On Finite Minimal Non-nilpotent Groups

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Abstract

A critical group for a class of groups $\mathcal{X}$ is a minimal non-$\mathcal{X}$-group. The critical groups are determined for various classes of finite groups. As a consequence, a classification of the minimal non-nilpotent groups (also called Schmidt groups) is given, together with a complete proof of Gol’fand’s theorem on maximal Schmidt groups.

1 Introduction

Given a class of groups $\mathcal{X}$, we say that a group $G$ is a minimal non-$\mathcal{X}$-group, or an $\mathcal{X}$-critical group, if $G \not\in \mathcal{X}$, but all proper subgroups of $G$ belong to $\mathcal{X}$. It is clear that detailed knowledge of the structure of minimal non-$\mathcal{X}$-groups can provide insight into what makes a group belong to $\mathcal{X}$. All groups considered in this paper are finite.

Minimal non-$\mathcal{X}$-groups have been studied for various classes of groups $\mathcal{X}$. For instance, minimal non-abelian groups were analysed by Miller and Moreno [10], while Schmidt [14] studied minimal non-nilpotent groups. The latter are now known as Schmidt groups. Itô [9] considered the minimal non-$p$-nilpotent groups for $p$ a prime, which turn out to be just the Schmidt groups. Finally, the third author [12] characterised the minimal non-$T$-groups ($T$-groups are groups in which normality is a transitive relation). He also
characterised in [13] the minimal non-PST-groups, where a PST-group is a group in which Sylow permutability is a transitive relation.

The aim of this paper is to give more precise information about the structure of Schmidt groups and show how to construct them in an efficient way. As a consequence of our study, a new proof of a classical theorem of Gol’fand is given.

Our approach depends on the classification of critical groups for the class of PST-groups given in [13]. Recall that a subgroup $H$ is said to be Sylow-permutable, or S-permutable, in a group $G$ if $H$ permutes with every Sylow subgroup of $G$. We mention a similar class $\mathcal{Y}_p$, which was introduced in [2].

If $p$ is a prime, a group $G$ belongs to the class $\mathcal{Y}_p$ if $G$ enjoys the following property: if $H$ and $K$ are $p$-subgroups of $G$ such that $H$ is contained in $K$, then $H$ is S-permutable in $N_G(K)$. Clearly every PST-group is a $\mathcal{Y}_p$-group.

There is a close relation between the class of groups just introduced and $p$-nilpotence, as in shown by the following result, which was proved in [2; Theorem 5].

**Theorem 1.** A group $G$ is a $\mathcal{Y}_p$-group if and only if either it is $p$-nilpotent or it has an abelian Sylow $p$-subgroup $P$ and every subgroup of $P$ is normal in $N_G(P)$.

Our first main result is:

**Theorem 2.** The minimal non-$\mathcal{Y}_p$-groups are just the minimal non-PST-groups with a non-trivial normal Sylow $p$-subgroup. Such groups are of the types described in I to IV below. Let $p$ and $q$ be distinct primes.

**Type I:** $G = [P]Q$, where $P = \langle a, b \rangle$ is an elementary abelian group of order $p^2$, $Q = \langle z \rangle$ is cyclic of order $q^r$, with $q$ a prime such that $q^r$ divides $p - 1$, $q^r > 1$ and $r \geq f$, and $a^z = a^i$, $b^z = b^{j+1}$, where $i$ is the least positive primitive $q^f$-th root of unity modulo $p$ and $j = 1 + qk^{f-1}$, with $0 < k < q$.

**Type II:** $G = [P]Q$, where $Q = \langle z \rangle$ is cyclic of order $q^r > 1$, with $q$ a prime not dividing $p - 1$ and $P$ an irreducible $Q$-module over the field of $p$ elements with centralizer $\langle z^q \rangle$ in $Q$.

**Type III:** $G = [P]Q$, where $P = \langle a_0, a_1, \ldots, a_{q-1} \rangle$ is an elementary abelian $p$-group of order $p^q$, $Q = \langle z \rangle$ is cyclic of order $q^r$, with $q$ a prime such that $q^r$ is the highest power of $q$ dividing $p - 1$ and $r > f$. Define $a_j^z = a_{j+1}$ for $0 \leq j < q - 1$ and $a^z_{q-1} = a_0$, where $i$ is a primitive $q^f$-th root of unity modulo $p$. 

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**Type IV:** \(G = [P]Q\), where \(P\) is a non-abelian special \(p\)-group of rank \(2m\), the order of \(p\) modulo \(q\) being \(2m\), \(Q = \langle z \rangle\) is cyclic of order \(q’ > 1\), \(z\) induces an automorphism in \(P\) such that \(P/\Phi(P)\) is a faithful irreducible \(Q\)-module, and \(z\) centralizes \(\Phi(P)\). Furthermore, \(|P/\Phi(P)| = p^{2m}\) and \(|P’| \leq p^m\).

Since a group is a soluble \(PST\)-group if and only if it belongs to \(\mathcal{Y}_p\) for all primes \(p\) [2; Theorem 4], Theorem 2 may be regarded as a local approach to the third author’s classification of minimal non-\(PST\)-groups [13].

An interesting consequence of Theorem 2 is the following classification of Schmidt groups. In order to describe the classification, we must introduce one further type of group:

**Type V:** \(G = [P]Q\), where \(P = \langle a \rangle\) is a normal subgroup of order \(p\), \(Q = \langle z \rangle\) is cyclic of order \(q’ > 1\), and \(a^i = z \cdot a^i\), where \(i\) is the least primitive \(q\)-th root of unity modulo \(p\).

Our main result can now be stated as:

**Theorem 3.** The Schmidt groups are exactly the groups of Type II, Type IV and Type V.

Our next result shows that \(p\)-soluble groups with Sylow \(p\)-subgroups isomorphic to a normal subgroup of a minimal non-\(\mathcal{Y}_p\)-group have a restricted structure.

**Theorem 4.** Let \(G\) be a \(p\)-soluble group with a Sylow \(p\)-subgroup \(P\). If \(P\) is isomorphic to a non-trivial normal Sylow subgroup of a minimal non-\(\mathcal{Y}_p\)-group, then \(G\) has \(p\)-length 1.

In [4] Gol’fand stated the following result:

**Theorem 5.** Let \(p\) and \(q\) be distinct primes, let \(r\) be a given positive integer, and let \(a\) be the order of \(p\) modulo \(q\). Then there is a unique minimal non-\(p\)-nilpotent group \(G_0\) of order \(p^{a_0}q^r\), where \(a_0 = a\) if \(a\) is odd and \(a_0 = 3a/2\) if \(a\) is even, such that all minimal non-\(p\)-nilpotent groups of order \(p^i q^r\) are isomorphic to quotients of \(G_0\) by central subgroups.

Only a sketch of proof of this theorem is given in Gol’fand’s article. In Section 3, we show how to construct the Schmidt groups of Gol’fand and we also give a complete proof of Theorem 5. We remark that Rédei [11] has given another construction of the Schmidt groups of maximum order.
2 Proofs of Theorems 2, 3 and 4

Proof of Theorem 2. Assume that $G$ is a minimal non-$\mathcal{Y}_p$-group and let $P$ be a Sylow $p$-subgroup of $G$. Since $G$ does not belong to $\mathcal{Y}_p$, there exist subgroups $H$ and $K$ of $P$ such that $H \leq K$ and $H$ is not $S$-permutable in $N_G(K)$. Consequently there is an element $z \in N_G(K)$ such that $z$ does not normalise $H$. Here it can be assumed that $z$ has order $q^r$ for some prime $q \neq p$. Then $G = K \langle z \rangle$ because $G$ is a minimal non-$\mathcal{Y}_p$-group. This implies that $K$ is a normal Sylow $p$-subgroup of $G$ and $Q = \langle z \rangle$ is a cyclic Sylow $q$-subgroup of $G$. Then $G$ is not a $PST$-group, yet every proper subgroup has $\mathcal{Y}_p$ and $\mathcal{Y}_q$, and thus is a $PST$-group by [2].

Conversely, if $G$ is a minimal non-$PST$-group, then $G$ does not have $\mathcal{Y}_p$ for some prime $p$. Since all its proper subgroups satisfy $\mathcal{Y}_p$, the group $G$ is a minimal non-$\mathcal{Y}_p$-group. The classification of minimal non-$PST$-groups given in [13] completes the proof. (Notice that the groups of Types IV and V of [13] are both of Type IV above).

Proof of Theorem 3. Let $G$ be a minimal non-nilpotent group. Then $G$ is a minimal non-$p$-nilpotent group for some prime $p$. Suppose that $G$ is not a $\mathcal{Y}_p$-group, so that $G$ is a minimal non-$\mathcal{Y}_p$-group. By Theorem 2, the group $G$ is of one of the types I–IV. By examining the group structure, we see that groups of Type I and III are not minimal non-$p$-nilpotent. Therefore $G$ must be of Type II or IV.

Assume now that $G$ belongs to $\mathcal{Y}_p$. Then by [1; Theorem A] and [3; VII, 6.18], the $p$-nilpotent residual $P$ of $G$ is an abelian minimal normal Sylow subgroup which is complemented in $G$ by a cyclic Sylow $q$-subgroup $Q$. Moreover $Q$ normalizes each subgroup of $P$. This implies that $P$ is cyclic of order $p$, say $P = \langle a \rangle$. In addition, $a^i = a^j$ for some $0 < i < j$ and $z^q$ centralizes $a$. This implies that $i$ must be a primitive $q$-th root of unity modulo $p$ and, by taking a suitable power of $z$ as a generator of $Q$, we can assume that $i$ is the least such positive integer. Hence $G$ is of Type V.

Proof of Theorem 4. Assume that $G$ is a $p$-soluble group with $p$-length $> 1$ and $G$ has least order subject to possessing a Sylow $p$-subgroup $P$ which is isomorphic to a non-trivial normal Sylow subgroup of a Schmidt group. By [6; VI, 6.10], we conclude that $P$ is not abelian. Thus $P$ is a Sylow $p$-subgroup of a group of Type IV in Theorem 2. By minimality of order $O_p'(G) = 1$ and $O_p^p(G) = G$. In addition, since the class of groups of $p$-length at most 1 is a saturated formation, we have $\Phi(G) = 1$ and hence $G$ has a unique minimal normal subgroup which is an elementary abelian $p$-group. Let $D = O_p(G)$; then $D$ is a non-trivial elementary abelian group and $C_G(D) = D$. Moreover $\Phi(P) = Z(P) \leq D$ and so $P/D$ is elementary abelian.
Let $T$ be the subgroup defined by $T/D = O_p'(G/D)$. Since $P/D$ is an elementary abelian $p$-group, $G/D$ has $p$-length at most 1 by [6; VI, 6.10]. It follows that $(T/D)(P/D)$ is a normal subgroup of $G/D$. Therefore $TP$ is a normal subgroup of $G$. Assume that $TP$ is a proper subgroup of $G$. Now $O_p'(TP) \leq O_p'(G) = 1$, so $P$ is a normal subgroup of $TP$ and hence of $G$, a contradiction which shows that $G = TP$.

Assume now that $P/D$ is a non-cyclic elementary abelian group. By [8; X, 1.9], we have $T/D = \langle C_{T/D}(xD) \mid xD \in P/D, xD \neq D \rangle$. Let $x \in P D$. Since $P/D$ centralizes $xD$, we have $P/D \leq N_{G/D}(C_{T/D}(xD))$. Let $T_x/D = C_{T/D}(xD)$. Assume that $PT_x = G$; then $T_x = T$ is a normal subgroup of $G$ and thus $O_p'(G/D) = T_x/D$. This implies that $(x)D/D \leq Z(G/D)$ and $(x)D$ is a normal $p$-subgroup of $G$, so that $(x)D$ is contained in $D$, a contradiction. Consequently $PT_x$ is a proper subgroup of $G$ for all $1 \neq xD \in P/D$. Hence $PT_x$ has $p$-length at most 1 by minimality of $G$. Since $C_G(D) = D$ and $O_p'(PT_x)$ centralizes $D$, we conclude that $O_p'(PT_x) = 1$. Therefore $P$ is a normal subgroup of $PT_x$, which shows that $T$ normalizes $P$ and thus $P$ is a normal subgroup of $G$. This contradiction shows that $P/D$ is cyclic.

Since $P$ has class 2, we see from [7; IX, 5.5] that, if $p > 3$, then $G$ has $p$-length at most 1. Therefore $p \leq 3$. Let $X$ be a minimal non-$\mathcal{Y}_p$-group such that $P$ is a Sylow $p$-subgroup of $X$. Note that $P/\Phi(P)$ is an irreducible $X$-module. In particular $D$, the subgroup of the previous paragraphs, is not normal in $X$ and so $P = DD^g$ for some $g \in X$. Since $D$ is abelian, $D \cap D^g \leq Z(P) = \Phi(P)$, and it follows that $P/\Phi(P)$ has order $p^2$. This implies that $P$ is an extra-special group of order $p^3$. If $p = 2$, then, since $C_G(D) = D$, we see that $G$ must be a symmetric group of degree 4. Hence $P$ is dihedral of order 8, which cannot lead to a group of Type IV since $\text{Aut}(P)$ is a 2-group. Hence $p = 3$. But a non-abelian group of order $3^3$ cannot occur as the normal Sylow 3-subgroup of a Schmidt group, because the only prime divisor of $3^2 - 1$ is 2 and the order of 3 modulo 2 is 1. This contradiction completes the proof of the theorem.

\[\square\]

3 The Construction of Gol’fand’s Groups and a Proof of Gol’fand’s Theorem

We begin by constructing groups of Type IV with a Sylow $p$-subgroup $P$ of order $p^{3m}$ and $|P/\Phi(P)| = p^{2m}$. These groups were constructed in [13] by a different method, but the present approach is more convenient when $p = 2$. We will use the following result on linear operators.
Lemma 6. Let $p$ be a prime and let $r$ be a positive integer such that $\gcd(p, r) = 1$. Let $\beta$ be a linear operator of order $p^ru$ on a vector space $V$ over the field of $p$-elements, where $u$ is a non-negative integer. If $\beta$ has irreducible minimum polynomial $f$, then $\beta^{pu}$ also has minimum polynomial $f$.

Proof. Let $g$ be the minimum polynomial of $\beta^{pu}$. Now $f(\beta^{pu}) = f(\beta)^{pu} = 0$, so that $g$ divides $f$. Since $f$ is irreducible, $f = g$. \hfill $\square$

Construction 7. Let $p$ and $q$ be distinct primes such that the order of $p$ modulo $q$ is $2m$, $m \geq 1$. Let $F$ be the free group with basis $\{f_0, f_1, \ldots, f_{2m-1}\}$. Write $R = F^p F$ and $R^* = [F, R] R^p$. Then $F/R$ is an elementary abelian $p$-group of order $p^{2m}$ and $H = F/R^*$ is a $p$-group such that $R/R^* = \Phi(H)$ is an elementary abelian $p$-group contained in $\mathbb{Z}(H)$. Moreover $H$ is a non-abelian group because an extra-special group of order $p^{2m+1}$ is an epimorphic image of $H$.

Denote by $g_i$ the image of $f_i$ under the natural epimorphism of $F$ onto $H = F/R^*$, $0 \leq i \leq 2m-1$. Since $H$ has class $2$, we know that $\Phi(H)$ is generated by all $[g_i, g_j]$, with $i < j$, and $g_{i}^{p}$. Therefore $\Phi(H)$ has dimension as $\text{GF}(p)$-vector space at most $\frac{1}{2}(2m(2m - 1) + 2m = m(2m + 1))$. Assume that the dimension is less than $m(2m + 1)$. Then there exists an element

$$
r = \prod_{j} (f_{j}^{p})^{\lambda_{j}} \prod_{j<k} [f_{j}, f_{k}]^{\mu_{jk}} \in R^*
$$

with some $\lambda_j$ or $\mu_{jk}$ not divisible by $p$. It is clear that $p \mid \lambda_{j}$ for all $j$ since $F^{p} F'/F'$ is a free abelian group with basis $\{f_{j}^{p} F' \mid 0 \leq j \leq 2m-1\}$. Suppose that $p \nmid \mu_{ik}$ for some $i < k$ and let $\rho_{i}$ be the endomorphism of $F$ defined by $\rho_{i}^{p} f = f_{i}^{p}$, $f_{i}^{p} = f_{i}$ for $l \neq i$. Then $\rho_{i}^{p} R^* = R^*$ and so $\rho_{i}^{p} R^* = R^*$. This implies that

$$
w = \prod_{j<i} [f_{j}, f_{i}]^{\mu_{ji}} \prod_{i<l} [f_{i}, f_{l}]^{\mu_{il}} \in R^*.
$$

On the other hand, by applying $\rho_{k}$ we find that

$$
w^{\rho_{k} w^{-1} R^*} = [f_{i}, f_{k}]^{\mu_{ik}} R^* = R^*.
$$

Since $p \nmid \mu_{ik}$, it follows that $\mu_{ik}$ has an inverse modulo $p$. This means that $[f_{i}, f_{k}] \in R^*$. Now since permutations of the generators of $F$ induce endomorphisms in $F$ and $R^*$ is fully invariant, it follows that $F' \leq R^*$ and $H$ is abelian, a contradiction. Therefore $\Phi(H)$ has dimension $m(2m + 1)$ and so $|\Phi(H)| = p^{m(2m+1)}$.

Let $f(t) = c_{0} + c_{1} t + \cdots + c_{2m-1} t^{2m-1} + t^{2m}$ be an irreducible factor of the cyclotomic polynomial of order $q$ over $\text{GF}(p)$ and let $\alpha$ be the endomorphism
of $F$ given by $f_i^α = f_{i+1}$ for $0 ≤ i ≤ 2m - 2$, $f_{2m-1}^α = f_0^{-c_0}f_1^{-c_1} \cdots f_{2m-1}^{-c_{2m-1}}$. Since $R^α$ is a fully invariant subgroup of $F$, it follows that $α$ induces an endomorphism $β$ on $H = F/R^α$. In turn, $β$ induces an automorphism $\bar{β}$ on $H/Φ(H)$. Since $H/Φ(H) = (H/Φ(H))^{\bar{β}} \leq H^{β}Φ(H)/Φ(H)$, it follows that $H = H^{β}Φ(H)$, whence $H = H^β$. Consequently $β$ is an automorphism of $H$.

It is clear that $β$ induces the linear operator $\bar{β}$, with minimum polynomial $f$, on the vector space $H/Φ(H)$. Now by [6; III, 3.18], we conclude that $β^q$ has order $p^m$ for some $u$ and hence $β$ has order $p^μq$. By Lemma 6, there is a $GF(p)$-basis $\{g_0, g_1, \ldots, g_{2m-1}\}$ of $H/Φ(H)$, where $g_i^α = g_iΦ(H)$, such that $g_i^{β^q} = g_i^{q^2}$ for $0 ≤ i ≤ 2m - 2$ and $g_{2m-1}^{β^q} = g_0^{-c_0}g_1^{-c_1} \cdots g_{2m-1}^{-c_{2m-1}}$. Hence we can replace $β$ by $β^{p^m}$ and assume without loss of generality that $β$ has order $q$.

It follows that $Φ(H)$ is a $GF(p)T$-module, where $T = \langle β \rangle$ is a cyclic group of order $q$. By Maschke’s Theorem $Φ(H)$ is a direct sum of irreducible $T$-modules. Let $N$ be the sum of all non-trivial irreducible submodules in the direct decomposition and write $P = H/N$. It is clear that $N$ is $β$-invariant and therefore $β$ induces an automorphism $γ$ of order $q$ in $P$. Let $Q = \langle z \rangle$ be a cyclic group of order $q^r$ acting on $P$ via $z \mapsto γ$. Let $G = [P]Q$ be the corresponding semidirect product.

It is easily checked that $G$ is a Schmidt group. Next we show that $P$ has order $p^{3m}$. From Theorem 3 we see that $Φ(P)$ has order at most $p^m$, where $|P/Φ(P)| = p^{2m}$. On the other hand, $|Φ(H)| = p^m(2m+1)$, and $N$ has order a power of $p^{2m}$ because every faithful irreducible $\langle β \rangle$-module over $GF(p)$ has dimension $2m$. Therefore $|Φ(P)| = p^m$.

Remark 8. In the group of Construction 7, we may assume that $\bar{g}_{2m-1}^z = \bar{g}_0^{-c_0}\bar{g}_1^{-c_1} \cdots \bar{g}_{2m-1}^{-c_{2m-1}}$, where $g_i = g_iN$.

Proof. We know that $\bar{g}_{2m-1}^z = \bar{g}_0^{-c_0}\bar{g}_1^{-c_1} \cdots \bar{g}_{2m-1}^{-c_{2m-1}}$ where $\bar{w} ∈ Φ(P)$. Since $f(t)$ is irreducible, 1 is not a root of $f(t)$ and it follows that $c = c_0 + c_1 + \cdots + c_{2m-1} + 1 ≠ 0 (mod p)$. Consequently there exists an integer $d$ such that $cd ≡ -1 (mod p)$. Put $w_0 = \bar{w}^d$ and consider the automorphism $δ$ of $P$ defined by $\bar{g}_i^δ = \bar{g}_i w_0$ for $0 ≤ i ≤ 2m - 1$. If we write $γ_0 = δγδ^{-1}$, it is easily checked by an elementary calculation that $\bar{g}_i^{γ_0} = \bar{g}_{i+1}$ for $0 ≤ i ≤ 2m - 2$, and $\bar{g}_{2m-1}^{γ_0} = \bar{g}_0^{-c_0}\bar{g}_1^{-c_1} \cdots \bar{g}_{2m-1}^{-c_{2m-1}}$. Let $\langle z_0 \rangle$ be a cyclic group of order $q^r$, with $z_0$ acting on $P$ via $z_0 \mapsto γ_0$. Since $\langle z_0 \rangle$ and $\langle z \rangle$ are conjugate in $Aut(P)$, it follows by [3; B, 12.1] that the groups $P\langle z \rangle$ and $P\langle z_0 \rangle$ are isomorphic.

Remark 9. The group in Construction 7 does not depend on the choice of irreducible factor $f(t)$.
Proof. Assume that the group $G_1 = [P_1]\langle z_1 \rangle$ has been constructed by using another irreducible factor $g(t)$ of the cyclotomic polynomial of order $q$ over $\text{GF}(p)$. Since $G$ and $G_1$ have the same order, it will be enough to find a set of generators of $G_1$ for which the relations of $G$ hold. Since $z$ centralizes $\Phi(P)$ and $z_1$ centralizes $\Phi(P_1)$, we have $G/\Phi(P) \cong [P/\Phi(P)]\langle z \rangle$ and $G_1/\Phi(P_1) \cong [P_1/\Phi(P_1)]\langle z_1 \rangle$. But $P/\Phi(P)$ and $P_1/\Phi(P_1)$ are faithful irreducible modules for a cyclic group of order $q$. Therefore $[P/\Phi(P)]((z)/(z^q))$ is isomorphic to $[P_1/\Phi(P_1)]((z_1)/(z_1^q))$ by [3; B, 12.4]. Let $\phi$ be an isomorphism between these groups. Then it is clear that $\phi$ induces an isomorphism $\psi$ between $G/\Phi(P)$ and $G_1/\Phi(P_1)$.

Let $\tilde{h}_i = h_i \Phi(P)$, $0 \leq i \leq 2m - 1$. Put $\tilde{k}_i = \tilde{h}_i^\psi$ and $\tilde{u} = \tilde{z}^\psi$. We show how to extend the isomorphism $\psi$ to an isomorphism between $G$ and $G_1$. In order to do so, we choose representatives $k_i$ of $\tilde{k}_i$ and $u$ of $\tilde{u}$ such that the order of $u$ is $q'$. There is no loss of generality in assuming that $k_i^u = k_{i+1}^u$ for $0 \leq i \leq 2m - 2$: indeed, if $k_i^u = k_{i+1}^u w_{i+1}$, with $w_{i+1} \in \Phi(P_1)$, then $k_i^u = k_i w_i \cdots w_1$ for $1 \leq i \leq 2m - 1$, $k_0' = k_0$ are representatives of $k_i$ and $k_i'^u = k_{i+1}'^u$ for $1 \leq i \leq 2m - 1$ because $u$ centralizes $\Phi(P_1)$. By using the same argument as in Remark 8, we may also assume $k_i'^u = k_0'^u k_1'^u \cdots k_{2m-1}'^u$. Therefore $G$ and $G_1$ satisfy the same relations and by Von Dyck’s theorem they are isomorphic.

Remark 10. In Construction 7, it is not necessary to assume that $\beta$ has order $q$. Indeed, it can be proved that $\beta^q$ fixes all elements of $\Phi(H)$ and that the automorphism $\gamma$ induced by $\beta$ in $H/N$ has order $q$.

Gol’fand’s result (Theorem 5) can be recovered with the help of Construction 7 and Theorem 3.

Proof of Theorem 5. Let $p$ and $q$ be distinct primes and let $a$ be the order of $p$ modulo $q$. Then $a$ is the dimension of each non-trivial irreducible module for a cyclic group of order $q$ over $\text{GF}(p)$. Assume that $a$ is odd. Then every Schmidt group $G$ with a normal Sylow $p$-subgroup $P$ such that $|P/\Phi(P)| = p^a$ is of Type II or Type V. Then the theorem holds in this case because all Schmidt groups of the same type with isomorphic Sylow $q$-subgroups are actually isomorphic.

Assume now that $a$ is even, with say $a = 2m$. Then we are dealing with Schmidt groups of Type II or Type IV. Let $G_0$ be the group of Construction 7. Then $|G_0| = p^{3m} q^r$ and $|P_0/\Phi(P_0)| = p^{2m}$, where $P_0$ is a normal Sylow $p$-subgroup of $G_0$. It is clear that $G_0/\Phi(P_0)$ is a Schmidt group of Type II. Therefore, if $G$ is a Schmidt group of Type II with order $p^t q^r$ and a normal Sylow $p$-subgroup, then $G \cong G_0/\Phi(P_0)$ and $\Phi(P_0) \leq Z(G_0)$. Consequently, we need only show that all Schmidt groups of Type IV and order $p^t q^r$, $t \leq 3m$,
which have a normal Sylow-$p$-subgroup are isomorphic to quotients of $G_0$ by central subgroups.

Let $\overline{G}$ be a Schmidt group of Type IV and order $p'q^r$ with a normal Sylow $p$-subgroup $P$. Then $G_0/\Phi(P_0)$ and $\overline{G}/\Phi(\overline{P})$ are isomorphic. Let us choose generators $z$ and $\bar{z}$ of Sylow $q$-subgroups $Q$ of $G_0$ and $\overline{Q}$ of $\overline{G}$ such that the minimum polynomial of the actions of $z$ on $P_0/\Phi(P_0)$ and $\bar{z}$ on $\overline{P}/\Phi(\overline{P})$ coincide. Also choose generators $g_0, g_1, \ldots, g_{2m-1}$ of the Sylow $p$-subgroup $P_0$ of $G_0$ and generators $\bar{g}_0, \bar{g}_1, \ldots, \bar{g}_{2m-1}$ of the Sylow $p$-subgroup $\overline{P}$ of $\overline{G}$ such that $g_j^z = g_{j+1}$ and $\bar{g}_j^\bar{z} = \bar{g}_{j+1}$ for $0 \leq j \leq 2m - 2$. Since $\Phi(P_0) = P'_0$ and $\Phi(\overline{P}) = \overline{P}'$, and both $P_0$ and $\overline{P}$ have class 2, the subgroup $\Phi(P_0)$ can be generated by the commutators $[g_i, g_j]$, while $\Phi(\overline{P})$ is generated by the commutators $[\bar{g}_i, \bar{g}_j]$. On the other hand, if $u_i = [g_0, g_i^z]$, we have $u_i = u_{i+1} = [g_k, g_k^z]$. It is easy to see that $u_i = [g_0, g_i^z] = [g_0^q, g_0^z] = u_{q-i}$.

Observe that $q$ is odd since $2m$ divides $q - 1$: write $q = 2s + 1$. By definition of the $g_i$ and $u_i$, and use of the minimum polynomial of the action of $z$ on $P_0/\Phi(P_0)$, it may be shown that for $l \geq 1$

$$u_{s+m+l} = u_{s-m+l}u_{s-m+l+1}^{-1} \cdots u_{s+m+l-2}^{-1}u_{s+m+l-1}^{-1}.$$

Now this formula and the relations $u_i = u_{q-i}^{-1}$ allow us to show by induction that each $u_{s+m+l}$ can be expressed in terms of elements of the set $B = \{u_{s-m+l}, u_{s-m+2}, \ldots, u_s\}$. Since $\Phi(P_0)$ has dimension $m$ over $GF(p)$, this expression is unique. It follows that each $u_j$ can be uniquely expressed in terms of the elements of $B$, and so this is also true for each generator of $\Phi(P_0)$. The same argument shows that the generators of $\Phi(\overline{P})$ have a similar unique expression subject to the same relations.

The arguments of Remark 9 allow us to assume that

$$\bar{g}_{2m-1} = \bar{g}_0^{-c_0}\bar{g}_1^{-c_1} \cdots \bar{g}_{2m-1}^{-c_{2m-1}} \text{ and } g_{2m-1} = g_0^{-c_0}g_1^{-c_1} \cdots g_{2m-1}^{-c_{2m-1}}.$$

Consequently, all relations of $G_0$ are satisfied by $\overline{G}$. By Von Dyck’s theorem, it follows that $\overline{G}$ is an epimorphic image of $G_0$ by a central subgroup of $G_0$.

\section*{References}


