This paper has been published in *Journal of Algebra*, 299(2):602–615 (2006).

http://dx.doi.org/10.1016/j.jalgebra.2006.01.003

http://www.sciencedirect.com/science/article/pii/S0021869306000068

Products of formations of finite groups

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Abstract

In this paper criteria for a product of formations to be $\mathfrak{X}$-local, $\mathfrak{X}$ a class of simple groups, are obtained. Some classical results on products of saturated formations appear as particular cases.

Key words: Finite groups, $\mathfrak{X}$-local formation, ω-local formation, formation products
1991 MSC: 20D10, 20F17

1 Introduction

All groups considered in this paper are tacitly assumed to be finite.

Recall that a formation $\mathfrak{F}$ is a class of groups which is closed under taking homomorphic images and subdirect products. The second condition ensures the existence of the $\mathfrak{F}$-*residual* $U^{\mathfrak{F}}$ of each group U, that is, the smallest normal

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subgroup of U whose factor group is in $\mathfrak{F}$. A formation $\mathfrak{F}$ is said to be *saturated* if $U \in \mathfrak{F}$ whenever the Frattini factor group $U/\Phi(U)$ is in $\mathfrak{F}$. Gaschütz introduced the concept of *local formation*, which allows us to construct saturated formations. In fact, the Gaschütz-Lubeseder-Schmid theorem states that the family of local formations coincides with the one of saturated formations (see [1, Section IV] for details). It was proved by Gaschütz and Lubeseder in the soluble universe, and by Schmid in the general one. Baer gave an alternative generalization of the theorem of Gaschütz and Lubeseder to the general universe. He studied a different concept of local formation, using the simple components of a chief factor to label it, rather than the primes dividing its order (see [1, Section IV]). He found a family of formations, the *Baer-local* formations, which contains the family of local ones and coincides with it in the soluble universe. Baer-local formations are exactly the ones that are closed under extensions through the Frattini subgroup of the soluble radical, i. e., the solubly saturated ones. This is known as Baer's theorem (see [1, IV, 4.17]). Another approach to the Gaschütz-Lubeseder theorem in the finite universe is due to Shemetkov. He uses functions assigning a certain formation to each group (he recently calls them *satellites*) and introduces the notion of *composition formation* (see the recent survey [2]). It turns out that composition formations are exactly the Baer-local ones.

Förster introduced in [3] the concept of $\mathfrak{X}$-local formation, where $\mathfrak{X}$ is a class of simple groups with a completeness property. The main aim of his work was to present a common extension of the Gaschütz-Lubeseder-Schmid and Baer theorems. Förster also defined a Frattini-like subgroup $\Phi^*_{\mathfrak{X}}(U)$ for each group U, which enables him to introduce the concept of $\mathfrak{X}$-saturation. He proves that $\mathfrak{X}$-saturated formations are exactly the $\mathfrak{X}$-local ones. In [4], an alternative $\mathfrak{X}$-Frattini subgroup $\Phi_{\mathfrak{X}}(U)$ of a group U is introduced. Except for a particular case, it is defined as $\Phi\big(\mathrm{O}_{\mathfrak{X}}(U)\big)$, where $\mathrm{O}_{\mathfrak{X}}(U)$ is the largest normal subgroup of U whose composition factors belong to $\mathfrak{X}$. It was proved in [4] that the new $\mathfrak{X}$-saturated formations also coincide with the $\mathfrak{X}$-local ones.

A local approach to the saturation is the *ω-saturation*, where ω is a non-empty set of primes. We say that a formation $\mathfrak{F}$ is *ω-saturated* if the condition $U/\big(\Phi(U) \cap \mathrm{O}_\omega(U)\big) \in \mathfrak{F}$ always implies that $U \in \mathfrak{F}$. This concept arises spontaneously when the saturation of formation products is considered (see [5]).

Given two classes $\mathfrak{Y}$ and $\mathfrak{Z}$ of groups, a product class can be defined by setting

$$\mathfrak{Y}\mathfrak{Z} = (U \in \mathfrak{E} \mid \text{there is a normal subgroup } V \text{ of } U \text{ such that } V \in \mathfrak{Y} \text{ and } U/V \in \mathfrak{Z}),$$

However this class is not in general a formation when $\mathfrak{Y}$ and $\mathfrak{Z}$ are formations. But there is a way of modifying the above definition to ensure that the class product of two formations is again a formation. If $\mathfrak{F}$ and $\mathfrak{G}$ are formations, the *formation product* or *Gaschütz product* of $\mathfrak{F}$ and $\mathfrak{G}$ is the class $\mathfrak{F} \circ \mathfrak{G}$ defined by

$$\mathfrak{F} \circ \mathfrak{G} := (U \in \mathfrak{E} \mid U^{\mathfrak{G}} \in \mathfrak{F}).$$

$\mathfrak{F} \circ \mathfrak{G}$ is again a formation and if $\mathfrak{F}$ is closed under taking subnormal subgroups, then $\mathfrak{F}\mathfrak{G} = \mathfrak{F} \circ \mathfrak{G}$ (see [1, IV, 1.7 and 1.8]).

It is well known that the formation product of two local formations is again a local formation (see [1, IV, 3.13 and 4.8]). However, the formation product of two $\mathfrak{X}$-local formations is not in general an $\mathfrak{X}$-local formation, as it is shown in Example 6. Taking this into account, the following question arises:

> *Which are the precise conditions on two $\mathfrak{X}$-local formations $\mathfrak{F}$ and $\mathfrak{G}$ to ensure that $\mathfrak{F} \circ \mathfrak{G}$ is an $\mathfrak{X}$-local formation?*

This question was studied by Salomon in [6] for Baer-local formations. We present a complete answer in Section 3. We prove that the formation product of a local formation and an $\mathfrak{X}$-local one is $\mathfrak{X}$-local. In particular, [1, IV, 3.13 and 4.8] follow from our results. In Section 4, which is independent of Section 3, we study when the product of two arbitrary formations $\mathfrak{F}$ and $\mathfrak{G}$ is $\mathfrak{X}$-local.

On the other hand, Shemetkov posed the following question in *The Kourovka Notebook* ([7]):

> Question 10.72 (Shemetkov). *To prove indecomposability of $\mathfrak{S}_p$, p a prime, into a product of two non-trivial subformations.*

This question was solved positively by Shemetkov and Skiba in [8]. In Section 5 we deal with ω-saturated formations and we prove a general version of this conjecture as a corollary of a more general result.

2 Preliminaries

We begin with the concept of $\mathfrak{X}$-local formation. It was introduced by Förster in [3].

Let $\mathfrak{J}$ denote the class of all simple groups. For any subclass $\mathfrak{Y}$ of $\mathfrak{J}$, we write $\mathfrak{Y}' := \mathfrak{J} \setminus \mathfrak{Y}$. Denote by $\mathrm{E}\,\mathfrak{Y}$ the class of groups whose composition factors belong to $\mathfrak{Y}$. A chief factor which belongs to $\mathrm{E}\,\mathfrak{Y}$ is called a $\mathfrak{Y}$-chief factor. If p is a prime, we write $\mathfrak{Y}_p$ to denote the class of all simple groups $S \in \mathfrak{Y}$ such that $p \in \pi(S)$. In the sequel it will be convenient to identify the prime p with the cyclic group C_p of order p. The class of all simple abelian groups is denoted by $\mathbb{P}$. The class of all π-groups, where π is a set of primes, is denoted by $\mathfrak{E}_\pi$. If $\pi = \{p\}$, then $\mathfrak{E}_\pi = \mathfrak{S}_p$.

Throughout this paper, $\mathfrak{X}$ denotes a fixed class of simple groups satisfying that $\pi(\mathfrak{X}) = \mathrm{char}(\mathfrak{X})$, where

$$\pi(\mathfrak{X}) := \{p \in \mathbb{P} \mid \text{there exists } S \in \mathfrak{X} \text{ such that } p \text{ divides } |S|\}$$

and $\mathrm{char}(\mathfrak{X}) := \{p \in \mathbb{P} \mid C_p \in \mathfrak{X}\}$.

Definition 1 ([3]). An $\mathfrak{X}$-*formation function* f associates with each $X \in \mathrm{char}(\mathfrak{X}) \cup \mathfrak{X}'$ a formation $f(X)$ (possibly empty). If f is an $\mathfrak{X}$-formation function, then $\mathrm{LF}_{\mathfrak{X}}(f)$ is the class of all groups U satisfying the following two conditions:

(1) If V/W is an $\mathfrak{X}_p$-chief factor of U, then $U/\mathrm{C}_U(V/W) \in f(p)$.
(2) If U/L is a monolithic quotient of U such that $\mathrm{Soc}(U/L)$ is an $\mathfrak{X}'$-chief factor of U, then $U/L \in f(E)$, where E is the composition factor of $\mathrm{Soc}(U/L)$.

The class $\mathrm{LF}_{\mathfrak{X}}(f)$ is a formation ([3]). A formation $\mathfrak{F}$ is said to be $\mathfrak{X}$-*local* if there exists an $\mathfrak{X}$-formation function f such that $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f)$. In this case we say that f is an $\mathfrak{X}$-*local definition* of $\mathfrak{F}$ or that f *defines* $\mathfrak{F}$. The $\mathfrak{X}$-formation function f is *full* if $\mathfrak{S}_p f(p) = f(p)$ for every $p \in \mathrm{char}(\mathfrak{X})$ and f is *integrated* if $f(S) \subseteq \mathfrak{F}$ for every simple group $S \in \mathrm{char}(\mathfrak{X}) \cup \mathfrak{X}'$.

Examples 2.

(1) Each formation $\mathfrak{F}$ is $\mathfrak{X}$-local for $\mathfrak{X} = \emptyset$ because $\mathfrak{F} = \mathrm{LF}_{\mathfrak{X}}(f)$, where $f(S) = \mathfrak{F}$ for each $S \in \mathfrak{J}$.
(2) If $\mathfrak{X} = \mathfrak{J}$, the class of all simple groups, an $\mathfrak{X}$-formation function is simply a formation function and the $\mathfrak{X}$-local formations are exactly the local formations.
(3) If $\mathfrak{X} = \mathbb{P}$, the class of all abelian simple groups, then an $\mathfrak{X}$-formation function is a Baer function and the $\mathfrak{X}$-local formations are exactly the Baer-local ones.

We will need the following results throughout the paper.

Lemma 3. *Let $\mathfrak{F}$ be an $\mathfrak{X}$-local formation and let f be an $\mathfrak{X}$-formation function defining $\mathfrak{F}$.*

(1) If V is a normal subgroup of U such that $V \in \mathrm{E}\mathfrak{X}$, $U/V \in \mathfrak{F}$, and $U/\mathrm{C}_U(V) \in f(p)$ for every $p \in \pi(V)$, then $U \in \mathfrak{F}$.
(2) If f is integrated, then $\mathfrak{S}_p f(p) \subseteq \mathfrak{F}$ for every $p \in \mathrm{char}(\mathfrak{X})$.

If $\mathfrak{K}$ is a class of groups and $p \in \operatorname{char} \mathfrak{X}$, denote

$$K_{\mathfrak{X}}(p) := \mathfrak{S}_p \operatorname{Q} \operatorname{R}_0\Big(U/\operatorname{C}_U(V/W) \mid U \in \mathfrak{K} \text{ and } V/W \text{ is an } \mathfrak{X}_p\text{-chief factor of } U\Big),$$

taking into account that $K_{\mathfrak{X}}(p) = \emptyset$ if there does not exist any group $U \in \mathfrak{K}$ with an $\mathfrak{X}_p$-chief factor. We write $\operatorname{form}_{\mathfrak{X}}(\mathfrak{K})$ to denote the smallest $\mathfrak{X}$-local formation containing $\mathfrak{K}$, i. e., the intersection of all $\mathfrak{X}$-local formations containing $\mathfrak{K}$. If $\mathfrak{X} = \mathfrak{J}$, we write $\operatorname{lform}(\mathfrak{K})$ to denote $\operatorname{form}_{\mathfrak{X}}(\mathfrak{K})$. The following theorem describes a full and integrated $\mathfrak{X}$-formation function defining $\operatorname{form}_{\mathfrak{X}}(\mathfrak{K})$.

Theorem 4 ([4] and [9]).

(1) Let $\mathfrak{K}$ be a class of groups. Then $\operatorname{form}_{\mathfrak{X}}(\mathfrak{K}) = \operatorname{LF}_{\mathfrak{X}}(K)$, where K is the following $\mathfrak{X}$-formation function:

$$\begin{cases} K(p) = K_{\mathfrak{X}}(p) & \text{if } p \in \operatorname{char}(\mathfrak{X}) \\ K(E) = \operatorname{Q}\operatorname{R}_0(\mathfrak{K}) & \text{if } E \in \mathfrak{X}' \end{cases}$$

Moreover, K is full and integrated. We say that K is the canonical $\mathfrak{X}$-local definition *of* $\operatorname{form}_{\mathfrak{X}}(\mathfrak{K})$.

(2) If $\mathfrak{X}$ and $\overline{\mathfrak{X}}$ are two classes of simple groups such that $\overline{\mathfrak{X}} \subseteq \mathfrak{X}$ and $\mathfrak{F}$ is an $\mathfrak{X}$-local formation, then $\mathfrak{F}$ is $\overline{\mathfrak{X}}$-local.

(3) If $\mathfrak{F}$ is an $\mathfrak{X}$-local formation defined by a full and integrated $\mathfrak{X}$-formation function f, then $f(p) = (U \mid C_p \wr U \in \mathfrak{F})$ for every $p \in \operatorname{char}(\mathfrak{X})$.

In particular, if $\mathfrak{K}$ is a class of groups, then $K_{\mathfrak{X}}(p) = \big(U \mid C_p \wr U \in \operatorname{form}_{\mathfrak{X}}(\mathfrak{K})\big)$ for every $p \in \operatorname{char}(\mathfrak{X})$.

Lemma 5. *Consider $p \in \operatorname{char} \mathfrak{X}$. If $\mathfrak{K}$ is a quotient-closed class of groups, then*

$$K_{\mathfrak{X}}(p) := \mathfrak{S}_p \operatorname{Q} \operatorname{R}_0\Big(U/\operatorname{C}_U(V) \mid U \text{ is monolithic, } U \in \mathfrak{K}, \text{ and } V = \operatorname{Soc}(U) \in \operatorname{E}\mathfrak{X}_p\Big).$$

PROOF. Consider a group $L \in \mathfrak{K}$ and let M/N be an $\mathfrak{X}_p$-chief factor of L. Take R maximal among the normal subgroups T of L such that $M \cap T = N$. Then the quotient group $U = L/R \in \mathfrak{K}$ is monolithic and its minimal normal subgroup is $V = MR/R \cong M/M\cap R = M/N$ (see [9, Lemma 1.30]). Moreover, $R \leq \operatorname{C}_L(M/N)$ and $L/\operatorname{C}_L(M/N) \cong U/\operatorname{C}_U(V)$.

Notation which is not explained here is consistent with the one in [1].

3 Products of $\mathfrak{X}$-local formations

The following example shows that the formation product of two $\mathfrak{X}$-local formations is not in general an $\mathfrak{X}$-local formation.

Example 6 ([6]). Consider $\mathfrak{F} = \mathrm{D}_0(1, A_5)$, the formation composed of all groups that are direct products of copies of A_5 together with the trivial group, and $\mathfrak{G} = \mathfrak{S}_2$. It is clear that $\mathfrak{F}$ and $\mathfrak{G}$ are Baer formations, that is, $\mathfrak{X}$-local where $\mathfrak{X} = \mathbb{P}$. Assume that $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$ is a $\mathbb{P}$-local formation. By Theorem 4, we have that $\mathfrak{H} = \mathrm{LF}_{\mathbb{P}}(H)$, where

$$\begin{cases} H(p) = H_{\mathbb{P}}(p) & \text{if } p \in \mathbb{P} \\ H(E) = \mathfrak{H} & \text{if } E \in \mathbb{P}' \end{cases}$$

Since $\mathfrak{G} \subseteq \mathfrak{H}$, it follows that $H(2) \neq \emptyset$. Consider $U = \mathrm{SL}(2, 5)$. Then $U/\operatorname{Z}(U) \in \mathfrak{H}$ and $U/\operatorname{C}_U\big(\operatorname{Z}(U)\big) \in H(2)$. Applying Lemma 3, we have that $U \in \mathfrak{H}$. This is not true. Hence $\mathfrak{H}$ is not a Baer-local formation.

Taking the above example into account, it is natural to study conditions on two non-empty $\mathfrak{X}$-local formations $\mathfrak{F}$ and $\mathfrak{G}$ to ensure that $\mathfrak{F} \circ \mathfrak{G}$ is an $\mathfrak{X}$-local formation.

In the following $\mathfrak{F}$ and $\mathfrak{G}$ are non-empty formations and $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$.

The next theorem provides an $\mathfrak{X}$-local definition of $\mathrm{form}_{\mathfrak{X}}(\mathfrak{H})$.

Theorem 7. *Assume that $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$, where $\mathfrak{F}$ and $\mathfrak{G}$ are non-empty formations, and $\mathfrak{F}$ is an $\mathfrak{X}$-local formation. Then the smallest $\mathfrak{X}$-local formation* $\mathrm{form}_{\mathfrak{X}}(\mathfrak{H})$ *containing $\mathfrak{H}$ is $\mathfrak{X}$-locally defined by the $\mathfrak{X}$-formation function h given by*

$$h(p) = \begin{cases} F_{\mathfrak{X}}(p) \circ \mathfrak{G} & \text{if } \mathfrak{S}_p \subseteq \mathfrak{F} \\ G_{\mathfrak{X}}(p) & \text{if } \mathfrak{S}_p \nsubseteq \mathfrak{F} \end{cases} \qquad p \in \operatorname{char} \mathfrak{X}$$

$$h(S) = \mathfrak{H} \qquad S \in \mathfrak{X}'$$

PROOF. We know by Theorem 4 that $\mathrm{form}_{\mathfrak{X}}(\mathfrak{H}) = \mathrm{LF}_{\mathfrak{X}}(H)$, where H is the $\mathfrak{X}$-formation function defined by

$$\begin{cases} H(p) = H_{\mathfrak{X}}(p) & \text{if } p \in \operatorname{char}(\mathfrak{X}) \\ H(S) = \mathfrak{H} & \text{if } S \in \mathfrak{X}' \end{cases}$$

If we prove that $H(p) = h(p)$ for every $p \in \operatorname{char}(\mathfrak{X})$, then the result is clear.

By Lemma 5, we have that

$$H(p) := \mathfrak{S}_p \operatorname{Q} \operatorname{R}_0\Big(U/\operatorname{C}_U(V) \mid U \text{ is monolithic}, U \in \mathfrak{H}, \\ \text{and } V = \operatorname{Soc}(U) \in \operatorname{E}\mathfrak{X}_p\Big).$$

Assume that U is a monolithic group in $\mathfrak{H}$, where $V = \operatorname{Soc}(U) \in \operatorname{E}\mathfrak{X}_p$, and consider $A = U^{\mathfrak{G}}$. If $A = 1$, it follows that $U \in \mathfrak{G}$. If $\mathfrak{S}_p \subseteq \mathfrak{F}$, then $h(p) = F_{\mathfrak{X}}(p) \circ \mathfrak{G}$ and, therefore, $U \in h(p)$. If $\mathfrak{S}_p \not\subseteq \mathfrak{F}$, then $U/\operatorname{C}_U(V) \in G_{\mathfrak{X}}(p) = h(p)$. Now suppose that $A \neq 1$. Since $V \leq A$, applying [1, A, 4.13], it follows that $V = V_1 \times \cdots \times V_n$, where V_i is a minimal normal subgroup of A, $1 \leq i \leq n$. Since $A \in \mathfrak{F}$, it follows that $A/\operatorname{C}_A(V_i) \in F_{\mathfrak{X}}(p)$, for all $i \in \{1, \ldots, n\}$, and $p \mid |V|$. Consequently $\big(U/\operatorname{C}_U(V)\big)^{\mathfrak{G}} \cong A/\operatorname{C}_A(V) \in \operatorname{R}_0 F_{\mathfrak{X}}(p) = F_{\mathfrak{X}}(p)$ and so $U/\operatorname{C}_U(V) \in F_{\mathfrak{X}}(p) \circ \mathfrak{G} = h(p)$ for all $p \mid |N|$. It follows that $H(p) = H_{\mathfrak{X}}(p) \subseteq \mathfrak{S}_p \operatorname{Q} \operatorname{R}_0 h(p) = h(p)$.

Now we prove that $h(p) \subseteq H(p) = H_{\mathfrak{X}}(p)$. If $\mathfrak{S}_p \not\subseteq \mathfrak{F}$, then clearly $h(p) = G_{\mathfrak{X}}(p) \subseteq H_{\mathfrak{X}}(p)$. Suppose that $\mathfrak{S}_p \subseteq \mathfrak{F}$, that is, $h(p) = F_{\mathfrak{X}}(p) \circ \mathfrak{G}$. Consider a group $U \in F_{\mathfrak{X}}(p) \circ \mathfrak{G}$. Then the wreath product $C_p \wr U \in \mathfrak{S}_p(F_{\mathfrak{X}}(p) \circ \mathfrak{G}) \subseteq \mathfrak{S}_p F_{\mathfrak{X}}(p) \circ \mathfrak{G}$. By Theorem 4, we know that $\mathfrak{S}_p F_{\mathfrak{X}}(p) \subseteq \mathfrak{F}$ and, hence, $C_p \wr U \in \mathfrak{F} \circ \mathfrak{G} = \mathfrak{H} \subseteq \operatorname{form}_{\mathfrak{X}}(\mathfrak{H})$. Then Theorem 4 shows that $U \in H_{\mathfrak{X}}(p)$. This proves that $h(p) \subseteq H(p)$.

The following definition was introduced in [6] for Baer-local formations.

Definition 8. Consider $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$, where $\mathfrak{F}$ and $\mathfrak{G}$ are non-empty formations. We say that the boundary $\operatorname{b}(\mathfrak{H})$ of $\mathfrak{H}$ is $\mathfrak{X}\mathfrak{G}$*-free* if every group $U \in \operatorname{b}(\mathfrak{H})$ such that $\operatorname{Soc}(U)$ is a p-group for some prime $p \in \operatorname{char}\mathfrak{X}$ satisfies that $U/\operatorname{C}_U\big(\operatorname{Soc}(U)\big) \notin G_{\mathfrak{X}}(p)$.

Remark 9. Note that in Example 6, $\operatorname{b}(\mathfrak{H})$ is not $\mathbb{P}\mathfrak{G}$-free.

Lemma 10. *If $\mathfrak{K}$ is a formation and $U \in \operatorname{b}(\mathfrak{K}) \cap \operatorname{form}_{\mathfrak{X}}(\mathfrak{K})$, with $V = \operatorname{Soc}(U)$, then V is an abelian p-group for a prime $p \in \operatorname{char}(\mathfrak{X})$.*

PROOF. According to Theorem 4, $\operatorname{form}_{\mathfrak{X}}(\mathfrak{K}) = \operatorname{LF}_{\mathfrak{X}}(K)$, where K is the following $\mathfrak{X}$-formation function:

$$\begin{cases} K(p) = K_{\mathfrak{X}}(p) & \text{if } p \in \operatorname{char}(\mathfrak{X}) \\ K(E) = \mathfrak{K} & \text{if } E \in \mathfrak{X}' \end{cases}$$

Clearly, V is a minimal normal subgroup of U. If V were an $\mathfrak{X}'$-group, we would have that $U \in K(E)$ for some $E \in \mathfrak{X}'$. This would imply that $U \in \mathfrak{K}$, contrary to supposition. Hence V is an $\mathfrak{X}$-chief factor of U. Let p be a prime dividing $|V|$. Since $p \in \operatorname{char} \mathfrak{X}$, it follows that $U/\operatorname{C}_U(V) \in K(p)$. Since $K(p) = K_{\mathfrak{X}}(p) \subseteq \mathfrak{S}_p\mathfrak{K}$ and $\operatorname{O}_p\big(U/\operatorname{C}_U(V)\big) = 1$, it follows that $U/\operatorname{C}_U(V) \in \mathfrak{K}$. Therefore, $\operatorname{C}_U(V) \neq 1$ and so V is an abelian p-group.

The next result provides a test for $\mathfrak{X}$-locality of $\mathfrak{H}$ in terms of its boundary.

Theorem 11. *Assume that $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$, where $\mathfrak{F}$ and $\mathfrak{G}$ are non-empty formations, and $\mathfrak{F}$ is $\mathfrak{X}$-local. Then $\mathfrak{H}$ is an $\mathfrak{X}$-local formation if and only if* $\operatorname{b}(\mathfrak{H})$ *is $\mathfrak{X}\mathfrak{G}$-free.*

PROOF. Suppose that $\mathfrak{H}$ is $\mathfrak{X}$-local. Then $\mathfrak{H} = \operatorname{LF}_{\mathfrak{X}}(H)$, where H is the canonical $\mathfrak{X}$-local definition of $\mathfrak{H}$. Let U be a group in $\operatorname{b}(\mathfrak{H})$ such that $\operatorname{Soc}(U)$ is a p-group for some $p \in \operatorname{char} \mathfrak{X}$. If $U/\operatorname{C}_U\big(\operatorname{Soc}(U)\big)$ were in $G_{\mathfrak{X}}(p)$, then we would have that $U/\operatorname{C}_U\big(\operatorname{Soc}(U)\big) \in H_{\mathfrak{X}}(p) = H(p)$, since $\mathfrak{G} \subseteq \mathfrak{H}$. By Lemma 3, it would imply that $U \in \mathfrak{H}$. This would be a contradiction. Therefore $U/\operatorname{C}_U\big(\operatorname{Soc}(U)\big) \notin G_{\mathfrak{X}}(p)$ and $\operatorname{b}(\mathfrak{H})$ is $\mathfrak{X}\mathfrak{G}$-free.

Conversely, suppose that $\operatorname{b}(\mathfrak{H})$ is $\mathfrak{X}\mathfrak{G}$-free. By Theorem 7, $\operatorname{form}_{\mathfrak{X}}(\mathfrak{H}) = \operatorname{LF}_{\mathfrak{X}}(h)$, where

$$\begin{aligned} h(p) &= \begin{cases} F_{\mathfrak{X}}(p) \circ \mathfrak{G} & \text{if } \mathfrak{S}_p \subseteq \mathfrak{F} \\ G_{\mathfrak{X}}(p) & \text{if } \mathfrak{S}_p \nsubseteq \mathfrak{F} \end{cases} && p \in \operatorname{char} \mathfrak{X} \\ h(S) &= \mathfrak{H} && S \in \mathfrak{X}' \end{aligned}$$

We shall prove that $\mathfrak{H} = \operatorname{form}_{\mathfrak{X}}(\mathfrak{H})$. Assume that this is not the case and choose a group U of minimal order in $\operatorname{form}_{\mathfrak{X}}(\mathfrak{H}) \setminus \mathfrak{H}$. Then $U \in \operatorname{b}(\mathfrak{H})$ and, according to Lemma 10, $V = \operatorname{Soc}(U)$ is an abelian p-group for a prime $p \in \operatorname{char}(\mathfrak{X})$. Therefore, $U/\operatorname{C}_U(V) \in h(p)$. Assume that $\mathfrak{S}_p$ is not contained in $\mathfrak{F}$. Then $h(p) = G_{\mathfrak{X}}(p)$. We conclude that $\operatorname{b}(\mathfrak{H})$ is not $\mathfrak{X}\mathfrak{G}$-free. This contradiction shows that $\mathfrak{S}_p$ is contained in $\mathfrak{F}$. Then $U/\operatorname{C}_U(V) \in F_{\mathfrak{X}}(p) \circ \mathfrak{G}$. It follows that $U^{\mathfrak{G}}/\operatorname{C}_{U^{\mathfrak{G}}}(V) \in F_{\mathfrak{X}}(p)$. Since $U^{\mathfrak{G}}/V \in \mathfrak{F}$, we can apply Lemma 3 to conclude that $U^{\mathfrak{G}} \in \mathfrak{F}$, that is, $U \in \mathfrak{H}$. This contradiction shows that $\operatorname{form}_{\mathfrak{X}}(\mathfrak{H})$ is contained in $\mathfrak{H}$ and, therefore, $\mathfrak{H}$ is $\mathfrak{X}$-local.

Example 12. Let S be a non-abelian simple group with trivial Schur multiplier. Consider $\mathfrak{F} = \operatorname{D}_0(1, S)$, the formation of all groups which are a direct product of copies of S together with the trivial group. Let $\mathfrak{X}$ be a class of simple groups such that $\pi(\mathfrak{X}) = \operatorname{char} \mathfrak{X}$ and $S \notin \mathfrak{X}$. Note that $\mathfrak{F}$ is $\mathfrak{X}$-local. Let $\mathfrak{G}$ be any formation. Suppose that $U \in \operatorname{b}(\mathfrak{H})$, $V = \operatorname{Soc}(U)$ is the minimal normal subgroup of U, and V is a p-group for some $p \in \operatorname{char} \mathfrak{X}$. If $U/\operatorname{C}_U(V) \in G_{\mathfrak{X}}(p)$, then $V \leq \operatorname{Z}(U^{\mathfrak{G}})$ because $1 \neq U^{\mathfrak{G}} \leq \operatorname{C}_U(V)$. Since $U/V \in \mathfrak{H}$, it follows that $U^{\mathfrak{G}}/V \in \mathfrak{F}$. Assume that $U^{\mathfrak{G}} \neq V$. This implies that $U^{\mathfrak{G}}/V$, a direct product of copies of S, has non-trivial Schur multiplier, contrary to [10, Exercise 4 (c), page 265]. Thus $U^{\mathfrak{G}} = V$ and then $U \in \mathrm{form}_{\mathfrak{X}}(\mathfrak{G})$ by Lemma 3 and Theorem 4. Assume, in addition, that $\mathrm{form}_{\mathfrak{X}}(\mathfrak{G}) \subseteq \mathfrak{N}_{p'}\mathfrak{G}$ for all primes $p \in \mathrm{char}(\mathfrak{X})$. It follows then that $U \in \mathfrak{G}$ and this contradicts our choice of U. Hence, $\mathrm{b}(\mathfrak{H})$ is $\mathfrak{X}\mathfrak{G}$-free and $\mathfrak{H}$ is $\mathfrak{X}$-local by Theorem 11. Consequently, $\mathfrak{H}$ is $\mathfrak{X}$-local for all formations $\mathfrak{G}$ satisfying that $\mathrm{form}_{\mathfrak{X}}(\mathfrak{G}) \subseteq \mathfrak{N}_{p'}\mathfrak{G}$ for all primes $p \in \mathrm{char}(\mathfrak{X})$.

We bring this section to a close with an application of Theorem 11 and some consequences.

Theorem 13. *Consider $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$, where $\mathfrak{F}$ and $\mathfrak{G}$ are non-empty formations, $\mathfrak{F}$ is $\mathfrak{X}$-local and one of the following two conditions is satisfied:*

(1) $\mathfrak{G}$ is $\mathfrak{X}$-local.
(2) $\mathfrak{S}_p\mathfrak{G} = \mathfrak{G}$ when $p \in \mathrm{char}\,\mathfrak{X}$ and $F_{\mathfrak{X}}(p) = \emptyset$.

Assume also that

$$\text{If } p \in \mathrm{char}\,\mathfrak{X},\ F_{\mathfrak{X}}(p) = \emptyset \text{ and } \mathfrak{S}_p \subseteq \mathfrak{G}, \text{ then } \mathfrak{F} \subseteq \mathfrak{E}_{p'}. \qquad (1)$$

Then $\mathfrak{H}$ is $\mathfrak{X}$-local.

PROOF. By Theorem 11, it suffices to prove that $\mathrm{b}(\mathfrak{H})$ is $\mathfrak{X}\mathfrak{G}$-free. Consider $U \in \mathrm{b}(\mathfrak{H})$ such that $V = \mathrm{Soc}(U)$ is a p-group for a prime $p \in \mathrm{char}\,\mathfrak{X}$ and assume that $U/\mathrm{C}_U(V) \in G_{\mathfrak{X}}(p)$. Consider $W = U^{\mathfrak{G}}$. Since $G_{\mathfrak{X}}(p) \subseteq \mathfrak{S}_p\mathfrak{G}$ and $\mathrm{O}_p\big(U/\mathrm{C}_U(V)\big) = 1$, it follows that $U/\mathrm{C}_U(V) \in \mathfrak{G}$, which implies that $W \leq \mathrm{C}_U(V)$. If $W = 1$, then $U \in \mathfrak{G} \subseteq \mathfrak{H}$, which contradicts the fact that $U \in \mathrm{b}(\mathfrak{H})$. Therefore $W \neq 1$ and, hence, $V \leq W$.

Now we aim to verify the hypotheses of (1). Since $U/V \in \mathfrak{H}$, it follows that $W/V \in \mathfrak{F}$. If $F_{\mathfrak{X}}(p) \neq \emptyset$, then $\mathrm{C}_W(V) = 1 \in F_{\mathfrak{X}}(p)$ and Lemma 3 implies that $W \in \mathfrak{F}$, which means that $U \in \mathfrak{H}$, contradicting the choice of U. Therefore $F_{\mathfrak{X}}(p) = \emptyset$. Since $U/\mathrm{C}_U(V) \in G_{\mathfrak{X}}(p)$, it is clear that $G_{\mathfrak{X}}(p) \neq \emptyset$ so, if Condition 1 holds, it follows that $\mathfrak{S}_p \subseteq \mathfrak{G}$. On the other hand, if Condition 2 holds, then $\mathfrak{S}_p\mathfrak{G} = \mathfrak{G}$ and hence $\mathfrak{S}_p \subseteq \mathfrak{G}$. Now we can deduce that $\mathfrak{F} \subseteq \mathfrak{E}_{p'}$.

We know that $W/V \in \mathfrak{F}$, so W/V is a p'-group and it follows from the Schur-Zassenhaus Theorem ([1, A, 11.3]) that W has a subgroup Y which complements V. Since $W \leq \mathrm{C}_U(V)$, this means that $W = V \times Y$ and $Y = \mathrm{O}_{p'}(W) \trianglelefteq U$. Moreover, U is monolithic, so we deduce that $Y = 1$ and $W = V$. This means that $U/V \in \mathfrak{G}$. Since $U/\mathrm{C}_U(V) \in G_{\mathfrak{X}}(p)$, if Condition 1 holds, then it follows from Lemma 3 that $U \in \mathfrak{G}$. If Condition 2 holds, then $U \in \mathfrak{S}_p\mathfrak{G} = \mathfrak{G}$. Thus in both cases $U \in \mathfrak{G} \subseteq \mathfrak{H}$, which gives the final contradiction.

Since local formations are $\mathfrak{X}$-local for every class of simple groups $\mathfrak{X}$ (see Theorem 4), we obtain as a special case of Theorem 13 the following results:

Corollary 14. *Suppose that either of the following conditions is fulfilled:*

(1) $\mathfrak{F}$ *is local and* $\mathfrak{G}$ *is* $\mathfrak{X}$*-local.*
(2) $\mathfrak{F}$ *is local and* $\mathfrak{S}_p\mathfrak{G} = \mathfrak{G}$ *for all* $p \in \operatorname{char}\mathfrak{X}$ *such that* $F_{\mathfrak{X}}(p) = \emptyset$*.*

Then $\mathfrak{H}$ *is an* $\mathfrak{X}$*-local formation.*

PROOF. If $\mathfrak{F}$ is local and $p \in \pi(\mathfrak{F})$, then $F_{\mathfrak{X}}(p) \neq \emptyset$. Therefore, Condition (1) in Theorem 13 is satisfied.

Corollary 15 ([1, IV, 3.13 and 4.8]). $\mathfrak{H}$ *is a local formation if either of the following conditions is satisfied:*

(1) $\mathfrak{F}$ *and* $\mathfrak{G}$ *are both local.*
(2) $\mathfrak{F}$ *is local and* $\mathfrak{S}_p\mathfrak{G} = \mathfrak{G}$ *for all primes* p *such that* $F_{\mathfrak{J}}(p) = \emptyset$*.*

4 $\mathfrak{X}$-local products of formations

Example 12 shows that there are many cases in which a product of an $\mathfrak{X}$-local formation and a non $\mathfrak{X}$-local formation is $\mathfrak{X}$-local. This observation leads to the following question:

Are there $\mathfrak{X}$*-local products of non* $\mathfrak{X}$*-local formations?*

The local version of the above question is the one appearing in *The Kourovka Notebook* ([7]) as Question 9.58. It was posed by Shemetkov and Skiba and answered affirmatively in several papers (see [11–13]).

The above question has a affirmative answer when $|\operatorname{char}\mathfrak{X}| \geq 2$, as the next example shows.

Example 16 ([11]). Assume that p and q are different primes in $\operatorname{char}\mathfrak{X}$. Consider the formations $\mathfrak{F} = \mathfrak{S}_p\mathfrak{A}_q \cap \mathfrak{A}_q\mathfrak{S}_p$ and $\mathfrak{G} = \mathfrak{S}_q\mathfrak{A}_p$, where $\mathfrak{A}_r$ denotes the formation of all abelian r-groups for a prime r. $\mathfrak{F}$ is not (C_q)-local and $\mathfrak{G}$ is not (C_p)-local. Therefore, by Theorem 4, $\mathfrak{F}$ and $\mathfrak{G}$ are not $\mathfrak{X}$-local. However $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$ is local and so it is $\mathfrak{X}$-local.

Bearing in mind Example 16, the following question naturally arises:

Which are the precise conditions on two non-empty formations $\mathfrak{F}$ *and* $\mathfrak{G}$ *to ensure that* $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$ *is* $\mathfrak{X}$*-local?*

Our next results answer this question.

Theorem 17. *Let $\mathfrak{K}$ be a non-empty formation. Then $\mathfrak{K}$ is $\mathfrak{X}$-local if and only if the following conditions hold:*

(1) $K_{\mathfrak{X}}(p) \subseteq \mathfrak{K}$ *for all* $p \in \operatorname{char}\mathfrak{X}$.
(2) If $U \in \mathrm{b}(\mathfrak{K})$, $V = \operatorname{Soc}(U) \in \mathfrak{S}_p$ *with* $p \in \operatorname{char}\mathfrak{X}$, *and* W *is the natural semidirect product* $[V]\big(U/\operatorname{C}_U(V)\big)$, *then* $W \in \mathrm{b}(\mathfrak{K})$.

PROOF. Assume that $\mathfrak{K}$ is an $\mathfrak{X}$-local formation. Then $\mathfrak{K} = \mathrm{LF}_{\mathfrak{X}}(K)$, where K is the $\mathfrak{X}$-formation function defined in Theorem 4. Consider a prime $p \in \operatorname{char}(\mathfrak{X})$. By Theorem 4, it follows that $K_{\mathfrak{X}}(p) \subseteq \mathfrak{K}$ so Condition 1 holds. Suppose that $U \in \mathrm{b}(\mathfrak{K})$, where $V = \operatorname{Soc}(U)$ is a p-group with $p \in \operatorname{char}\mathfrak{X}$ and consider $W = [V]\big(U/\operatorname{C}_U(V)\big)$. If $W \in \mathfrak{K}$, we would have that $W/\operatorname{C}_W(V) \in K_{\mathfrak{X}}(p)$ and, therefore, $U/\operatorname{C}_U(V) \in K_{\mathfrak{X}}(p)$. Since $U/V \in \mathfrak{K}$, this would imply by Lemma 3 that $U \in \mathrm{LF}_{\mathfrak{X}}(K) = \mathfrak{K}$. This contradiction proves that $W \notin \mathfrak{K}$. On the other hand, since $V \leq \operatorname{C}_U(V)$, it is clear that $U/\operatorname{C}_U(V) \in \mathfrak{K}$. Therefore, $W/V \in \mathfrak{K}$. Since W is monolithic, it follows that $W \in \mathrm{b}(\mathfrak{K})$ and Condition 2 holds.

To prove the sufficiency, assume that $\mathfrak{K}$ satisfies Conditions 1 and 2. We will obtain a contradiction by supposing that $\mathrm{form}_{\mathfrak{X}}(\mathfrak{K}) \setminus \mathfrak{K}$ contains a group U of minimal order. Such a U has a unique minimal normal subgroup, V, and $U/V \in \mathfrak{K}$. This is to say that $U \in \mathrm{b}(\mathfrak{K})$. According to Lemma 10, V is an abelian p-group for a prime $p \in \operatorname{char}(\mathfrak{X})$. We have that $W = [V]\big(U/\operatorname{C}_U(V)\big) \in \mathrm{b}(\mathfrak{K})$. Since $U \in \mathrm{form}_{\mathfrak{X}}(\mathfrak{K})$, $U/\operatorname{C}_U(V) \in K_{\mathfrak{X}}(p)$. Consequently, $W/V \in K_{\mathfrak{X}}(p)$ and $W \in \mathfrak{S}_p K_{\mathfrak{X}}(p) = K_{\mathfrak{X}}(p)$. Since Condition 1 states that $K_{\mathfrak{X}}(p) \subseteq \mathfrak{K}$, it follows that $W \in \mathfrak{K}$, which is a contradiction.

Remark 18. If $\mathfrak{X} = \mathfrak{J}$, then Condition 1 implies Condition 2 in the above theorem.

PROOF. Assume that $\mathfrak{K}$ satisfies Condition 1. Consider $U \in \mathrm{b}(\mathfrak{K})$, where $V = \operatorname{Soc}(U)$ is a p-group with $p \in \operatorname{char}\mathfrak{X}$ and $W = [V]\big(U/\operatorname{C}_U(V)\big)$.

Suppose that $\Phi(U) = 1$. Then U is a primitive group, $\operatorname{C}_U(V) = V$, and U is isomorphic to $W = [V](U/V)$. Therefore, $W \in \mathrm{b}(\mathfrak{K})$. Now assume that $\Phi(U) \neq 1$. Consider $T/V := \operatorname{O}_{p'}(U/V)$. Since T/V is p-nilpotent and $V \leq \Phi(U)$, we have by [14, VI, 6.3] that T is p-nilpotent. This implies that $T = V$ because otherwise we would find a non-trivial normal p'-subgroup of U. Hence, $\operatorname{O}_{p'}(U/V) = 1$. Consequently, $U/V \in K_{\mathfrak{X}}(p)$ by [1, IV, 3.7] and, hence, $U \in \mathfrak{S}_p K_{\mathfrak{X}}(p) = K_{\mathfrak{X}}(p)$. By Condition 1 we conclude that $U \in \mathfrak{K}$, which contradicts our supposition.

If $\mathfrak{Y}$ is a class of groups, denote $\mathfrak{Y}^{\mathfrak{G}} = (Y^{\mathfrak{G}} \mid Y \in \mathfrak{Y})$. The following lemma can be deduced from the proof of [11, Theorem A].

Lemma 19. *If T is a group such that $T \notin \mathfrak{G}$ and $\mathfrak{S}_p(T) \subseteq \mathfrak{H}$ for some prime p, then $\mathfrak{S}_p(T^{\mathfrak{G}}) \subseteq \mathfrak{F}$.*

Lemma 20. *Suppose that $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$, where $\mathfrak{F}$ and $\mathfrak{G}$ are non-empty formations. Consider $p \in \operatorname{char} \mathfrak{X}$. Then the following conditions are equivalent:*

(1) $H_{\mathfrak{X}}(p) \subseteq \mathfrak{H}$.
(2) $H_{\mathfrak{X}}(p)^{\mathfrak{G}} \subseteq \mathfrak{F}$.
(3) Either $H_{\mathfrak{X}}(p) \subseteq \mathfrak{G}$ *or* $\mathfrak{S}_p H_{\mathfrak{X}}(p)^{\mathfrak{G}} \subseteq \mathfrak{F}$.

PROOF. It is clear that Conditions 1 and 2 are equivalent. Now assume that Condition 1 is satisfied. If $H_{\mathfrak{X}}(p) \not\subseteq \mathfrak{G}$, take a group $T \in H_{\mathfrak{X}}(p) \setminus \mathfrak{G}$. We have that $\mathfrak{S}_p(T) \subseteq \mathfrak{S}_p H_{\mathfrak{X}}(p) = H_{\mathfrak{X}}(p) \subseteq \mathfrak{H}$. Hence, by Lemma 19, we have that $\mathfrak{S}_p(T^{\mathfrak{G}}) \subseteq \mathfrak{F}$. Now consider a group U in $\mathfrak{S}_p H_{\mathfrak{X}}(p)^{\mathfrak{G}}$. Then U has a normal p-subgroup V such that $U/V \cong R^{\mathfrak{G}}$, where $R \in H_{\mathfrak{X}}(p)$. If $R^{\mathfrak{G}} \neq 1$, we have just proved that $\mathfrak{S}_p(R^{\mathfrak{G}}) \subseteq \mathfrak{F}$ and, therefore, $U \in \mathfrak{F}$. If $R^{\mathfrak{G}} = 1$, then $U \in \mathfrak{S}_p$. Consider the group $W := U \times T^{\mathfrak{G}}$. We have that $W \in \mathfrak{S}_p(T^{\mathfrak{G}}) \subseteq \mathfrak{F}$ and, therefore, $U \in \mathrm{Q}(\mathfrak{F}) = \mathfrak{F}$. We conclude that $\mathfrak{S}_p H_{\mathfrak{X}}(p)^{\mathfrak{G}} \subseteq \mathfrak{F}$ and Condition 3 holds.

Now suppose that Condition 3 is satisfied. If $H_{\mathfrak{X}}(p) \subseteq \mathfrak{G}$, it is clear that $H_{\mathfrak{X}}(p) \subseteq \mathfrak{H}$ and Condition 1 holds. If $\mathfrak{S}_p H_{\mathfrak{X}}(p)^{\mathfrak{G}} \subseteq \mathfrak{F}$, then $H_{\mathfrak{X}}(p)^{\mathfrak{G}} \subseteq \mathfrak{F}$ and Condition 2 is satisfied.

The following theorem can be deduced from Theorem 17 and Lemma 20.

Theorem 21. *Suppose that $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$, where $\mathfrak{F}$ and $\mathfrak{G}$ are non-empty formations. Then $\mathfrak{H}$ is $\mathfrak{X}$-local if and only if the following conditions hold:*

(1) If $p \in \operatorname{char} \mathfrak{X}$, then either $H_{\mathfrak{X}}(p) \subseteq \mathfrak{G}$ or $\mathfrak{S}_p H_{\mathfrak{X}}(p)^{\mathfrak{G}} \subseteq \mathfrak{F}$.
(2) If $U \in \mathrm{b}(\mathfrak{H})$, $V = \operatorname{Soc}(U) \in \mathfrak{S}_p$ with $p \in \operatorname{char} \mathfrak{X}$, and W is the natural semidirect product $[V]\big(U/\operatorname{C}_U(V)\big)$, then $W \in \mathrm{b}(\mathfrak{H})$.

Note that if Conditions 1 and 2 hold for $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$, then it is easy to deduce that $\mathrm{b}(\mathfrak{H})$ is $\mathfrak{X}\mathfrak{G}$-free. However it seems harder to prove directly that if $\mathfrak{F}$ is $\mathfrak{X}$-local and $\mathrm{b}(\mathfrak{H})$ is $\mathfrak{X}\mathfrak{G}$-free, then Conditions 1 and 2 are satisfied.

Corollary 22 ([11, Theorem A]). *A formation product $\mathfrak{H}$ of two non-empty formations $\mathfrak{F}$ and $\mathfrak{G}$ is local if and only if $\mathfrak{H}$ satisfies the following condition:*

- *If p is a prime, then either $H_{\mathfrak{X}}(p) \subseteq \mathfrak{G}$ or $\mathfrak{S}_p H_{\mathfrak{X}}(p)^{\mathfrak{G}} \subseteq \mathfrak{F}$.*

When a product $\mathfrak{H}$ of two non-empty formations $\mathfrak{F}$ and $\mathfrak{G}$ is $\mathfrak{X}$-local, the formation $\mathfrak{G}$ has a very nice property.

Corollary 23. *If* $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$ *is* $\mathfrak{X}$*-local, then* $\mathrm{form}_{\mathfrak{X}}(\mathfrak{G}) \subseteq \mathfrak{N}_{p'}\mathfrak{G}$ *for all primes* $p \in \mathrm{char}(\mathfrak{X}) \setminus \pi(\mathfrak{F})$.

PROOF. Let $p \in \mathrm{char}(\mathfrak{X}) \setminus \pi(\mathfrak{F})$. By Theorem 21, we have that $H_{\mathfrak{X}}(p) \subseteq \mathfrak{G}$. Consider the canonical $\mathfrak{X}$-formation function G defining $\mathrm{form}_{\mathfrak{X}}(\mathfrak{G})$ (see Theorem 4). Suppose that $\mathrm{form}_{\mathfrak{X}}(\mathfrak{G})$ is not contained in $\mathfrak{N}_{p'}\mathfrak{G}$, and let $U \in \mathrm{form}_{\mathfrak{X}}(\mathfrak{G}) \setminus \mathfrak{N}_{p'}\mathfrak{G}$ be a group of minimal order. Then U has a unique minimal normal subgroup, V say. Since $U \in \mathrm{form}_{\mathfrak{X}}(\mathfrak{N}_{p'}\mathfrak{G}) \cap \mathrm{b}(\mathfrak{N}_{p'}\mathfrak{G})$, it follows form Lemma 10 that $V \in \mathfrak{S}_q$ for a prime $q \in \mathrm{char}(\mathfrak{X})$. Assume that $\Phi(U) = 1$. Then U is a primitive group and $V = \mathrm{C}_U(V)$. Therefore $U \in G(q)$. If $p \neq q$, then $U \in \mathfrak{N}_{p'}\mathfrak{G}$ because $G(q) \subseteq \mathfrak{S}_q\mathfrak{G}$ and if $p = q$, then $U \in \mathfrak{S}_p H_{\mathfrak{X}}(p) = H_{\mathfrak{X}}(p) \subseteq \mathfrak{G}$. In both cases, we reach a contradiction. Hence V is contained in $\Phi(U)$. If $p \neq q$, then the Fitting subgroup $\mathbf{F}(U)$ of U is a p'-group and $U/\mathbf{F}(U) \cong (U/V)\big/\mathbf{F}(U/V) \in \mathfrak{G}$. Hence, $U \in \mathfrak{N}_{p'}\mathfrak{G}$, contrary to supposition. Assume that $p = q$. Then, since $U/V \in \mathfrak{N}_{p'}\mathfrak{G}$, it follows that $(U/V)^{\mathfrak{G}} = U^{\mathfrak{G}}/V$ is a p'-group. Thus $U^{\mathfrak{G}}/V$ is contained in $\mathrm{O}_{p'}(U/V) = 1$ by [14, VI, 6.3]. Hence $V = U^{\mathfrak{G}}$. Since $U \in \mathfrak{H}$, we have that $U^{\mathfrak{G}} = V \in \mathfrak{F}$ and $p \in \pi(\mathfrak{F})$. This final contradiction proves that $\mathrm{form}_{\mathfrak{X}}(\mathfrak{G}) \subseteq \mathfrak{N}_{p'}\mathfrak{G}$.

If $\mathfrak{X} = \mathfrak{J}$, we have:

Corollary 24 ([15]). *If* $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$ *is local, then* $\mathrm{lform}(\mathfrak{G})$ *is contained in* $\mathfrak{N}_{p'}\mathfrak{G}$ *for all primes* $p \notin \pi(\mathfrak{F})$.

5 p-saturated products of formations

Corollary 24 motivates the following definition.

Definition 25. [15] Let ω be a non-empty set of primes, and let $\mathfrak{F}$ be a formation. We say that $\mathfrak{F}$ is ω*-local* if $\mathrm{lform}(\mathfrak{F})$ is contained in $\mathfrak{N}_{\omega'}\mathfrak{F}$.

When $\omega = \{p\}$, we shall say p*-local* instead of $\{p\}$-local. It is clear that a formation $\mathfrak{F}$ is ω-local if and only if $\mathfrak{F}$ is p-local for all $p \in \omega$. In particular, $\mathfrak{F}$ is local if and only if $\mathfrak{F}$ is p-local for every prime p.

Shemetkov and Skiba ([5, Theorem 1]) proved that a formation $\mathfrak{F}$ is ω-saturated if and only if $\mathfrak{F}$ is ω-local and use what they call ω*-local satellites* to study the structure of ω-local formations.

In [16], it is shown that ω-saturated formations are $\mathfrak{X}_\omega$-local, where $\mathfrak{X}_\omega$ is the class of all simple ω-groups. However, the converse does not hold.

The following lemma gives a characterization of p-saturated formations.

Lemma 26. *Let $\mathfrak{K}$ be a non-empty formation. Then $\mathfrak{K}$ is p-saturated if and only if $K_{\mathfrak{J}}(p) \subseteq \mathfrak{K}$.*

PROOF. Suppose that $\mathfrak{K}$ is a p-saturated formation, where p is a prime. Then $\operatorname{lform}(\mathfrak{K}) \subseteq \mathfrak{N}_{p'}\mathfrak{K}$. By Theorem 4, we know that $K_{\mathfrak{J}}(p) \subseteq \operatorname{lform}(\mathfrak{K})$ and, therefore, $K_{\mathfrak{J}}(p) \subseteq \mathfrak{N}_{p'}\mathfrak{K}$. This implies that $K_{\mathfrak{J}}(p) \subseteq \mathfrak{K}$.

Now suppose that $\mathfrak{K}$ is not p-saturated and $K_{\mathfrak{J}}(p) \subseteq \mathfrak{K}$. Let U be a group of minimal order satisfying $U/\big(\Phi(U) \cap \mathrm{O}_p(U)\big) \in \mathfrak{K}$ and $U \notin \mathfrak{K}$. Then U is a monolithic group and $V := \mathrm{Soc}(U) \leq \Phi(U) \cap \mathrm{O}_p(U)$. We have that $\mathrm{O}_{p',p}(U/V) = \mathrm{O}_{p',p}(U)/V$, since $V \leq \Phi(U)$. Moreover, $U/V \in \mathfrak{K}$ and, therefore, $U/\mathrm{O}_{p',p}(U) \in K_{\mathfrak{J}}(p)$, bearing in mind that $p \in \pi(U/V)$. Since $\mathrm{O}_{p',p}(U) = \mathrm{O}_p(U)$, $U \in K_{\mathfrak{J}}(p) \subseteq \mathfrak{K}$. This is not possible.

Theorem 27. *Suppose that $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$, where $\mathfrak{F}$ and $\mathfrak{G}$ are non-empty formations, and let p be a prime. Then the following statements are equivalent:*

(1) $\mathfrak{H}$ is a p-saturated formation.
(2) Either $H_{\mathfrak{J}}(p) \subseteq \mathfrak{G}$ or $\mathfrak{S}_p H_{\mathfrak{J}}(p)^{\mathfrak{G}} \subseteq \mathfrak{F}$.

PROOF. It is clear applying Lemma 26 and Lemma 20 for $\mathfrak{X} = \mathfrak{J}$.

Theorem 27 also confirms a more general version of the abovementioned conjecture of Shemetkov concerning the non-decomposability of the formation of all p-groups (p a prime) as formation product of two non-trivial subformations.

Corollary 28. *Suppose that $\mathfrak{H} = \mathfrak{F} \circ \mathfrak{G}$, where $\mathfrak{F}$ and $\mathfrak{G}$ are non-empty formations, and $\mathfrak{H}$ is p-saturated. If $\mathfrak{F} \subseteq \mathfrak{S}_p$ and $\mathfrak{F} \neq \mathfrak{S}_p$, then $\mathfrak{H} = \mathfrak{G}$.*

PROOF. If $H_{\mathfrak{J}}(p) = \emptyset$, it follows that $\mathfrak{H} \subseteq \mathfrak{E}_{p'}$. In this case, we have that $\mathfrak{H} \subseteq \mathfrak{E}_{p'} \cap (\mathfrak{S}_p \circ \mathfrak{G})$. Therefore, $\mathfrak{H} \subseteq \mathfrak{G}$. If $H_{\mathfrak{J}}(p) \neq \emptyset$, we have that $\mathfrak{H} \subseteq \mathfrak{E}_{p'} H_{\mathfrak{J}}(p)$. If $H_{\mathfrak{J}}(p)$ is contained in $\mathfrak{G}$, then $\mathfrak{H} \subseteq \big(\mathfrak{E}_{p'} H_{\mathfrak{J}}(p)\big) \cap (\mathfrak{S}_p \mathfrak{G}) \subseteq (\mathfrak{E}_{p'} \mathfrak{G}) \cap (\mathfrak{S}_p \mathfrak{G}) = \mathfrak{G}$ and the result holds. Suppose that $H_{\mathfrak{J}}(p)$ is not contained in $\mathfrak{G}$. Then $\mathfrak{S}_p H_{\mathfrak{J}}(p)^{\mathfrak{G}}$ is contained in $\mathfrak{F}$ by Theorem 27. In particular, $\mathfrak{S}_p \subseteq \mathfrak{F}$, and we have a contradiction.

Acknowledgements

The authors are indebted to the referee for his/her helpful comments, which have improved the exposition of the paper.

This research is supported by Grant MTM2004-08219-C02-02, from MEC (Spain) and FEDER (European Union).

The second author has been supported by a predoctoral grant from MEC (Spain).

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