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A NEW LATTICE ACTION FOR STUDYING TOPOLOGICAL CHARGE

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Abstract

We propose a new lattice action for non-abelian gauge theories, which will reduce short-range lattice artifacts in the computation of the topological susceptibility. The standard Wilson action is replaced by the Wilson action of a gauge covariant interpolation of the original fields to a finer lattice. If the latter is fine enough, the action of all configurations with non-zero topological charge will satisfy the continuum bound. As a simpler example we consider the $O(3)$ σ -model in two dimensions, where a numerical analysis of discretized continuum instantons indicates that a finer lattice with half the lattice spacing of the original is enough to satisfy the continuum bound.

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Field configurations with non-zero topological charge are expected to have a strong influence on the dynamics of asymptotically free theories. In QCD, such configurations are responsible for breaking axial symmetry and resolving the U(1) problem [1]. The study of these effects however requires non-perturbative techniques and one would expect that ultimately Monte Carlo methods on the lattice would be best suited to it. The observable to consider is inspired by the classic large- N_c analyses of Witten and of Veneziano [2], which showed that,

$$m_{\eta'}^2 + m_{\eta}^2 - 2 m_K^2 = \frac{6 \chi_t}{f_{\pi}^2}, \quad (1)$$

where χ_t is the topological susceptibility. In the continuum it is given by,

$$\chi_t \equiv \int d^4x \langle q(x)q(0) \rangle |_{no\ quarks} \quad (2)$$

with $q(x)$ being the topological charge density,

$$q(x) = \frac{1}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} Tr[F_{\mu\nu} F_{\rho\sigma}]. \quad (3)$$

The topological charge, $Q \equiv \int q(x)$, is an integer if the field strength vanishes at infinity or if (euclidean) space-time is compact. A continuum analysis also shows that the action of any configuration with non-zero topological charge must satisfy the following bound,

$$S \geq \frac{8\pi^2|Q|}{g_0^2}. \quad (4)$$

A big effort has been devoted to the study of the topological susceptibility on the lattice. There are several choices for the operator $q(x)$. The naive discretization of (3) does not yield integer values for Q and requires renormalization factors [3]. On the other hand, the cleaner geometrical definition due to Lüscher [4] gives an integer-valued topological charge and does not require renormalization. Here we will deal only with a geometrical definition very similar to Lüscher's. The geometrical topological susceptibility is then obtained by computing,

$$\chi_t = \frac{\langle Q^2 \rangle}{V}, \quad (5)$$

where V is the volume of the lattice. This definition is clearly equivalent to (2) in finite volume. From the continuum formula (1), it follows that the

topological susceptibility in QCD should scale as $(mass)^4$ in the continuum limit, namely

$$\chi_t \sim (b^{-1} \exp^{-\frac{1}{2\beta_0 g_0^2}})^4, \quad (6)$$

where b is the lattice spacing and $\beta(g_0) \simeq -\beta_0 g_0^3$ is the leading term of the beta function.

However, it was found, first in the $O(3)$ σ -model in two space-time dimensions [5][6] and then in non-abelian gauge theories in four [7] [8], that the standard actions in both cases give rise to short-range fluctuations with non-zero geometrical topological charge and smaller action than the continuum bound. These fluctuations, often referred to as “dislocations”, overwhelm the contribution of the slowly varying fields, which would otherwise dominate in the continuum limit, and are expected to destroy the scaling of the topological susceptibility. On the other hand, if the bound is satisfied, the semiclassical continuum analysis of non-abelian gauge theories indicates that the susceptibility should indeed be ultraviolet finite, and therefore scale. Satisfying the continuum bound is thus a sufficient condition for scaling in non-abelian gauge theories [6][7][8][10] †. This suggests that eq. (4) can be satisfied on the lattice by giving dislocations a larger action.

Several proposals to solve this problem have been considered in the literature, like the cooling method [8] and the use of improved actions [7] [10]. In this paper we propose to use a new action which satisfies the continuum bound. It is related in spirit to the actions proposed in [10], but may be simpler to implement.

The idea behind our new action is easy to understand once one realizes the mismatch between the geometrical definition of topological charge [4] and the Wilson action for gauge fields which allows dislocations to arise. Consider a continuum instanton $A_\mu(y)$ which saturates the bound (4), and discretize it on a lattice of spacing b ,

$$U_\mu(s) \equiv P \exp(i \int_{sb}^{sb+b\hat{\mu}} dy A_\mu(y)), \quad (7)$$

where s b are the sites of the b lattice. The geometrical definition of topological charge assigns a non-zero value even to a lattice configuration (7)

†It has been argued by Lüscher[6] that in the case of the $O(3)$ σ -model, the failure in finding scaling is not related to the existence of dislocations and is an essential problem of this model: the partition functional in the instanton sector shows a logarithmic ultraviolet divergence in the continuum analysis.

obtained from very small instantons, of order of the lattice spacing b . On the other hand, it is clear that the Wilson action very poorly approximates the continuum action of the original instanton of $O(b)$, and in fact it is generically smaller and therefore does not satisfy the bound. Such rough configurations are dislocations and can destroy the scaling of the susceptibility. An important observation is that the geometrical topological charge assigned to a lattice configuration can be understood as the naive topological charge of a *continuum* configuration obtained by smoothly interpolating the lattice configuration. Then it is clear that if, instead of using the standard Wilson action of the original lattice configuration, we use the continuum action of the interpolated configuration, the continuum bound is necessarily satisfied, as first suggested in [11].

More concretely, in [12] we described a procedure to obtain a continuum gauge field $a_\mu(y)$ which interpolates any b -lattice configuration[§]. The interpolation is local and gauge covariant, i.e. for a b -lattice gauge transformation $\Omega(s)$,

$$a_\mu[U^\Omega] = a_\mu^\omega[U], \quad (8)$$

where ω is a gauge transformation in the continuum. (Other interpolations can also be used in this context [11].) A geometrical topological charge of the b -lattice configuration is defined as the one associated to the interpolated field [12],

$$Q = \frac{1}{32\pi^2} \int d^4y \operatorname{Tr}[\tilde{f}_{\mu\nu}(y)f_{\mu\nu}(y)]. \quad (9)$$

This definition has the same properties as Lüscher's original definition [4] and requires roughly the same computational effort. Now, it is clear that replacing the standard Wilson action by the continuum action of a_μ , i.e.

$$S^{cont} = \frac{1}{2g_0^2} \int d^4y \operatorname{Tr}[f_{\mu\nu}(y)f_{\mu\nu}(y)], \quad (10)$$

insures that the continuum bound is satisfied for the same reason as it is in the continuum. Notice that this is a perfectly gauge invariant action for the lattice field $U_\mu(s)$, by eq. (8).

[§]The continuum field is differentiable inside each b -hypercube and transversely continuous across the b -boundaries, and when discretized according to (7) gives back $U_\mu(s)$ [12].

However, for numerical purposes the continuum action is impractical [11]. The central observation of this paper is that it is not necessary to interpolate all the way to the continuum, but just to a finer lattice, with lattice spacing f (we will take b/f to be integer). From now on, we refer to x as the sites on the f -lattice and s as the sites on the b -lattice ($x_\mu = s_\mu + m_\mu \frac{f}{b}$, $m_\mu = 0, \dots, \frac{b}{f} - 1$). The interpolation procedure in [12] gives a set of link variables $u_\mu[U](x)$, such that

$$-\frac{i}{f} \log(u_\mu(x)) = a_\mu(x) + O(f/b^2), \quad (11)$$

where $a_\mu(x)$ is the continuum interpolation discussed above, at point x . On the f lattice, we can simply choose the standard Wilson action. The partition functional will then have the form,

$$\mathcal{Z} = \int \prod_s \mathcal{D}U(s) e^{-S_{wilson}^f[u[U]]}, \quad (12)$$

where, DU is the usual Haar measure for non-abelian gauge fields on the b -lattice, and the Wilson action in terms of the interpolated link variables $u[U]$ is given by,

$$\begin{aligned} S_{wilson}^f[u[U]] &= \frac{1}{g_0^2} \sum_x \sum_{\mu \neq \nu} (I - u_{\mu\nu}[x] + h.c.), \\ u_{\mu\nu}[x] &\equiv Tr[u_\mu(x)u_\nu(x + \hat{\mu})u_\mu^\dagger(x + \hat{\nu})u_\nu^\dagger(x)]. \end{aligned} \quad (13)$$

Again, this action is gauge invariant, because the functional $u[U]$ is gauge covariant [12]. From eq. (11) it then follows that,

$$S_{wilson}^f = \frac{1}{2g_0^2} \int d^4y Tr[f_{\mu\nu}(y)f_{\mu\nu}(y)] + O(f/b). \quad (14)$$

Although the $O(f/b)$ terms are not necessarily positive definite, clearly by taking f/b small enough we can come arbitrarily close to satisfying the continuum bound, so that χ_t shows scaling. Determining how small this ratio must be in practice requires a numerical analysis, which is beyond the scope of this letter. However, we have carried out a numerical analysis of this issue in a simplified model in two dimensions, which shares many features with four dimensional Yang-Mills theories and our results encourage us to believe that $\frac{f}{b}$ need not be very small in order to recover scaling of χ_t .

1 O(3) σ -Model in 2D

The O(3) σ -model in two dimensions [13] [14] is the simplest asymptotically free field theory and, as is well-known, it has instanton solutions [13]. A continuum bound on the action exists for configurations with non-zero topological charge. We will show that, while in the standard lattice formulation of this model [5], there are topologically non-trivial configurations with a smaller action than the continuum bound, this is not the case when we use an improved action along the lines described above. For a similar discussion in the context of renormalization improved actions see [15]. We will not address in this paper the interesting question of whether short-range configurations satisfying the continuum bound in this model can still dominate in the continuum limit, as argued by Lüscher [6].

The O(3) σ -model in the continuum is defined by the action,

$$S = \frac{1}{2g_0^2} \int d^2x \sum_{\mu} (\partial_{\mu} \vec{N}(x))^2, \quad (15)$$

where \vec{N} is a 3-component real field satisfying the constraint $\vec{N}^2 = 1$ (which defines the surface of a sphere of unit radius). The continuum topological charge in this model is given by,

$$Q = \frac{1}{8\pi} \int d^2x \epsilon_{\mu\nu} \vec{N} \cdot (\partial_{\mu} \vec{N} \times \partial_{\nu} \vec{N}), \quad (16)$$

which is the number of times that space-time wraps around the \vec{N} -sphere. Using the identity [16],

$$\frac{1}{4} (\partial_{\mu} \vec{N} + \epsilon_{\mu\nu} \vec{N} \times \partial_{\nu} \vec{N})^2 = \frac{1}{2} (\partial_{\mu} \vec{N})^2 - \frac{1}{2} \epsilon_{\mu\nu} \vec{N} \cdot \partial_{\mu} \vec{N} \times \partial_{\nu} \vec{N}, \quad (17)$$

it follows that,

$$S = \frac{4\pi Q}{g_0^2} + \frac{1}{4g_0^2} \int d^2x (\partial_{\mu} \vec{N} + \epsilon_{\mu\nu} [\vec{N} \times \partial_{\nu} \vec{N}])^2, \quad (18)$$

and, since the second term is positive definite,

$$S \geq \frac{4\pi|Q|}{g_0^2}. \quad (19)$$

Continuum instantons saturate this bound [13].

In a standard lattice treatment the action is,

$$S^b = \frac{1}{2g_0^2} \sum_s \sum_\mu (\hat{\partial}_\mu \vec{N}(s))^2 \quad (20)$$

where s are the sites of a two dimensional lattice and $\hat{\partial}_\mu \vec{N} \equiv \vec{N}(s+\hat{\mu}) - \vec{N}(s)$. The topological charge as defined by Berg and Lüscher [6] is given by,

$$Q^b = \sum_s q(s), \quad (21)$$

$$q(s) = \frac{1}{2\pi} \{ \text{Im} [\text{Log} [f[s, s+\hat{\mu}, s+\hat{\mu}+\hat{\nu}] f[s, s+\hat{\mu}+\hat{\nu}, s+\hat{\nu}]]] \},$$

$$f[s_1, s_2, s_3] \equiv \frac{1 + \sum_{i<j} \vec{N}(s_i) \cdot \vec{N}(s_j) + i \vec{N}(s_1) \cdot [\vec{N}(s_2) \times \vec{N}(s_3)]}{[2 \prod_{i<j} (1 + \vec{N}(s_i) \cdot \vec{N}(s_j))]^{1/2}} \quad (22)$$

It is not hard to understand this formula. Consider the plaquette $(s, \hat{1}, \hat{2})$. The spin variables at the corners are four points on the sphere. We can form two triangles with corners at these points and with sides along geodesics, $T_1(s) = (\vec{N}(s), \vec{N}(s+\hat{1}), \vec{N}(s+\hat{1}+\hat{2}))$ and $T_2(s) = (\vec{N}(s), \vec{N}(s+\hat{1}+\hat{2}), \vec{N}(s+\hat{2}))$. $q(s)$ is simply the sum of the area of these two image triangles on the \vec{N} -sphere. If periodic boundary conditions are imposed, Q^b is necessarily an integer.

With these definitions of the action (20) and the topological charge (22), the analysis in [5][6] showed that there are dislocations, i.e. configurations with non-zero topological charge that have a smaller action than the continuum bound (19). We will show that this picture changes considerably when the action of an interpolation of \vec{N} is used.

We first interpolate the lattice spin variables to a finer lattice, by moving along geodesics on the sphere. The interpolation will be done locally, i.e. the interpolated fields within a plaquette $(s, \hat{1}, \hat{2})$ of the b lattice will only depend on the four spins associated with this plaquette, i.e. $\vec{N}(s), \vec{N}(s+\hat{1}), \vec{N}(s+\hat{2}), \vec{N}(s+\hat{1}+\hat{2})$. The points of the finer lattice contained in the plaquette at s , are $x = s + t_1 \hat{1} + t_2 \hat{2}$, with $0 \leq t_{1,2} \leq 1$ and multiples of f/b . The spin fields at these points will be denoted by $\vec{n}(t_1, t_2)$. We first define the interpolated spin fields at the corners in the obvious way,

$$\vec{n}(t_1, t_2) = \vec{N}(s + t_1 \hat{1} + t_2 \hat{2}) \quad t_1, t_2 = 0, 1. \quad (23)$$

We now interpolate along the one dimensional boundaries of the plaquette, by moving along geodesics on the sphere. For $0 < t_1 < 1$,

$$\begin{aligned}\vec{n}(t_1, 0) &= \frac{\sin[\theta(0, 0; 1, 0)(1 - t_1)]}{\sin[\theta(0, 0; 1, 0)]}\vec{n}(0, 0) + \frac{\sin[\theta(0, 0; 1, 0) t_1]}{\sin[\theta(0, 0; 1, 0)]}\vec{n}(1, 0) \\ \vec{n}(t_1, 1) &= \frac{\sin[\theta(0, 1; 1, 1)(1 - t_1)]}{\sin[\theta(0, 1; 1, 1)]}\vec{n}(0, 1) + \frac{\sin[\theta(0, 1; 1, 1) t_1]}{\sin[\theta(0, 1; 1, 1)]}\vec{n}(1, 1),\end{aligned}\tag{24}$$

where we have defined,

$$\theta(t_1, t_2; t'_1, t'_2) \equiv d[\vec{n}(t_1, t_2), \vec{n}(t'_1, t'_2)] = \arccos[\vec{n}(t_1, t_2) \cdot \vec{n}(t'_1, t'_2)].\tag{25}$$

The expression for spins on the boundaries along direction $\hat{2}$ are analogous so we skip the formulae.

We now define the spins in the interior. In order to do this we first interpolate along the diagonal geodesic connecting $\vec{N}(s)$ and $\vec{N}(s + \hat{1} + \hat{2})$,

$$\vec{n}(t, t) = \frac{\sin[\theta(0, 0; 1, 1)(1 - t)]}{\sin[\theta(0, 0; 1, 1)]}\vec{n}(0, 0) + \frac{\sin[\theta(0, 0; 1, 1)t]}{\sin[\theta(0, 0; 1, 1)]}\vec{n}(1, 1).\tag{26}$$

For the interior points with $t_1 \geq t_2$, we obtain the spin by simply moving a fraction t_2 of the distance along the geodesic from $\vec{n}(t_1, 0)$ to $\vec{n}(t_1, t_1)$. Similarly for points with $t_2 \geq t_1$, the spin variable is obtained by moving a distance t_2 along the geodesic linking $\vec{n}(t_1, 1)$ and $\vec{n}(t_1, t_1)$. This corresponds to separately interpolating the interior points of the two image triangles $T_1(s)$ and $T_2(s)$. Defining,

$$\alpha(t_1) \equiv \theta(t_1, 0; t_1, t_1) \quad \beta(t_1) \equiv \theta(t_1, t_1; t_1, 1),\tag{27}$$

the final expression is,

$$\begin{aligned}\vec{n}(t_1, t_2) &= \frac{\sin[\alpha (1 - t_2/t_1)]}{\sin[\alpha]}\vec{n}(t_1, 0) + \frac{\sin[\alpha t_2/t_1]}{\sin[\alpha]}\vec{n}(t_1, t_1) \quad t_1 \geq t_2 \\ \vec{n}(t_1, t_2) &= \frac{\sin[\beta (1 - \tilde{t}_2/\tilde{t}_1)]}{\sin[\beta]}\vec{n}(t_1, 1) + \frac{\sin[\beta \tilde{t}_2/\tilde{t}_1]}{\sin[\beta]}\vec{n}(t_1, t_1) \quad t_2 \geq t_1\end{aligned}\tag{28}$$

with $\tilde{t}_i \equiv 1 - t_i$.

The interpolation (28) is now a configuration on an f -lattice and we define the improved action to be,

$$S^f = \frac{1}{2g_0^2} \sum_x \sum_{\mu=1,2} (\hat{\partial}_\mu \vec{n}(x))^2, \quad (29)$$

where x is any point on the f -lattice and $\hat{\partial}_\mu \vec{n}(x) \equiv \vec{n}(x + f/b\hat{\mu}) - \vec{n}(x) = O(f/b)$.

Now, we can easily derive a bound on the action, noticing that by construction,

$$Q^f = Q^b, \quad (30)$$

where Q^f is the topological charge for the f -lattice configuration: it is given by (22), simply substituting \vec{N} by \vec{n} and s by x . Now we can expand Q^f in $\hat{\partial} \vec{n} \sim O(f)$ and find that,

$$Q^f = \frac{1}{8\pi} \sum_x \epsilon_{\mu\nu} \vec{n}(x) \cdot (\hat{\partial}_\mu \vec{n}(x) \times \hat{\partial}_\nu \vec{n}(x)) + O(f/b). \quad (31)$$

Using (17) one finds,

$$S^f = \frac{4\pi Q^f}{g_0^2} + \frac{1}{4g_0^2} \sum_x (\hat{\partial}_\mu \vec{n} + \epsilon_{\mu\nu} [\vec{n} \times \hat{\partial}_\nu \vec{n}])^2 + O(f/b). \quad (32)$$

Then,

$$S^f \geq \frac{4\pi Q^b}{g_0^2} + O(f/b). \quad (33)$$

Since it is not possible to prove that the $O(f/b)$ terms are positive definite, we do not know how small an f/b we need to be sufficiently close to the continuum bound.

In order to address this question, we considered the discretization of a continuum instanton configuration with unit topological charge given by [16],

$$n_1 + in_2 = 2w/(1 + |w|^2) \quad n_3 = (1 - |w|^2)/(1 + |w|^2)$$

$$w(x_1 + ix_2) \equiv \frac{x_1 + ix_2 - a(1 + i)}{x_1 + ix_2 - c(1 + i)} \quad (34)$$

where $\vec{n} = (n_1, n_2, n_3)$. We take a and c to be real and define,

$$a = (r_0 + r/2\sqrt{2}) \quad c = (r_0 - r/2\sqrt{2}) \quad (35)$$

The radius of the instanton is proportional to $r = |a - c|$, while the center is located at (r_0, r_0) , with $r_0 = |a + c|/2$. Generically there is always a critical, r_c , below which $Q^b = 0$. We expect, and find numerically, that $r_c \sim b$. We consider the center to be situated at the center of the volume to reduce finite volume effects. Obviously, the continuum instanton is not periodic. We impose periodic boundary conditions by defining,

$$\vec{N}(s_1, s_2) \equiv \vec{n}(z_1, z_2), \quad z_i = \frac{2L_{max}}{\pi} \frac{\sin(\pi(1 - s_i/L_{max}))}{1 - \cos(\pi(1 - s_i/L_{max}))}, \quad (36)$$

where z_i are coordinates on the infinite plane and $s_i \in (-L_{max}, L_{max})$ are the coordinates on the lattice (torus). $(2L_{max})^2$ is then the number of lattice sites. The connection between these two sets of coordinates is established through stereographic projection. This deformation of the infinite volume instanton is small if the instanton size is much smaller than L_{max} . For such configurations therefore, the continuum action nearly saturates the bound. An alternative procedure would be to discretize the continuum instanton solutions of this model in a torus [15].

Figure 1 summarizes our results. It represents the action of the discretized instanton configuration as the radius is varied. The continuous line corresponds to the standard action (20), while the dashed lines correspond to the improved action for different values of the ratio f/b . It is clear that the standard action is problematic, since for $r > r_c$ the action is smaller than the continuum bound. For the new action however the situation is different. Not only is the continuum bound satisfied, but also as the instanton is shrunk to sizes of $O(b)$, a small barrier develops, separating the $Q^b = 0$ and $Q^b = 1$ sectors. The existence of this barrier is easy to understand. The new action is the action of the instanton which has been first discretized on the b -lattice and then interpolated. For a large instanton (compared to b), the interpolation should recover approximately the original configuration and so the action must be near that of a continuum instanton and show scale invariance. On the other hand, for an instanton of size $\sim b$ a lot of information is lost in the discretization and the interpolation is not expected to give a configuration similar to the original continuum instanton. In general, the interpolation of small discretized instantons of $O(b)$ will then be some other configuration that need not be even approximately an instanton, and

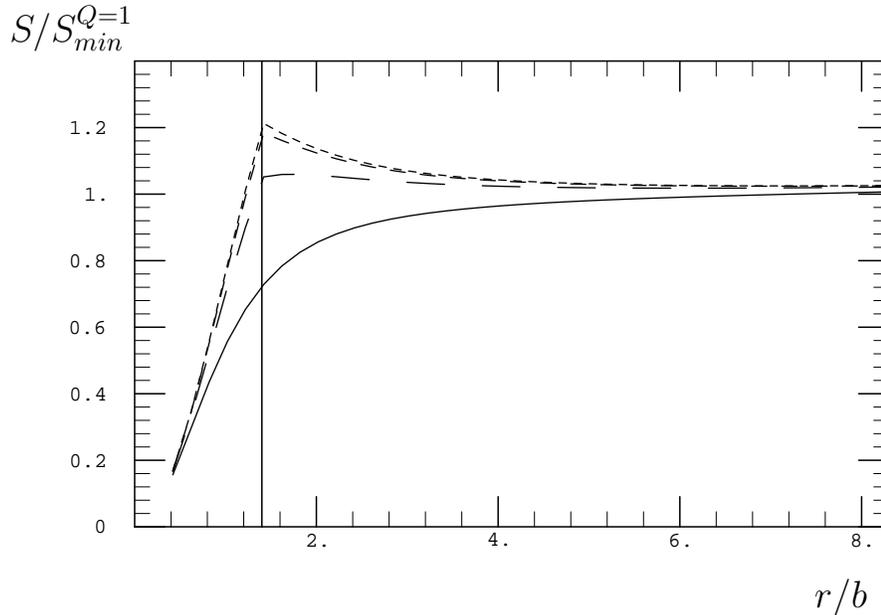


Figure 1: Action of a discretized instanton of the O(3) model (normalized to the continuum one-instanton bound (19)) as a function of its radius, r . The full line is the standard action in a 100×100 lattice and the dashed lines correspond respectively to $f/b = 1/2, 1/4, 1/6$ (smaller f/b , smaller dashing). The vertical line at $r_c \sim 1.4b$ separates the $Q^b = 0, 1$ sectors.

consequently its action will be larger than the bound, since only instantons saturate the bound. Thus the action must increase as we decrease r near r_c , as is clearly seen in Fig. 1. The other important point to notice is that the continuum bound (19) is satisfied even for a ratio f/b as large as $1/2$. This indicates that the extra effort required to use the improved action is clearly manageable in this case.

2 Conclusions

We have presented a new action for non-abelian gauge theories which is better suited to studying topology on the lattice than the standard Wilson action. The idea is to use the Wilson action of an interpolation to a finer

lattice of the original lattice configuration. In this way, if the ratio of lattice spacings is small enough, the continuum bound on configurations with non-zero topological charge is satisfied. In non-abelian gauge theories this is a sufficient condition for proper scaling of the topological susceptibility. We also considered a new action for the $O(3)$ σ -model in two dimensions along the same lines. A numerical study of discretized continuum instantons in this model indicates that already for a ratio of lattice spacings of $1/2$, the continuum bound is satisfied. Although the results for the $O(3)$ model are very promising, a separate numerical analysis is needed in the Yang-Mills case to determine the ratio there.

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