experiment. We show below that these catastrophic consequences are intimately connected with the undepleted-pump approximation (UPA), which is a widely used approximation that usually leads to sensible results, but not for photorefractive four-wave mixing. The connection with the UPA appears when one realizes that the perturbative treatment of the full model naturally leads to evolution equations under the UPA, and it is the UPA that introduces the already mentioned spurious solutions.

The situation can be summarized as follows: (i) the full model has steady states that fit with experimental results. In particular, these steady states correctly predict that a photorefractive degenerate four-wave mixing cavity like the one shown in figure 1 can work without one of its mirrors (the right one in figure 1) but not without the other (the left one in figure 1). The difference between removing one or the other mirror is due to the existence of a c^+ -axis in the crystal, which introduces an asymmetry that implies that there is phase conjugation gain in only one of the resonator arms. (ii) In contrast, the UPA model has steady states that are symmetric with respect to the cavity mirror reflectivities, hence it fails to correctly describe the asymmetry captured by the full model. (iii) The linear stability analysis of the UPA model is in agreement with its steady states, hence it is in contradiction with experiment. (iv) The linear stability analysis of the full model is exactly the same as that of the UPA model, which is an astonishing result. This means that the linear stability analysis of the full model cannot correctly capture the emission properties, since it predicts destabilization of the trivial state to a nontrivial state when a nontrivial state does not exist. In other words, even if the full model has the correct steady states, the

destabilization of the trivial state cannot be correctly captured by its linear stability analysis.

Finally, we conclude that the perturbative treatment of the full model leads to incorrect predictions regarding the emission properties of the degenerate four-wave mixing cavity. Therefore, the derivation of a mean-field model seems impossible for this particular system, and we find no alternative to the use of the full model equations in order to study the system's nonlinear dynamics.

Below, we demonstrate the above assertions both theoretically and experimentally. In making our analysis, we use the standard theory of photorefractive four-wave mixing, which has been well established since the 1980s [16, 17], see also [18–20]. Below, we closely follow section 4.1 of Pochi Yeh's monograph [20] (with minor changes in the notation), even reproducing some of its results for the purpose of clarity. We also extend Yeh's treatment by further taking into account the time evolution of the refractive index grating, which allows us to study the linear stability analysis of the system's trivial solution. Even if this is the usual way to understand how emission starts in nonlinear optical cavities [1, 2], it is precisely this linear stability analysis that shows, in a striking way, the previously mentioned difficulties in predicting emission under conditions where a nontrivial steady state does not exist, which we consider to be a remarkable failure.

After this introduction, the rest of the article is organized as follows: in section 2, we present the standard model of a photorefractive degenerate four-wave mixing oscillator with transmission gratings [20], then briefly consider two-wave mixing and present experimental results. In section 3, we analyze the UPA of the four-wave mixing problem, which we show provides some unphysical solutions incompatible with experiment. Then, in section 4, we analyze the complete model (its nontrivial solutions as well as the stability analysis of the trivial state) and show that the perturbative treatment of this complete model irremediably leads back to the UPA, with its unphysical predictions; we also connect this fact with the breaking of energy conservation during propagation. Finally, in section 5, we present our main conclusions.

2. Model of photorefractive four-wave mixing oscillator with transmission gratings

Consider the situation depicted in figure 1 in which degenerate four-wave mixing occurs via transmission gratings. It is well established that, for a sufficiently slow photorefractive medium, the dynamical equations for the system can be written as [20],

$$\frac{\partial}{\partial z} \begin{pmatrix} A_1 \\ A_2^* \end{pmatrix} = \mp G \begin{pmatrix} A_2 \\ A_1^* \end{pmatrix}, \quad (1a)$$

$$\frac{\partial}{\partial z} \begin{pmatrix} A_3 \\ A_4^* \end{pmatrix} = \pm G \begin{pmatrix} A_4 \\ A_3^* \end{pmatrix}, \quad (1b)$$

$$\frac{1}{\gamma} \frac{\partial}{\partial t} G = -G + \frac{\Gamma}{2} \frac{A_1 A_2^* + A_3 A_4^*}{I_0}, \quad (1c)$$

where $A_1(z, t)$ and $A_4(z, t)$ are the slowly varying field amplitudes of the intracavity fields, and $A_2(z, t)$ and $A_3(z, t)$ are those of the pumping fields, all of which have the same optical frequency, $I_0 = \sum_{i=1}^4 |A_i|^2$, and $G(z, t)$ is proportional to the complex amplitude of the refractive index grating. Note that our G is that used in Yeh's monograph [20] multiplied by $\frac{\Gamma}{2}$. In the above equations, Γ is the coupling constant and γ is the grating decay constant, which are both real for a photorefractive crystal working by diffusion only and become complex when a bias field is applied.

It is worth stressing that, in deriving the above model equations, the existence of a c^+ -axis in the crystal is a vital ingredient, characteristic of the photorefractive effect, which is responsible for the different role played by the pumping fields with respect to the signal ones. Note, in particular, that while the field A_3 effectively pumps the intracavity field A_4 (and hence G is a positive gain for field A_4), field A_2 does not provide direct gain for field A_1 (and hence G is now a positive gain for field A_2 and a negative one for A_1).

It can be shown that equations (1) have a number of conserved quantities with respect to z , namely,

$$c_1 = |A_1|^2 + |A_2|^2, \quad c_2 = |A_3|^2 + |A_4|^2, \quad (2a)$$

$$\sigma = c_1 - c_2, \quad I_0 = c_1 + c_2, \quad (2b)$$

$$c = A_1 A_4 + A_2 A_3, \quad s = \sqrt{\sigma^2 + 4|c|^2}, \quad (2c)$$

are all constant with respect to z . This is particularly relevant when finding the nontrivial steady state [20], see section 4 below.

In order to incorporate the presence of the cavity mirrors, the model equations must be complemented with appropriate boundary conditions that, for a Fabry–Perot cavity, can be written as,

$$A_4\left(\frac{L}{2}, t\right) = R_+ e^{i\delta_+} A_1\left(\frac{L}{2}, t - t_+\right), \quad (3a)$$

$$A_1\left(-\frac{L}{2}, t\right) = R_- e^{i\delta_-} A_4\left(-\frac{L}{2}, t - t_-\right), \quad (3b)$$

with t_+ (t_-) the time that light takes to go from $z = -L/2$ to the left mirror (from $z = L/2$ to the right mirror) and back, see figure 1, and $\delta_{\pm}$ the correspondingly acquired phases.

2.1. Two-wave mixing

It is instructive to briefly consider two-wave mixing in order to gain some physical insight into the problem. Equations (1) contain two embedded two-wave mixing problems, one for $A_1 = A_2 = 0$ and another for $A_3 = A_4 = 0$ (see figure 1). However, the two subsystems cannot be more different. When $A_1 = A_2 = 0$, pumping field A_3 effectively pumps the intracavity field A_4 (i.e. G is a positive gain for field A_4 . Note that A_4 travels along the negative z -direction, see equation (1b)). In contrast, in the case $A_3 = A_4 = 0$, field A_2 does not pump the

intracavity field A_1 but gets energy from it, as G is a positive gain for A_2 and a negative one for A_1 , see equation (1a). This difference is obviously due to the different orientation of the crystal c^+ -axis with respect to the cavity mode directions.

2.2. Experimentals results

From the above simple considerations, one would expect that in the case of intracavity four-wave mixing, as in figure 1, the system could still work if one removes the right cavity mirror because, even without it, the intracavity field A_4 would get gain from four-wave mixing and from pumping field A_3 through two-wave mixing. One would expect that the cavity cannot work at all without the left mirror, because field A_2 does not provide two-wave mixing gain for A_1 , and any four-wave mixing gain for the A_2 field could hardly compensate for this fact. Even if this way of thinking is quite naive, it agrees with experiment. While the right mirror in figure 1 can be removed and the cavity still emits, the left mirror cannot be removed without turning the cavity off.

We have tested experimentally the above predictions by using a degenerate four-wave mixing photorefractive oscillator very similar to that described in [7], where all technical aspects are described in detail. Our experimental results are shown in the video accompanying this paper, see figure 2.

3. Undepleted-pump four-wave mixing model

Before analyzing the complete four-wave mixing model, it is most convenient to consider a simplified version for later use. In the limit of large pump intensities and small gain for the signal fields, at first sight it would seem reasonable to assume that there is no significant pump depletion, i.e. taking A_2 and A_3 as constants. One is then left with a much simpler model:

$$\frac{\partial}{\partial z} \begin{pmatrix} A_1 \\ A_4^* \end{pmatrix} = -G \begin{pmatrix} A_2 \\ A_3^* \end{pmatrix}, \quad (4a)$$

$$\frac{1}{\gamma} \frac{\partial}{\partial t} G = -G + \frac{\Gamma}{2} \frac{A_1 A_2^* + A_3 A_4^*}{I_0}, \quad (4b)$$

where, as stated, A_2 and A_3 are now constants. This is the UPA degenerate four-wave mixing model. Note that these are linear equations, and they can be used to understand the behavior of the system only for low gain.

3.1. Single-pass steady states

We closely follow [20] in this subsection. Equations (4) possess the trivial steady state $A_1 = A_4^* = G = 0$, and a nontrivial steady state ($\partial_t G = 0$). Here, we study the propagation of the fields through the nonlinear crystal by solving,

$$\frac{\partial}{\partial z} A_1 = -\frac{\Gamma |A_2|^2}{2I_0} A_1 - \frac{\Gamma A_2 A_3}{2I_0} A_4^*, \quad (5a)$$

$$\frac{\partial}{\partial z} A_4^* = -\frac{\Gamma |A_3|^2}{2I_0} A_4^* - \frac{\Gamma A_2^* A_3^*}{2I_0} A_1, \quad (5b)$$

whose solution reads,

$$A_1(z) = \bar{A}_{1L} f_+(z) - \bar{A}_{4R}^* \frac{A_3}{A_2^*} g_-(z), \quad (6a)$$

$$A_4^*(z) = \bar{A}_{4R}^* g_+(z) - \bar{A}_{1L} \frac{A_3^*}{A_2^*} f_-(z), \quad (6b)$$

with

$$f_{\pm} = \frac{q e^{-\frac{\Gamma}{2}L} \pm e^{-\frac{\Gamma}{2}(z+\frac{L}{2})}}{1 + q e^{-\frac{\Gamma}{2}L}}, g_{\pm} = \frac{1 \pm e^{-\frac{\Gamma}{2}(z+\frac{L}{2})}}{1 + q e^{-\frac{\Gamma}{2}L}}, \quad (7)$$

where we introduced the notation,

$$\bar{A}_{1L} = A_1(-L/2), \quad (8a)$$

$$\bar{A}_{4R}^* = A_4^*(L/2), \quad (8b)$$

$$q \equiv |A_3|^2 / |A_2|^2, \quad (8c)$$

for the signal fields at the left and right crystal faces, as well as the defined quotient of pump intensities. Let us analyze this result in two special cases.

- (i) On the one hand, when $\bar{A}_{4R}^* = 0$, we can readily obtain the *reflectivity* of the left side of the cavity:

$$\rho_L = \frac{A_4(-L/2)}{A_1^*(-L/2)} = \frac{A_3}{A_2^*} \frac{1 - e^{-\frac{\Gamma}{2}L}}{1 + q e^{-\frac{\Gamma}{2}L}}, \quad (9)$$

which is the phase-conjugation reflectivity. Note that $\bar{A}_{4R}^* = 0$ corresponds to the removal of the *right* mirror in figure 1 and the result (9) implies that the cavity could work without this right mirror.

- (ii) On the other hand, and this case is not considered in [20], one could also consider the special case $\bar{A}_{1L} = 0$, in which case one obtains the reflectivity of the right side of the cavity. The result reads as follows:

$$\rho_R = \frac{A_1(L/2)}{A_4^*(L/2)} = -\rho_L, \quad (10)$$

with ρ_L given by (9). However, $\bar{A}_{1L} = 0$ corresponds to the removal of the *left* mirror in figure 1. Hence (10) is an astonishing result because one suspects that the cavity cannot work without the left mirror because the intracavity field has no two-wave mixing gain, as has already been discussed in section 2. This reflectivity ρ_L in the absence of the left mirror should be identically null, as observed in the experiment, see figure 2.

In order to gain a little more insight into solution (6), we represent, in figure 3, the intensities $I_1 = |A_1|^2$ and $I_4 = |A_4|^2$ from (6). Note that the intensities only depend on ΓL , on the quotient of pump intensities q , and on the boundary conditions and that they can be represented as a dimensional quantity. In the figure, we choose $q = 1$ and $\Gamma L = 5$, but this is

not critical. In figure 3(a), we see how the intensities evolve inside the crystal when there is no injection from the right side, $A_{4R} = 0$, and there is a small injection on the left side, $A_{1L} = 0.1$. This would correspond to the removal of the right mirror in figure 1. Note that I_1 decreases when propagating to the right, while I_4 increases when propagating to the left. This is satisfactory because field A_4 has a positive two-wave mixing gain, while A_1 has not, and it is consistent with experiment. Now, in figure 3(b), we consider the same parameters but now with $A_{1L} = 0$ and $A_{4R} = 0.1$, which would correspond to the removal of the left mirror in figure 1. We see that now, different to figure 3(a), the two intracavity fields grow monotonically when propagating, which looks unphysical and, in any case, is incompatible with experiment. (We show below that it is also incompatible with the analytical steady state of the full model.) In figure 4, we show the total intensity, i.e. $I_0 = I_1 + I_2 + I_3 + I_4$ corresponding to the same two cases, and we see that even if I_0 is not conserved in either of the cases, it is always larger for case $A_{1L} = 0$ (no left mirror) than for case $A_{4R} = 0$ (no right mirror). The non-depletion of the pump as the signals increase allows for an unrealistic amount of energy, which permits the existence of a second, unphysical solution.

3.2. Stability of the trivial solution

Even if one of the propagating solutions just seen seems to be incorrect, it is worth performing the stability analysis of the trivial solution of the simplified model, which will be of later use. For that, we write,

$$G(z, t) = \bar{G}(z) + \delta G(z, t), \quad (11)$$

with δG a small perturbation, and take the intracavity field amplitudes as small perturbations too, i.e. $A_{1,4}(z) = \delta A_{1,4}(z)$. After linearizing (4) by neglecting products of perturbations, one readily obtains $\delta G(z, t) = e^{\lambda t} \delta G(z)$, with,

$$\delta G = \frac{1}{2} \frac{\gamma \Gamma}{\gamma + \lambda} \frac{A_2^* \delta A_1 + A_3 \delta A_4^*}{I_2 + I_3}, \quad (12)$$

which substituted into (5) leads to,

$$\frac{d}{dz} \begin{pmatrix} \delta A_1 \\ \delta A_4^* \end{pmatrix} = \hat{\mathcal{L}} \begin{pmatrix} \delta A_1 \\ \delta A_4^* \end{pmatrix}, \quad (13a)$$

$$\hat{\mathcal{L}} = -K(\lambda) \begin{pmatrix} \sin^2 \theta & \frac{1}{2} \sin 2\theta \\ \frac{1}{2} \sin 2\theta & \cos^2 \theta \end{pmatrix}, \quad (13b)$$

$$K(\lambda) \equiv \frac{1}{2} \frac{\gamma \Gamma}{\gamma + \lambda}, \quad (13c)$$

where we have introduced the pumping parameters,

$$\sin \theta = \sqrt{\frac{I_2}{I_2 + I_3}}, \cos \theta = \sqrt{\frac{I_3}{I_2 + I_3}}. \quad (14)$$

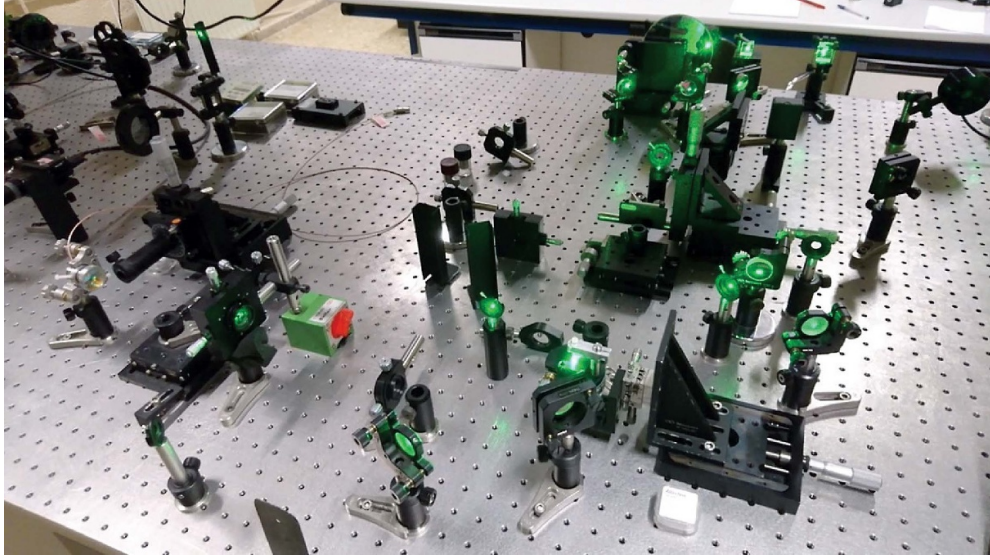


Figure 2. Video showing the different behavior of the photorefractive oscillator in response to blocking of the left or the right mirror. Experiment has been carried out with a linear cavity containing an SBN crystal pumped by a laser @ 532 nm. See supplementary material. For details about the setup, cavity stabilization procedure, interferometry and data processing, see [14].

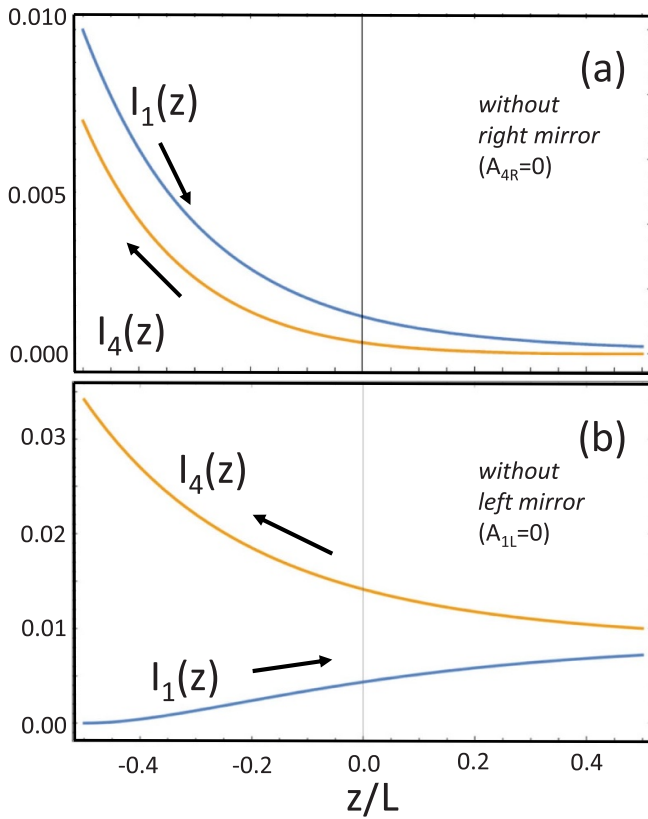


Figure 3. Evolution of the undepleted-pump model signal fields' intensities (6) along z for $\Gamma L = 5$, $I_2 = I_3 = 1$ and $A_{1L} = 0.1$ in (a) and $A_{4R}^* = 0.1$ in (b). Note that while in (a) the intensity I_1 decreases and the intensity I_4 increases during propagation, in (b) the two intensities increase during propagation, which is unphysical.

Equation (13) has the following solution:

$$\begin{pmatrix} \delta A_1(z) \\ \delta A_4^*(z) \end{pmatrix} = \hat{M}(z) \begin{pmatrix} \delta A_1(0) \\ \delta A_4^*(0) \end{pmatrix}, \quad (15)$$

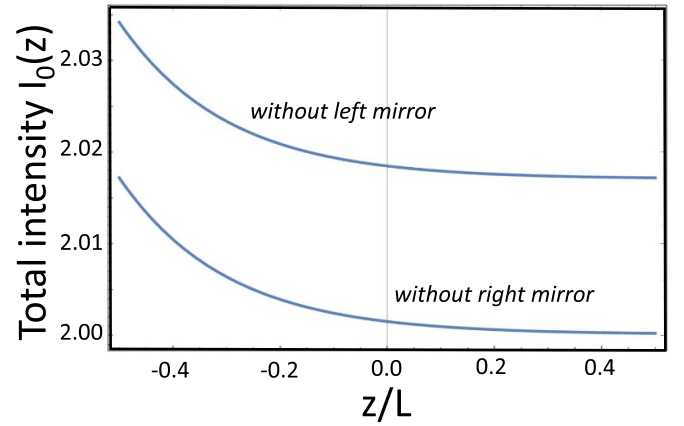


Figure 4. Evolution of the field intensities according to the undepleted-pump model, see 6, along z for $\Gamma L = 5$, $I_2 = I_3 = 1$, and $A_{1L} = 0.1$ (lower curve) and $A_{4R}^* = 0.1$ (upper curve).

where $\hat{M}(z) \equiv \exp \hat{\mathcal{L}}z$, whose matrix elements read,

$$M_{11} = e^{-K(\lambda)z} \sin^2 \theta + \cos^2 \theta, \quad (16a)$$

$$M_{22} = e^{-K(\lambda)z} \cos^2 \theta + \sin^2 \theta, \quad (16b)$$

$$M_{12} = M_{21} = \frac{1}{2} \left[e^{-K(\lambda)z} - 1 \right] \sin 2\theta. \quad (16c)$$

From (15), one easily obtains,

$$\begin{pmatrix} \delta A_1(L/2) \\ \delta A_4^*(L/2) \end{pmatrix} = \hat{M}(L) \begin{pmatrix} \delta A_1(-L/2) \\ \delta A_4^*(-L/2) \end{pmatrix}. \quad (17)$$

For equations that, complemented with the boundary conditions (3) applied to δA_i , form a linear system of eight equations

for the perturbations $(\delta A_1, \delta A_1^*, \delta A_4, \delta A_4^*)$, both at $z = L/2$ and $z = -L/2$, their solvability condition can be shown to be,

$$0 = a_1 \exp[-(x + \tilde{x})] + a_2 \exp(-x) + a_2^* \exp(-\tilde{x}) + a_3, \quad (18)$$

with

$$x = -K(\lambda)L, \quad (19a)$$

$$\tilde{x} = -K^*(\lambda^*)L, \quad (19b)$$

and coefficients,

$$a_1 = (\cos^2 \theta - R_-^2 \sin^2 \theta) (\cos^2 \theta - R_+^2 \sin^2 \theta), \quad (20a)$$

$$a_2 = \frac{1}{2} (1 + R_-^2) (1 + R_+^2) \sin^2 2\theta - R_+ R_- e^{i\delta}, \quad (20b)$$

$$a_3 = (\sin^2 \theta - R_-^2 \cos^2 \theta) (\sin^2 \theta - R_+^2 \cos^2 \theta), \quad (20c)$$

where $\delta = \delta_+ + \delta_-$, see (3), is the cavity detuning. Equation (18) is the characteristic equation of the eigenvalues λ .

We see that the solvability condition (18) is symmetric under the swapping $R_+ \longleftrightarrow R_-$, which is consistent with the nontrivial steady states that we have discussed above, but is as inconsistent with experiments as those incorrect steady states. Hence, the undepleted-pump model cannot correctly describe the physics of the system, and we must consider the full model.

4. Analysis of the complete model

Next, we turn back to the complete model (1) and perform a linear stability analysis of its trivial solution. As before, we consider perturbations $A_{1,4}(z) = \delta A_{1,4}(z)$ and also $A_{2,3}(z) = \bar{A}_{2,3} + \delta A_{2,3}(z)$ together with (11). Then, substituting into equations (1), at order δA_i one obtains... exactly the same equations as in the UPA, equations (13). In other words the perturbations of the pump fields do not enter into the linearized evolution of the intracavity modes and the linear stability of the complete model turns out to be the same as that of the UPA model of the previous section.

However, this unexpected result (that seems to confirm the unrealistic predictions of the undepleted-pump model), the nontrivial solution of the complete model agrees with the observations, as we intend to show.

The nontrivial solution of equations (1) can be written in the form [20],

$$\tanh^{-1} \frac{2c^* u(z) - \sigma}{s} = \frac{s\Gamma^*}{4I_0} \left(z + \frac{L}{2} \right) + D_1, \quad (21a)$$

$$\tanh^{-1} \frac{2c^* v(z) + \sigma}{s} = -\frac{s\Gamma^*}{4I_0} \left(z + \frac{L}{2} \right) + D_2, \quad (21b)$$

$$v(z) = \frac{A_4(z)}{A_1^*(z)}, \quad u(z) = \frac{A_2(z)}{A_3^*(z)}, \quad (21c)$$

where c, s, σ and I_0 are the constants defined in equations (2), and D_1 and D_2 are integration constants to be fixed by boundary conditions. By plugging these solutions into the boundary conditions (3), one determines the integration constants D_1 and D_2 and then one determines the steady states, which we do not give explicitly.

Instead, we now analyze this result in the same two special cases we considered with the UPA model.

- (i) First, when $A_{4R}^* = 0$ (that applies when the *right* mirror in figure 1 is removed) we see that $v(L/2) = A_4(L/2)/A_1^*(L/2) = 0$, and by solving (21b) for $z = L/2$ one obtains the integration constant,

$$D_2 = \frac{s\Gamma^*L}{4I_0} + \tanh^{-1} \frac{\sigma}{s}, \quad (22)$$

so that the reflectivity on the left side $\rho = v(-L/2)$ finally reads [20],

$$\rho = \frac{A_4(-L/2)}{A_1^*(-L/2)} = \frac{2c \tanh \kappa L}{s + \sigma \tanh \kappa L}, \quad \kappa = \frac{s\Gamma^*}{4I_0}, \quad (23)$$

which can be easily shown to consistently reduce to (9) in the undepleted-pump limit.

- (ii) Consider now the case $A_{1L} = 0$ (that applies when the left mirror in figure 1 is removed). We first rewrite equation (21b) in the form,

$$2c^* = \frac{s \tanh D_2 - \sigma}{v(-L/2)} = \frac{(s \tanh D_2 - \sigma) A_1^*(-L/2)}{A_4(-L/2)}, \quad (24)$$

and then conclude that $c = 0$ when $A_1^*(-L/2) = A_{1L} = 0$, since the first term on the right-hand side is bounded. However, $c = 0$ implies $A_1 = A_4 = 0$, so that a nontrivial steady state does not exist when $A_{1L} = 0$. This result agrees with experiment and is in contradiction with both the UPA model and the linear stability analysis of the full model.

Thus, the steady state of the complete model is in agreement with experiment, as it should be, and in disagreement with the undepleted-pump model so that the latter gets definitively disproved. However, and most surprisingly, there remains a contradiction in the full model between the steady state and the linear stability of the trivial state. We find that this is a most remarkable result. The linearization of the trivial solution predicts that the four-wave mixing oscillator can work without the left mirror, but there is no steady state to which the trivial state can evolve. The linear stability analysis makes a flawed prediction for photorefractive degenerate four-wave mixing.

5. Discussion

The results just presented require a short enunciation; the linearization of the fully degenerate four-wave mixing model leads to unphysical predictions. The physical rationale behind this failure is that the linearization of the full model leads to the undepleted-pump model (5), see equations (13), as the fluctuations of the pumping fields are not retained by the linearization. Then, as in the UPA model, energy is not conserved, which is at variance with the full model. The non-depletion of the pumps leads to an unrealistic high gain for mode A_1 , which is just an artifact due to linearization. One is then forced to conclude that the evolution of the fields inside the nonlinear medium cannot be correctly described within the UPA, and consequently that the laser stability analysis cannot correctly capture the stability properties of the system. In the aftermath of this peculiarity of photorefractive four-wave mixing, it is not possible to reduce the system of partial differential equations in (z, t) of the complete model to a system of ordinary differential equations in t , something that is perfectly possible for photorefractive two-wave mixing oscillators. The impossibility of capturing pump depletion through perturbative techniques, seems to prevent the derivation of mean field models for photorefractive degenerate four-wave mixing, at least in the usual way. We conclude that the full model equations must be used to make sensible predictions regarding the system's nonlinear dynamics.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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