

## $\langle VAP \rangle$ Green Function in the Resonance Region\*

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### Abstract

We analyse the  $\langle VAP \rangle$  three-point function of vector, axial-vector and pseudoscalar currents. In the spirit of large  $N_C$ , a resonance dominated Green function is confronted with the leading high-energy behaviour from the operator product expansion. The matching is shown to be fully compatible with a chiral resonance Lagrangian and it allows to determine some of the chiral low-energy constants of  $\mathcal{O}(p^6)$ .

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1. Following the work of Moussallam [1] and of Knecht and Nyffeler [2], we reanalyse the three-point function of vector, axial-vector and pseudoscalar currents. The procedure of matching between low and high energies is especially transparent for Green functions like the  $\langle VAP \rangle$  correlator that are order parameters of chiral symmetry breaking. Working in the chiral limit and at leading order in the  $1/N_C$  expansion, the matching is performed by saturating the operator product expansion (OPE) with a certain number of resonance multiplets in accordance with the Minimal Hadronic Ansatz [3]. An alternative way consists in using directly a chiral resonance Lagrangian to perform the matching, getting thereby also information on some of the resonance couplings of such a Lagrangian.

In Ref. [2] it was claimed that, unlike the situation at  $\mathcal{O}(p^4)$  [4, 5], a minimal chiral resonance Lagrangian using the Proca formalism for spin-1 fields [6] is unable to recover the asymptotic behaviour of the Green functions considered, in particular the  $\langle VAP \rangle$  correlator. This claim should be contrasted with the results of Ref. [7] for the  $\langle VVP \rangle$  correlator where a chiral resonance Lagrangian, using the antisymmetric tensor formalism of spin-1 fields, was found to be consistent with the leading asymptotic behaviour of QCD.

An important bonus of the matching procedure is the prediction of resonance contributions to couplings of the Chiral Perturbation Theory (CHPT) Lagrangian at  $\mathcal{O}(p^6)$  [8–11]. With 90 such low-energy constants (LECs) in the even-intrinsic parity Lagrangian for three flavours, such predictions are indispensable at present to do phenomenology to next-to-next-to-leading order in the low-energy expansion. Compatibility between the constraints from QCD at high energies and a suitable chiral resonance Lagrangian, as already established at  $\mathcal{O}(p^4)$  [4, 5], would allow for a consistent Lagrangian treatment of resonance contributions at least up to  $\mathcal{O}(p^6)$ . With this motivation in mind, we first show that a chiral resonance Lagrangian with an appropriate set of resonance fields is fully consistent with QCD constraints for the  $\langle VAP \rangle$  Green function to be specified below. We then go on to determine six of the LECs of  $\mathcal{O}(p^6)$  from  $V$ ,  $A$  and  $P$  resonance contributions.

The approach employed in this letter should be seen as an approximation to large- $N_C$  QCD. The approximation consists in the choice of a hadronic ansatz and in a set of QCD short-distance constraints to be satisfied. We adopt the following guidelines:

- *Hadronic ansatz:* We include all lowest-lying resonance multiplets that can contribute to the given Green function. This choice is motivated by the well-founded assumption that the low-lying hadronic spectrum determines the chiral LECs that govern the low-energy behaviour of the Green function.
- *Short-distance constraints:* Working with a finite number of resonance multiplets, conflicts may arise between different types of asymptotic constraints [12]. We assign the highest priority to the constraints dictated by the OPE to leading order in inverse powers of large momenta. We then consider the constraints implied by the high-momentum behaviour of all hadronic form factors (as predicted by the quark counting rules [13–15]) with on-shell Goldstone modes and photons. We do not consider form factors with external resonance states.

2. The  $\langle VAP \rangle$  three-point function in momentum space is defined as

$$(\Pi_{VAP})_{\mu\nu}^{abc}(p, q) = \int d^4x \int d^4y e^{i(p \cdot x + q \cdot y)} \langle 0 | T \{ V_\mu^a(x) A_\nu^b(y) P^c(0) \} | 0 \rangle, \quad (1)$$

with  $SU(3)$  octet vector, axial-vector and pseudoscalar currents

$$V_\mu^a = \bar{\psi} \gamma_\mu \frac{\lambda^a}{2} \psi, \quad A_\mu^a = \bar{\psi} \gamma_\mu \gamma_5 \frac{\lambda^a}{2} \psi, \quad P^a = \bar{\psi} i \gamma_5 \frac{\lambda^a}{2} \psi. \quad (2)$$

It satisfies the chiral Ward identities [1, 16]

$$\begin{aligned} p^\mu (\Pi_{VAP})_{\mu\nu}^{abc}(p, q) &= \langle \bar{\psi} \psi \rangle_0 f^{abc} \left[ \frac{q_\nu}{q^2} - \frac{(p+q)_\nu}{(p+q)^2} \right], \\ q^\nu (\Pi_{VAP})_{\mu\nu}^{abc}(p, q) &= \langle \bar{\psi} \psi \rangle_0 f^{abc} \frac{(p+q)_\mu}{(p+q)^2}, \end{aligned} \quad (3)$$

where  $\langle \bar{\psi} \psi \rangle_0$  denotes the quark condensate in the chiral limit. The general solution of these Ward identities, taking into account the QCD symmetries  $SU(3)_V$ , parity and time reversal, is [1, 2]

$$\begin{aligned} (\Pi_{VAP})_{\mu\nu}^{abc}(p, q) &= f^{abc} \left\{ \langle \bar{\psi} \psi \rangle_0 \left[ \frac{(p+2q)_\mu q_\nu}{q^2(p+q)^2} - \frac{g_{\mu\nu}}{(p+q)^2} \right] \right. \\ &\quad \left. + P_{\mu\nu}(p, q) \mathcal{F}(p^2, q^2, (p+q)^2) + Q_{\mu\nu}(p, q) \mathcal{G}(p^2, q^2, (p+q)^2) \right\}. \end{aligned} \quad (4)$$

The transverse tensors  $P_{\mu\nu}$  and  $Q_{\mu\nu}$  are defined as

$$\begin{aligned} P_{\mu\nu}(p, q) &= q_\mu p_\nu - (p \cdot q) g_{\mu\nu}, \\ Q_{\mu\nu}(p, q) &= p^2 q_\mu q_\nu + q^2 p_\mu p_\nu - (p \cdot q) p_\mu q_\nu - p^2 q^2 g_{\mu\nu}. \end{aligned} \quad (5)$$

The behaviour of the invariant functions  $\mathcal{F}$  and  $\mathcal{G}$  at small momentum transfers is governed by the contributions from Goldstone boson intermediate states. As one-particle exchange dominates in the limit  $N_C \rightarrow \infty$ , we only need to keep the corresponding Goldstone boson poles and the polynomial terms involving the LECs. In the basis of Ref. [9] for the LECs of  $\mathcal{O}(p^6)$  one finds [2] ( $F$  is the pion decay constant in the chiral limit)

$$\begin{aligned} \mathcal{F}^{\text{CHPT}}(p^2, q^2, (p+q)^2) &= \frac{4 \langle \bar{\psi} \psi \rangle_0}{F^2 (p+q)^2} \left[ L_9 + L_{10} + \left( C_{78} - \frac{5}{2} C_{88} - C_{89} + 3C_{90} \right) p^2 \right. \\ &\quad \left. + \left( C_{78} - 2C_{87} + \frac{1}{2} C_{88} \right) q^2 + \left( C_{78} + 4C_{82} - \frac{1}{2} C_{88} \right) (p+q)^2 \right] + \mathcal{O}(p^8), \\ \mathcal{G}^{\text{CHPT}}(p^2, q^2, (p+q)^2) &= \frac{4 \langle \bar{\psi} \psi \rangle_0}{F^2 q^2 (p+q)^2} \left[ L_9 + 2(-C_{88} + C_{90}) p^2 \right. \\ &\quad \left. + (2C_{78} - C_{89} + C_{90}) q^2 - 2C_{90} (p+q)^2 \right] + \mathcal{O}(p^8). \end{aligned} \quad (6)$$

**3.** We next recall the properties of the  $\langle VAP \rangle$  correlator at short distances. Following Refs. [1, 2], the analysis will be restricted to the leading term in the OPE. Being an order parameter of chiral symmetry breaking, the behaviour of the  $\langle VAP \rangle$  Green function at short distances is smoother than expected from naive power counting arguments. The leading contributions at large momenta are proportional to the quark condensate, which has the same anomalous dimension as the pseudoscalar current. Hence the corresponding Wilson coefficient in the OPE of the  $\langle VAP \rangle$  correlator does not have an anomalous dimension and, therefore, QCD corrections should modify the following results only mildly.

As shown by Knecht and Nyffeler [2], two short-distance limits are of interest here. In the first case, the two momenta  $p$  and  $q$  in the correlator (1) become simultaneously large. In position space this amounts to the situation where the space-time arguments of the three operators tend towards the same point at the same rate ( $x \sim y \sim 0$ ). Restricting the discussion as always to the leading term in the OPE, one obtains [1, 2]

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} (\Pi_{VAP})_{\mu\nu}^{abc}(\lambda p, \lambda q) &= \frac{\langle \bar{\psi}\psi \rangle_0}{\lambda^2} f^{abc} \frac{1}{p^2 q^2 (p+q)^2} \left\{ p^2 (p+2q)_\mu q_\nu \right. \\ &\quad \left. - g_{\mu\nu} p^2 q^2 + \frac{1}{2} (p^2 - q^2 - (p+q)^2) P_{\mu\nu} - Q_{\mu\nu} \right\} + \mathcal{O}\left(\frac{1}{\lambda^4}\right) \end{aligned} \quad (7)$$

and therefore

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} \mathcal{F}((\lambda p)^2, (\lambda q)^2, (\lambda p + \lambda q)^2) &= \frac{\langle \bar{\psi}\psi \rangle_0}{2\lambda^4} \frac{p^2 - q^2 - (p+q)^2}{p^2 q^2 (p+q)^2} + \mathcal{O}\left(\frac{1}{\lambda^6}\right), \\ \lim_{\lambda \rightarrow \infty} \mathcal{G}((\lambda p)^2, (\lambda q)^2, (\lambda p + \lambda q)^2) &= -\frac{\langle \bar{\psi}\psi \rangle_0}{\lambda^6} \frac{1}{p^2 q^2 (p+q)^2} + \mathcal{O}\left(\frac{1}{\lambda^8}\right). \end{aligned} \quad (8)$$

The second situation of interest corresponds to the case where the relative distance between only two of the three operators involved becomes small. We refer to Ref. [2] for a complete discussion of the various cases where different two-point functions arise. It turns out that many of the resulting conditions are not independent when taken together with the constraint (7). Therefore, we only reproduce the following short-distance condition from Ref. [2], which, together with (7), leads to a complete set of leading-order high-energy constraints:

$$\lim_{\lambda \rightarrow \infty} (\Pi_{VAP})_{\mu\nu}^{abc}(\lambda p, q - \lambda p) = -\frac{1}{\lambda} f^{abc} \langle \bar{\psi}\psi \rangle_0 \frac{p_\mu q_\nu + p_\nu q_\mu - (p \cdot q) g_{\mu\nu}}{p^2 q^2} + \mathcal{O}\left(\frac{1}{\lambda^2}\right). \quad (9)$$

In terms of the invariant functions  $\mathcal{F}$  and  $\mathcal{G}$ , this asymptotic behaviour implies

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} \mathcal{F}((\lambda p)^2, (q - \lambda p)^2, q^2) &= \frac{\langle \bar{\psi}\psi \rangle_0}{\lambda^2 p^2} \left[ \mathcal{F}^{(0)}(q^2) + \frac{1}{\lambda} \frac{p \cdot q}{p^2} \mathcal{F}^{(1)}(q^2) + \mathcal{O}\left(\frac{1}{\lambda^2}\right) \right], \\ \lim_{\lambda \rightarrow \infty} \mathcal{G}((\lambda p)^2, (q - \lambda p)^2, q^2) &= \frac{\langle \bar{\psi}\psi \rangle_0}{(\lambda^2 p^2)^2} \left[ \mathcal{G}^{(0)}(q^2) + \frac{1}{\lambda} \frac{p \cdot q}{p^2} \mathcal{G}^{(1)}(q^2) + \mathcal{O}\left(\frac{1}{\lambda^2}\right) \right], \end{aligned} \quad (10)$$

with

$$\mathcal{F}^{(0)}(q^2) - \mathcal{G}^{(0)}(q^2) = \frac{1}{q^2}, \quad \mathcal{F}^{(1)}(q^2) - \mathcal{G}^{(1)}(q^2) + \mathcal{G}^{(0)}(q^2) = \frac{2}{q^2}. \quad (11)$$

The  $\langle VAP \rangle$  correlator is also related to the  $\Gamma_{VA}$  and  $\Gamma_{VP}$  vertex functions [1,2]. The short-distance behaviour of these vertex functions gives additional constraints on the parameters of  $\mathcal{F}$  and  $\mathcal{G}$ . However, using the expansion of  $\Gamma_{VA}$  and  $\Gamma_{VP}$  consistently up to  $\mathcal{O}(1/\lambda)$ , those constraints are equivalent to the limit where one momentum of  $\langle VAP \rangle$  becomes large and therefore they do not provide new information at leading order.

In order to solve the short-distance conditions (7) and (9), we propose the following ansatz inspired by large  $N_C$  that is a generalization of the one used in Refs. [1,2]:

$$\begin{aligned} \mathcal{F}(p^2, q^2, (p+q)^2) &= \frac{\langle \bar{\psi}\psi \rangle_0}{(p^2 - M_V^2)(q^2 - M_A^2)} \\ &\times \left[ a_0 + \frac{b_1 + b_2 p^2 + b_3 q^2}{(p+q)^2} + \frac{c_1 + c_2 p^2 + c_3 q^2}{(p+q)^2 - M_P^2} \right], \\ \mathcal{G}(p^2, q^2, (p+q)^2) &= \frac{\langle \bar{\psi}\psi \rangle_0}{(p^2 - M_V^2)q^2} \left[ \frac{d_1 + d_2 q^2}{(p+q)^2(q^2 - M_A^2)} + \frac{f}{(p+q)^2 - M_P^2} \right]. \end{aligned} \quad (12)$$

This ansatz differs from the one in Refs. [1,2] by the inclusion of a nonet of pseudoscalar resonances with mass  $M_P$  (remember that we always work in the chiral limit). The ansatz of Refs. [1,2] is recovered in the limit  $M_P \rightarrow \infty$ , i.e. by dropping the parameters  $c_1, c_2, c_3, f$  that specify the contributions from pseudoscalar resonance exchange. While the ansatz of Refs. [1,2] was designed to match the leading short-distance constraints with the minimal resonance content, we include all lowest-lying resonance multiplets that can contribute to the LECs of  $\mathcal{O}(p^6)$ . Our approach appears more natural when attempting to construct an explicit Lagrangian realization for the resonance interactions (see discussion below).

The parameters in (12) fall into two classes:

- 1) The dimensionless parameters  $a_0, b_2, b_3, c_2, c_3, d_2, f$  are constrained by the OPE conditions (7) and (9).
- 2) The parameters  $b_1, c_1, d_1$  with squared mass dimension are not affected by the (leading-order) OPE conditions. As discussed in the next paragraph, they can be constrained by asymptotic conditions on various form factors when one or two pions are put on-shell [1,2].

The short-distance conditions (7) and (9) yield the following set of six linear equations for the seven parameters in class 1:

$$\begin{aligned} a_0 = -\frac{1}{2}, \quad b_2 + c_2 = \frac{1}{2}, \quad b_3 + c_3 = -\frac{1}{2}, \\ d_2 + f = -1, \quad b_2 + b_3 - d_2 = 1, \quad 2b_2 - d_2 = 2. \end{aligned} \quad (13)$$

We use these equations to express six of the parameters in terms of  $b_3$ :

$$\begin{aligned} a_0 &= -\frac{1}{2} , & b_2 &= 1 + b_3 , & d_2 &= 2b_3 , \\ c_2 &= -\frac{1}{2} - b_3 , & c_3 &= c_2 , & f &= 2c_2 . \end{aligned} \quad (14)$$

Setting the pseudoscalar exchange parameters  $c_2, c_3$  and  $f$  to zero (or, equivalently, letting  $M_P \rightarrow \infty$  in the ansatz (12)), we recover the solution of Refs. [1,2]:

$$a_0 = -\frac{1}{2} , \quad b_2 = \frac{1}{2} , \quad b_3 = -\frac{1}{2} , \quad d_2 = -1 . \quad (15)$$

4. Additional information on the parameters in Eq. (12) can be obtained by putting one or two momenta in the  $\langle VAP \rangle$  Green function (1) on the pion mass shell [1,2]. The form factors appearing in the resulting vertex functions are not directly constrained by QCD but there are strong theoretical arguments [13–15] for the form factors in question to fall off at least like  $1/q^2$  for large momentum transfers.

The two dimensional parameters  $b_1, d_1$  were determined in this way by Moussallam [1], making also use of the two Weinberg sum rules [17]. The inclusion of pseudoscalar resonances does not affect those results:

$$b_1 = M_A^2 - M_V^2 , \quad d_1 = 2M_A^2 . \quad (16)$$

Although we cannot determine  $c_1$  from consideration of a pionic form factor we can fix the remaining dimensionless parameter  $b_3$  in this way. For this purpose, we consider the axial form factor  $G_A(t)$  governing the matrix element  $\langle \gamma | A_\mu | \pi \rangle$  [16]. Extracting  $G_A(t)$  from the Green function (1) by setting  $p^2 = 0$  and  $(p+q)^2 = 0$  (massless pion), one finds in terms of the parameters defined in (12)

$$G_A(t) = \frac{F^2}{M_V^2} \frac{b_1 + b_3 t}{M_A^2 - t} . \quad (17)$$

Demanding that the form factor  $G_A(t)$  vanishes for large  $t$  [5, 13–15], we obtain

$$b_3 = 0 . \quad (18)$$

Therefore, the solution (15) [1,2] is not compatible with the asymptotic vanishing of  $G_A(t)$ . The value  $b_3 = -1/2$  in (15) is also at the origin of the very small partial width obtained [1,2] for the decay  $a_1 \rightarrow \pi\gamma$ . The decay matrix element is governed by the combination  $b_1 + b_3 M_A^2$ . With the solution (15), this matrix element is proportional to  $(M_A^2 - 2M_V^2)/2$  and therefore suppressed compared to our solution with  $b_3 = 0$  where the same matrix element is given by  $b_1 = M_A^2 - M_V^2$ . The numerical value of the decay width  $\Gamma(a_1 \rightarrow \pi\gamma)$  will be discussed later.

The  $\langle VAP \rangle$  Green function also contributes to the decay  $\tau \rightarrow 3\pi\nu_\tau$ . One can study the axial-vector form factor contributing to this process and require that it vanishes like  $1/q^2$  for large momentum transfer. This procedure provides the conditions

$$b_2 = 1 , \quad b_3 = 0 , \quad d_2 = 0 \quad (19)$$

that are consistent with the results in Eqs. (14) and (18).

The discussion above is related to a general point raised in Ref. [12]. There it was claimed that for a given three-point function that is an order parameter of chiral symmetry breaking, any large- $N_C$  inspired ansatz with a *finite number* of resonance multiplets will fail to reproduce simultaneously (i) the leading OPE constraints and (ii) the  $1/q^2$  asymptotic behaviour of *all* hadronic form factors (appearing as residues of two-particle poles in the Green function). Our explicit construction shows that with a reasonable number of resonance multiplets one can still fulfill both the leading OPE constraints and the correct asymptotic behaviour of form factors involving Goldstone modes and on-shell photons ( $F_V^\pi(t)$  and  $G_A(t)$  in our case).

**5.** We now turn to an explicit realization of our solutions (14), (16) and (18) in terms of a chiral resonance Lagrangian. Such a Lagrangian was introduced in Refs. [4, 5] to investigate the LECs of  $\mathcal{O}(p^4)$ . That Lagrangian has to be extended when going up to  $\mathcal{O}(p^6)$  where bilinear resonance couplings also contribute.

In the notation of Refs. [4, 5], the kinetic terms of the Lagrangian restricted to vector, axial-vector and pseudoscalar resonance fields ( $V(1^{--})$ ,  $A(1^{++})$  and  $P(0^{-+})$ ) are given by

$$\begin{aligned}\mathcal{L}_{\text{kin}}^R &= -\frac{1}{2}\langle\nabla^\lambda R_{\lambda\mu}\nabla_\nu R^{\nu\mu} - \frac{M_R^2}{2}R_{\mu\nu}R^{\mu\nu}\rangle, \quad R = V, A, \\ \mathcal{L}_{\text{kin}}^P &= \frac{1}{2}\langle\nabla^\mu P\nabla_\mu P - M_P^2 P^2\rangle.\end{aligned}\quad (20)$$

$V_{\mu\nu}$  and  $A_{\mu\nu}$  are antisymmetric tensor fields describing nonets of spin-1 mesons and  $P$  is a pseudoscalar (nonet) field. The brackets  $\langle\dots\rangle$  denote a three-dimensional trace in flavour space. In the large- $N_C$  limit where multiple trace terms are suppressed, the interaction terms linear in the resonance fields and with the minimal number of derivatives and mass insertions are given by [4]

$$\mathcal{L}_2^{V,A,P} = \frac{F_V}{2\sqrt{2}}\langle V_{\mu\nu}f_+^{\mu\nu}\rangle + i\frac{G_V}{\sqrt{2}}\langle V_{\mu\nu}u^\mu u^\nu\rangle + \frac{F_A}{2\sqrt{2}}\langle A_{\mu\nu}f_-^{\mu\nu}\rangle + i d_m \langle P\chi_-\rangle. \quad (21)$$

The chiral fields  $u^\mu$ ,  $f_+^{\mu\nu}$ ,  $f_-^{\mu\nu}$ ,  $\chi_-$  are defined as usual [4, 5] in terms of Goldstone fields and external fields. The coupling constants  $F_V$ ,  $G_V$ ,  $F_A$  and  $d_m$  are real.

To account for LECs of  $\mathcal{O}(p^6)$ , we must also include couplings of Goldstone bosons with two resonance fields. As for the linear couplings (21), we only include terms with the minimal number of derivatives and mass insertions. Although there is a priori no guarantee that the various asymptotic constraints discussed previously can be satisfied with such a minimal Lagrangian this approach proved to be successful at  $\mathcal{O}(p^4)$  [4, 5].

The vector-axial-vector bilinear terms were already introduced in Ref. [18]:

$$\begin{aligned}\mathcal{L}_2^{VA} &= \sum_{i=1}^5 \lambda_i^{VA} \mathcal{O}_i^{VA} \\ \mathcal{O}_1^{VA} &= \langle [V^{\mu\nu}, A_{\mu\nu}] \chi_- \rangle, & \mathcal{O}_2^{VA} &= i \langle [V^{\mu\nu}, A_{\nu\alpha}] h_\mu^\alpha \rangle, \\ \mathcal{O}_3^{VA} &= i \langle [\nabla^\mu V_{\mu\nu}, A^{\nu\alpha}] u_\alpha \rangle, & \mathcal{O}_4^{VA} &= i \langle [\nabla_\alpha V_{\mu\nu}, A^{\alpha\nu}] u^\mu \rangle, \\ \mathcal{O}_5^{VA} &= i \langle [\nabla_\alpha V_{\mu\nu}, A^{\mu\nu}] u^\alpha \rangle.\end{aligned}\quad (22)$$

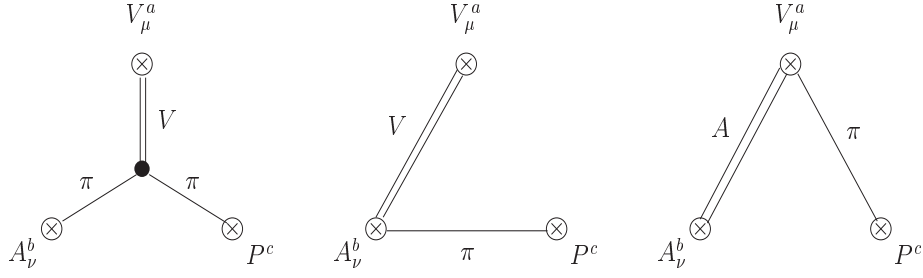


Figure 1: Single-resonance exchange:  $\pi$  denotes a Goldstone boson,  $V$  and  $A$  stand for vector and axial-vector resonances, respectively.

The pseudoscalar–vector bilinear couplings are given by

$$\begin{aligned} \mathcal{L}_2^{PV} &= \sum_{i=1}^2 \lambda_i^{PV} \mathcal{O}_i^{PV} \\ \mathcal{O}_1^{PV} &= i \langle [\nabla^\mu P, V_{\mu\nu}] u^\nu \rangle, & \mathcal{O}_2^{PV} &= i \langle [P, V_{\mu\nu}] f_-^{\mu\nu} \rangle. \end{aligned} \quad (23)$$

Finally, only one bilinear term with pseudoscalar and axial-vector fields contributes to our Green function:

$$\begin{aligned} \mathcal{L}_2^{PA} &= \lambda_1^{PA} \mathcal{O}_1^{PA} \\ \mathcal{O}_1^{PA} &= i \langle [P, A_{\mu\nu}] f_+^{\mu\nu} \rangle. \end{aligned} \quad (24)$$

Adding the lowest-order chiral Lagrangian [19], we obtain the following chiral resonance Lagrangian to be used for the calculation of the  $\langle VAP \rangle$  Green function:

$$\mathcal{L}_{\text{CHRL}} = \frac{F^2}{4} \langle u_\mu u^\mu + \chi_+ \rangle + \mathcal{L}_{\text{kin}}^{V,A,P} + \mathcal{L}_2^{V,A,P} + \mathcal{L}_2^{VA} + \mathcal{L}_2^{PV} + \mathcal{L}_2^{PA}. \quad (25)$$

**6.** Lowest-order Goldstone boson exchange provides the first two terms in the  $\langle VAP \rangle$  correlator (4) that drive the chiral Ward identities (3). Next we compute the diagrams shown in Fig. 1 where a single resonance is exchanged. There is no contribution from pseudoscalar resonance exchange in this case [4]. The diagrams in Fig. 1 give rise to

$$\begin{aligned} \mathcal{F}^{V,A}(p^2, q^2, (p+q)^2) &= \frac{\langle \bar{\psi}\psi \rangle_0}{(p+q)^2(p^2 - M_V^2)} \left( \frac{F_V^2 - 2F_V G_V}{F^2} - \frac{p^2 - M_V^2}{q^2 - M_A^2} \frac{F_A^2}{F^2} \right), \\ \mathcal{G}^{V,A}(p^2, q^2, (p+q)^2) &= \frac{\langle \bar{\psi}\psi \rangle_0}{q^2(p+q)^2(p^2 - M_V^2)} \frac{-2F_V G_V}{F^2}. \end{aligned} \quad (26)$$

The double-resonance contributions to the  $\langle VAP \rangle$  Green function are described by the diagrams in Fig. 2. Summing up single- and double-resonance exchange contributions, the



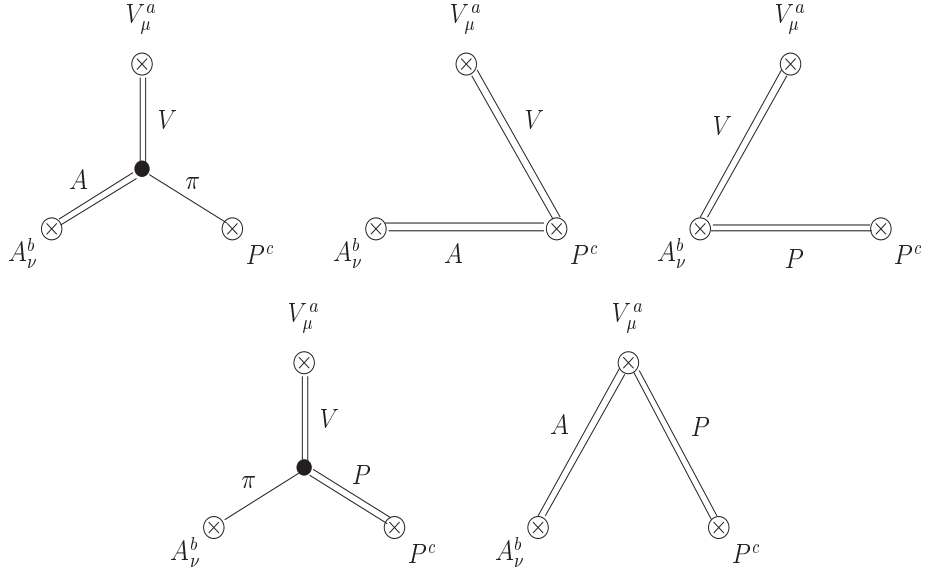


Figure 2: Double-resonance exchange contribution to the  $\langle VAP \rangle$  Green function;  $P$  denotes a pseudoscalar resonance.

final result can be given in terms of the parameters defined in the general ansatz (12):

$$\begin{aligned}
a_0 &= -2\sqrt{2}\frac{F_V F_A}{F^2}\lambda_0, & b_1 &= M_V^2\frac{F_A^2}{F^2} - M_A^2\frac{F_V^2 - 2F_V G_V}{F^2}, \\
b_2 &= -\frac{F_A^2}{F^2} + 2\sqrt{2}\frac{F_V F_A}{F^2}\lambda', & b_3 &= \frac{F_V^2 - 2F_V G_V}{F^2} + 2\sqrt{2}\frac{F_V F_A}{F^2}\lambda'', \\
c_1 &= -M_V^2 c_2 - M_A^2 c_3, & c_2 &= -8\sqrt{2}\frac{F_A d_m}{F^2}\lambda_1^{PA}, \\
c_3 &= 8\sqrt{2}\frac{F_V d_m}{F^2}\left(\frac{\lambda_1^{PV}}{2} + \lambda_2^{PV}\right), & d_1 &= \frac{2F_V G_V}{F^2}M_A^2, \\
d_2 &= -\frac{2F_V G_V}{F^2} + 2\sqrt{2}\frac{F_V F_A}{F^2}(\lambda' + \lambda''), & f &= 4\sqrt{2}\frac{F_V d_m}{F^2}\lambda_1^{PV},
\end{aligned} \tag{27}$$

where we have used the definitions [18]

$$\begin{aligned}
\sqrt{2}\lambda_0 &= -4\lambda_1^{VA} - \lambda_2^{VA} - \frac{\lambda_4^{VA}}{2} - \lambda_5^{VA}, \\
\sqrt{2}\lambda' &= \lambda_2^{VA} - \lambda_3^{VA} + \frac{\lambda_4^{VA}}{2} + \lambda_5^{VA}, \\
\sqrt{2}\lambda'' &= \lambda_2^{VA} - \frac{\lambda_4^{VA}}{2} - \lambda_5^{VA}.
\end{aligned} \tag{28}$$

As pointed out in Ref. [5], the short-distance structure of QCD can be used to constrain the couplings of the chiral resonance Lagrangian by matching the asymptotic behaviour of

two-point functions, form factors and scattering amplitudes with the results from resonance exchange (at leading order in the  $1/N_C$  expansion and assuming a single nonet of  $V$  and  $A$  resonances each). In this way one finds the relations [5, 17]

$$F_V G_V = F^2, \quad F_V^2 - F_A^2 = F^2, \quad F_V^2 M_V^2 = F_A^2 M_A^2, \quad (29)$$

allowing to express the couplings  $F_V$ ,  $G_V$  and  $F_A$  in terms of  $F$ ,  $M_V$  and  $M_A$ . From a similar joint analysis of the scalar form factor [20, 21] and the SS-PP sum rules [22] one gets, assuming again only one nonet of  $S$  and  $P$  resonances each [23],

$$d_m = \frac{F}{2\sqrt{2}}. \quad (30)$$

The ten relations in (27) can now be compared with the previous results (14), (16) and (18). From the equations for  $a_0, b_2, b_3$  we extract the combinations of coupling constants  $\lambda_0, \lambda', \lambda''$  that satisfy the relation  $4\lambda_0 = \lambda' + \lambda''$ . The equation for  $d_2$  is then automatically satisfied. The relations for  $c_2, c_3, f$  fix the coupling constants  $\lambda_1^{PV}, \lambda_2^{PV}, \lambda_1^{PA}$ . The equations for  $b_1, d_1$  are consistent with (16) and there is a new relation for the dimensional parameter  $c_1$ :

$$c_1 = \frac{1}{2}(M_V^2 + M_A^2). \quad (31)$$

The predictions of the chiral resonance Lagrangian (25) are fully consistent with the OPE and form factor constraints (14), (16) and (18). In other words, at leading order in  $1/N_C$  and considering for each current in the Green function only one multiplet of resonances with the same quantum numbers, the chiral resonance Lagrangian provides a  $\langle VAP \rangle$  Green function with the correct asymptotic behaviour dictated by QCD.

Using (29) and (30), we get the following final results for the coupling constants of the chiral Lagrangians (22), (23) and (24), depending only on the masses  $M_V$  and  $M_A$ :

$$\begin{aligned} \lambda' &= \frac{M_A}{2\sqrt{2}M_V}, & \lambda'' &= \frac{M_A^2 - 2M_V^2}{2\sqrt{2}M_V M_A}, & 4\lambda_0 &= \lambda' + \lambda'', \\ \lambda_1^{PV} &= -4\lambda_2^{PV}, & \lambda_2^{PV} &= \frac{\sqrt{M_A^2 - M_V^2}}{8M_A}, & \lambda_1^{PA} &= \frac{\sqrt{M_A^2 - M_V^2}}{8M_V}. \end{aligned} \quad (32)$$

We now come back to the axial form factor  $G_A(t)$  in the matrix element  $\langle \gamma | A_\mu | \pi \rangle$ . With single-resonance exchange only, this form factor is given by [5]

$$G_A(t) = \frac{2F_V G_V - F_V^2}{M_V^2} + \frac{F_A^2}{M_A^2 - t}. \quad (33)$$

Requiring  $G_A(t)$  to vanish for  $t \rightarrow \infty$  implies the relation  $F_V = 2G_V$ , one version of the so-called KSFR relation [24, 25]. The inclusion of bilinear resonance couplings modifies the

form factor as given in Eq. (17) with  $b_1 = M_A^2 - M_V^2$ ,  $b_3 = 0$ , and it induces a correction to the KSFR relation:

$$\frac{2F_V G_V - F_V^2}{2F^2} = 1 - \frac{F_V^2}{2F^2} = \frac{M_A^2 - 2M_V^2}{2(M_A^2 - M_V^2)} . \quad (34)$$

With  $M_A = 1.23$  GeV,  $M_V = 0.771$  GeV [26], the right-hand side takes the value  $\simeq 0.18$ . The partial decay width  $\Gamma(a_1 \rightarrow \pi\gamma)$  is now

$$\Gamma(a_1 \rightarrow \pi\gamma) = \frac{\alpha M_A}{24} \left( \frac{M_A^2}{M_V^2} - 1 \right)^3 \left( 1 - \frac{M_\pi^2}{M_A^2} \right)^3 . \quad (35)$$

With the same values for  $M_V, M_A$  and with the physical pion mass, we obtain  $\Gamma(a_1 \rightarrow \pi\gamma) = 1.33$  MeV, in reasonable agreement with the experimental value  $640 \pm 246$  keV [27]. It should be noted that the width is very sensitive to  $M_A$ . A value of  $M_A = \sqrt{2}M_V$ , as required by the KSFR relation, would give  $\Gamma(a_1 \rightarrow \pi\gamma) = 316$  keV and for  $M_A = 1.2$  GeV, a value extracted from  $\tau \rightarrow 3\pi\nu_\tau$  data [18], one obtains  $\Gamma(a_1 \rightarrow \pi\gamma) = 1.01$  MeV. In comparison, the decay width in the scenario of Refs. [1, 2] is strongly suppressed with respect to (35) by a factor

$$\frac{(M_A^2 - 2M_V^2)^2}{4(M_A^2 - M_V^2)^2} \simeq 0.03 . \quad (36)$$

Although the corresponding width is more than an order of magnitude smaller than the listed value [26] the experimental situation (discussed in Ref. [1]) remains to be settled. However, we are confident that future experiments will be able to decide between two predictions that differ by a factor 30.

As commented above, the relations (32) also have a bearing on the decays  $\tau \rightarrow 3\pi\nu_\tau$ . As shown in Ref. [18], the requirement that the  $J = 1$  axial spectral function vanishes for large momentum transfer implies certain values for  $\lambda', \lambda''$ . Those values coincide<sup>1</sup> with the corresponding results in Eq. (32). The coupling  $\lambda_0$  was extracted in that reference from a fit to the spectrum and branching ratio of the decay. However, the fitted value turns out to be too large and, as discussed in that reference, carries a big uncertainty due to the fact that in the  $\tau \rightarrow 3\pi\nu_\tau$  amplitude the coupling  $\lambda_0$  always appears multiplied by a factor  $M_\pi^2$ .

**7.** Having established the compatibility between the QCD short-distance constraints and the chiral resonance Lagrangian, we can now use the results (14), (16) and (18) (with the additional relation (31) for  $c_1$ ) to compare with the low-energy expansion (6) of the  $\langle VAP \rangle$  Green function. It turns out that all LECs appearing in (6) can be determined separately in

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<sup>1</sup>In Ref. [18] the KSFR relation  $F_V = 2G_V$  was adopted implying  $M_A^2 = 2M_V^2$ .

this way:

$$\begin{aligned}
L_9 &= \frac{F^2}{2M_V^2} , & L_{10} &= -\frac{F^2(M_A^2 + M_V^2)}{4M_V^2 M_A^2} , \\
C_{78} &= \frac{F^2(3M_A^2 + 4M_V^2)}{8M_V^4 M_A^2} - \frac{F^2}{16M_V^2 M_P^2} , & C_{82} &= -\frac{F^2(4M_A^2 + 5M_V^2)}{32M_V^4 M_A^2} - \frac{F^2}{32M_A^2 M_P^2} , \\
C_{87} &= \frac{F^2(M_A^4 + M_V^4 + M_A^2 M_V^2)}{8M_V^4 M_A^4} , & C_{88} &= -\frac{F^2}{4M_V^4} + \frac{F^2}{8M_V^2 M_P^2} , \\
C_{89} &= \frac{F^2(3M_A^2 + 2M_V^2)}{4M_V^4 M_A^2} , & C_{90} &= \frac{F^2}{8M_V^2 M_P^2} .
\end{aligned} \tag{37}$$

The results for  $L_9$  and  $L_{10}$ , the LECs of  $\mathcal{O}(p^4)$ , coincide with those in Ref. [5]. The LECs of  $\mathcal{O}(p^6)$  differ from the ones in Ref. [2], first of all by terms involving the mass  $M_P$  of the pseudoscalar resonance nonet. However, even in the limit  $M_P \rightarrow \infty$  a small difference remains for the LECs  $C_{78}$  and  $C_{82}$ . The reason for this difference is that the short-distance limit and the limit  $M_P \rightarrow \infty$  do not commute, as it is evident from the analysis of Eq. (13). Since  $M_P \simeq 1.3 \text{ GeV}$  is rather heavy, the pseudoscalar contributions to the LECs are not large, ranging from  $0.02 \times 10^{-4}$  to  $0.09 \times 10^{-4}$  for  $F^2 C_i$ . All contributions from pseudoscalar resonances originate from the  $f$  parameter in Eq. (12), except for  $C_{82}$  which also gets a contribution from the  $c_1$  term.

Our large- $N_C$  determination of these LECs cannot reproduce their scale dependence (a next-to-leading-order effect in the  $1/N_C$  counting). We have checked that in all cases the variation of the renormalized LECs between  $\mu = M_K$  and  $\mu = 1 \text{ GeV}$  remains within a range of 30%. Although there is no reason a priori that this result will be valid for all LECs our findings quantify the reliability of the estimates (37) for phenomenological applications.

**8.** We have extended the ansatz of Moussallam, Knecht and Nyffeler [1, 2] for the  $\langle VAP \rangle$  Green function in the intermediate energy region by including the lowest-lying nonet of pseudoscalar resonances. Since the model has more parameters it trivially satisfies all short-distance constraints discussed in Refs. [1, 2].

The distinctive features of our solution are the following:

- The axial form factor in the matrix element for the decay  $\pi \rightarrow e\nu_e\gamma$  vanishes for  $t \rightarrow \infty$ .
- We obtain a partial decay width  $\Gamma(a_1 \rightarrow \pi\gamma)$  in reasonable agreement with the experimental value [27] but more than a factor 30 bigger than the prediction of Refs. [1, 2].
- The asymptotic vanishing of the axial spectral function relevant for the decay  $\tau \rightarrow 3\pi\nu_\tau$  [18] is compatible with the model.
- The LECs of  $\mathcal{O}(p^6)$  determined by the matching procedure differ in general from the ones derived in Ref. [2] even in the limit  $M_P \rightarrow \infty$  for the mass of the pseudoscalar resonance nonet.

- The solution of the short-distance constraints is consistent with a minimal chiral resonance Lagrangian with vector, axial-vector and pseudoscalar resonances where the spin-1 mesons are described by antisymmetric tensor fields.

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