A Note on $k$-Generalized Projections

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Abstract

In this note, we investigate characterizations for $k$-generalized projections (i.e., $A^k = A^*$) on Hilbert spaces. The obtained results generalize those for generalized projections on Hilbert spaces in [Hong-Ke Du, Yuan Li, The spectral characterization of generalized projections, Linear Algebra and its Applications, 400, (2005), 313–318] and those for matrices in [J. Benitez, N. Thome, Characterizations and linear combinations of $k$-generalized projectors, Linear Algebra and its Applications, In Press].

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In [2], it was defined a generalized projection as a complex matrix $A$ satisfying $A^2 = A^*$. This concept was extended in [3] for infinite-dimensional Hilbert spaces. For a Hilbert space, we shall denote

$$\mathcal{B}(H) = \{A/ A \text{ is linear and bounded operator, } A : H \to H\}.$$

If $k$ is an integer greater than 1, we define a $k$-generalized projection as an element $A$ of $\mathcal{B}(H)$ such that $A^k = A^*$, where $A^*$ is the adjoint operator of $A$.

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Moreover, the $n \times n$ complex matrices such that $A^k = A^*$ (where $A^*$ denotes its conjugate transpose) were characterized in [1].

We recall that $A \in \mathcal{B}(H)$ is said to be normal if $AA^* = A^*A$, it is said to be orthogonal projection if $A^2 = A = A^*$, and $A$ is called $k$-potent if $A^k = A$. In particular, $A$ is a projection if $A^2 = A$ and $A$ is tripotent if $A^3 = A$. In addition, the spectrum of $A$ will be denoted by $\sigma(A)$.

The main purpose of this note is to give characterizations of the $k$-generalized projections by using the spectral theorem for normal operators on Hilbert spaces (see [4]). We quote this theorem for the sake of completeness.

**Theorem 1 ([4])** Let $H$ be a Hilbert space and $A \in \mathcal{B}(H)$. If $A$ is normal then there exists a unique resolution of the identity $E$ on the Borel subsets of $\sigma(A)$ which satisfies

\[ A = \int_{\sigma(A)} \lambda dE(\lambda), \]

where $E(\lambda)$ denotes the spectral projection associated with the spectral point $\lambda \in \sigma(A)$ and $E(\lambda) = 0$ if $\lambda \notin \sigma(A)$.

The main result of this note is the following.

**Theorem 2** Let $H$ be a Hilbert space and $A \in \mathcal{B}(H)$. Then the following statements are equivalent.

(a) $A$ is a $k$-generalized projection.

(b) $A$ is normal and $\sigma(A) \subseteq \{0\} \cup \left\{ k^{1/2} \right\}$, where $k^{1/2}$ denotes the unity roots of order $k + 1$.

(c) $A$ is normal and $(k + 2)$-potent.

In this case, one has

\[ A = \bigoplus_{\lambda \in \mathcal{R}} \lambda E(\lambda), \]

where $E(\lambda) = 0$ if $\lambda \notin \sigma(A)$ and $\oplus$ stands for the direct sum.

**Proof.** (a) $\Rightarrow$ (b). Suppose that $A^k = A^*$. It is evident that $AA^* = A^*A$, i.e., $A$ is normal. Theorem 1 assures that

\[ A = \int_{\sigma(A)} \lambda dE(\lambda) \]

where $E(\lambda) = 0$ if $\lambda \notin \sigma(A)$ and $\oplus$ stands for the direct sum.
and then \(0 = A^k - A^* = \int_{\sigma(A)} (\lambda^k - \overline{\lambda}) dE(\lambda)\), which implies \(\lambda^k - \overline{\lambda} = 0\) for all \(\lambda \in \sigma(A)\). The roots of this equation are 0 and \(\frac{k+1}{2}\) since if \(\lambda = re^{i\theta}\), with \(r > 0\) and \(-\pi \leq \theta < \pi\), then we get \(r^k e^{ik\theta} = re^{-i\theta}\) and so \(r = 1\) and \(e^{i(k+1)\theta} = 1\), i.e., \(\lambda = e^{i\theta} \in \frac{k+1}{2}\). From (2), it is clear that (1) holds.

(b) \(\Rightarrow\) (c). If \(A\) is normal and \(\sigma(A) \subseteq \{0\} \cup \frac{k+1}{2}\) then (1) is true from Theorem 1. Now, since \(\lambda^{k+2} = \lambda\) for all \(\lambda \in \sigma(A)\),

\[
A^{k+2} = \bigoplus_{\lambda \in \frac{k+1}{2}} \lambda^{k+2} E(\lambda) = \bigoplus_{\lambda \in \frac{k+1}{2}} \lambda E(\lambda) = A.
\]

(c) \(\Rightarrow\) (a). If \(A\) is normal, from Theorem 1 one has that

\[
A = \int_{\sigma(A)} \lambda dE(\lambda).
\]

From \(A^{k+2} = A\) we get that

\[
0 = A^{k+2} - A = \int_{\sigma(A)} (\lambda^{k+2} - \lambda) dE(\lambda).
\]

Hence, \(\lambda^{k+2} - \lambda = 0\) for all \(\lambda \in \sigma(A)\). Now, it is easy to deduce \(\lambda^k = \overline{\lambda}\) for all \(\lambda \in \sigma(A)\) and so, from (3) we obtain \(A^k = A^*\).

This completes the proof. \(\square\)

Theorem 2 in [3] and Theorem 2.1 in [1] can be obtained as corollaries of Theorem 2.

**Corollary 1** Let \(H\) be a Hilbert space and let \(A \in \mathcal{B}(H)\) be a \(k\)-generalized projection.

(I) If \(\sigma(A) \subseteq \mathbb{R}\) and

(a) \(k\) is even then \(A\) is a projection.

(b) \(k\) is odd then \(A\) is a tripotent operator.

(II) If \(\sigma(A) \subseteq i\mathbb{R}\) and

(a) \(k\) is a multiple of 4 then \(A^3 = -A\).

(b) \(k\) is not a multiple of 4 then \(A = O\).
**Proof.** By Theorem 2 we know that $A$ is normal and $\sigma(A) \subseteq \{0\} \cup k\sqrt{1}$.

(I) By hypothesis, $\sigma(A) \subseteq \{0\} \cup (k\sqrt{1} \cap \mathbb{R})$. If $k$ is even then $\sigma(A) \subseteq \{0, 1\}$, hence $A^2 = A$. If $k$ is odd then $\sigma(A) \subseteq \{-1, 0, 1\}$, hence $A^3 = A$.

(II) In this case, $\sigma(A) \subseteq i\mathbb{R} \cap (\{0\} \cup k\sqrt{1})$. If $k$ is a multiple of 4 then $i\mathbb{R} \cap (\{0\} \cup k\sqrt{1}) = \{0, i, -i\}$ and hence $A^3 + A = O$. If $k$ is not a multiple of 4 then $i\mathbb{R} \cap (\{0\} \cup k\sqrt{1}) = \{0\}$ and hence $A = O$. This conclude the proof.

It is well-known that: $A$ is normal and $\sigma(A) \subseteq \mathbb{R}$ if and only if $A = A^*$ (i.e., $A$ is self-adjoint). So, the hypothesis that “$A$ is a $k$-generalized projection and $\sigma(A) \subseteq \mathbb{R}$” is equivalent to “$A$ is a $k$-generalized projection and $A^* = A$”. Analogously, the hypothesis that “$A$ is a $k$-generalized projection and $\sigma(A) \subseteq i\mathbb{R}$” is equivalent to “$A$ is a $k$-generalized projection and $A^* = -A$” (i.e., $A$ is skew self-adjoint).

**Corollary 2** Let $H$ be a Hilbert space and let $A \in \mathcal{B}(H)$ be a $k$-generalized projection. The range of $A$ (denoted by $\mathcal{R}(A)$) is closed.

**Proof.** Since $A$ is a $k$-generalized projection, by Theorem 2 we get that $A$ is normal and its spectrum is finite, so 0 is not a limited point of the spectrum of the normal operator $A$, then $\mathcal{R}(A)$ is closed. This completes the proof.

A similar result to Theorem 2 can be established for matrices and it generalizes Corollary 4 in [3].

**Corollary 3** Let $H$ be a Hilbert space and let $A \in \mathcal{B}(H)$ be a $k$-generalized projection. Then $A^{k+1}$ is an orthogonal projection.

**Proof.** From Theorem 2, we get $A^{k+2} = A$ and then $(A^{k+1})^2 = A^{k+2}A^k = AA^k = A^{k+1}$. Moreover, $A^{k+1}$ is an orthogonal projection because

$$ (A^{k+1})^* - A^{k+1} = (A^kA)^* - A^kA = (A^*A)^* - A^*A = 0, $$

since $A^*A$ is self-adjoint. This completes the proof.

It is clear that Corollary 2 and Corollary 3 generalize the results given in Corollary 3 in [3].

**References**

