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Effective Field Theory Description of the Pion Form Factor*

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Abstract

Using our present knowledge on effective hadronic theories, short–distance QCD information, the $1/N_C$ expansion, analyticity and unitarity, we derive an expression for the pion form factor, in terms of m_π, m_K, M_ρ and f_π. This parameter–free prediction provides a surprisingly good description of the experimental data up to energies of the order of 1 GeV.

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1. Introduction

The QCD currents are a basic ingredient of the electromagnetic (vector) and weak (vector, axial, scalar, pseudoscalar) interactions. A good understanding of their associated hadronic matrix elements is then required to control the important interplay of QCD in electroweak processes. Given our poor knowledge of the QCD dynamics at low energies, one needs to resort to experimental information, such as $e^+e^- \to$ hadrons or semileptonic decays.

At the inclusive level, the analysis of two–point function correlators constructed from the T–ordered product of two currents, has been widely used to get a link between the short–distance description in terms of quarks and gluons and the hadronic long–distance world. In this way, information on fundamental parameters, such as quark masses, α_s or vacuum condensates, is extracted from the available phenomenological information on current matrix elements. The lack of good experimental data in a given channel translates then into unavoidable uncertainties on the obtained theoretical results.

At very low energies, the chiral symmetry constraints[1] are powerful enough to determine the hadronic matrix elements of the light quark currents.[2,3,4,5] Unfortunately, these chiral low–energy theorems only apply to the threshold region. At the resonance mass scale, the effective Goldstone chiral theory becomes meaningless and the use of some QCD–inspired model to obtain theoretical predictions seems to be unavoidable.

Obviously, to describe the resonance region, one needs to use an effective theory with explicit resonance fields as degrees of freedom. Although not so predictive as the standard chiral Lagrangian for the pseudo-Goldstone mesons, the resonance chiral effective theory[6] turns out to provide interesting results, once some short–distance dynamical QCD constraints are taken into account.[7]

We would like to investigate how well the resonance region can be understood, using our present knowledge on effective hadronic theories, short–distance QCD information and other important constraints such as analyticity and unitarity. Present experiments are providing a rich data sample (specially on hadronic τ decays), which can be used to test our theoretical skills in this important, but poorly understood, region.

Our goal in this first letter is to study one of the simplest current matrix elements, the pion form factor $F(s)$, defined (in the isospin limit) as

$$\langle \pi^0\pi^- | \bar{d}\gamma^\mu u | \emptyset \rangle \,=\, \sqrt{2}\, F(s)\, (p_{\pi^-} - p_{\pi^0})^\mu \,, \qquad (1)$$

with $s \equiv q^2 \equiv (p_{\pi^-} + p_{\pi^0})^2$. At $s > 0$, $F(s)$ is experimentally known from the decay $\tau^- \to \nu_\tau \pi^-\pi^0$ and (through an isospin rotation) from $e^+e^- \to \pi^+\pi^-$, while the elastic $e^-\pi^+$ scattering provides information at $s < 0$.

Theoretically, the pion form factor has been extensively investigated for many years. Thus, many of our results are not new. We want just to achieve a description of $F(s)$ as simple as possible, in order to gain some understanding which could be used in other more complicated current matrix elements.

2. Effective Lagrangian Results

Near threshold, the pion form factor is well described by chiral perturbation theory (ChPT). At one loop, it takes the form[3]:

$$F(s)^{\rm ChPT} = 1 + \frac{2L_9^r(\mu)}{f_\pi^2}\, s - \frac{s}{96\pi^2 f_\pi^2}\left[A(m_\pi^2/s, m_\pi^2/\mu^2) + \frac{1}{2}A(m_K^2/s, m_K^2/\mu^2)\right], \tag{2}$$

where the functions

$$A(m_P^2/s, m_P^2/\mu^2) \;=\; \ln\left(m_P^2/\mu^2\right) + \frac{8m_P^2}{s} - \frac{5}{3} + \sigma_P^3\, \ln\left(\frac{\sigma_P+1}{\sigma_P-1}\right) \tag{3}$$

contain the loop contributions, with the usual phase–space factor

$$\sigma_P \equiv \sqrt{1 - 4m_P^2/s}\,, \tag{4}$$

and $L_9^r(\mu)$ is an $O(p^4)$ chiral counterterm renormalized at the scale μ. The measured pion electromagnetic radius,[8] $\langle r^2\rangle^{\pi^\pm} = (0.439 \pm 0.008)$ fm^2, fixes $L_9^r(M_\rho) = (6.9 \pm 0.7) \times 10^{-3}$.

A two loop calculation in the $SU(2) \otimes SU(2)$ theory (i.e. no kaon loops) has been completed recently.[9,10]

Using an effective chiral theory which explicitly includes the lightest octet of vector resonances, one can derive[6,7] the leading effect induced by the ρ resonance:

$$F(s)^{\rm V} = 1 + \frac{F_V G_V}{f_\pi^2}\frac{s}{M_\rho^2 - s}\,, \tag{5}$$

where the couplings F_V and G_V characterize the strength of the $\rho\gamma$ and $\rho\pi\pi$ couplings, respectively. The resonance contribution appears first at the next-to-leading order in the chiral expansion. This tree–level result corresponds to the leading term in the $1/N_C$ expansion, with $N_C = 3$ the number of QCD colours. Comparing it with Eq. (2), it gives an explicit calculation of L_9 in the $N_C \to \infty$ limit; chiral loops (and the associated scale dependence) being suppressed by an additional $1/N_C$ factor.

Eq. (5) was obtained imposing the constraint that the short–distance behaviour of QCD allows at most one subtraction for the pion form factor.[7] However, all empirical evidence and theoretical prejudice suggests that $F(s)$ vanishes sufficiently fast for $s \to \infty$ to obey an unsubtracted dispersion relation. If this is the case, one gets the relation[7] (at leading order in $1/N_C$ and if higher–mass states are not considered) $F_V G_V/f_\pi^2 = 1$, which implies the well–known Vector

Meson Dominance (VMD) expression:

$$F(s)^{\mathrm{VMD}} = \frac{M_\rho^2}{M_\rho^2 - s}\,. \tag{6}$$

The resulting prediction for the $O(p^4)$ chiral coupling L_9,

$$L_9 = \frac{F_V G_V}{2M_\rho^2} = \frac{f_\pi^2}{2M_\rho^2} = 7.2 \times 10^{-3}\,, \tag{7}$$

is in very good agreement with the phenomenologically extracted value. This shows explicitly that the ρ contribution is the dominant physical effect in the pion form factor.

Combining Eqs. (2) and (6), one gets an obvious improvement of the theoretical description of $F(s)$:

$$F(s) = \frac{M_\rho^2}{M_\rho^2 - s} - \frac{s}{96\pi^2 f_\pi^2}\left[A(m_\pi^2/s, m_\pi^2/M_\rho^2) + \frac{1}{2}A(m_K^2/s, m_K^2/M_\rho^2)\right]. \tag{8}$$

The VMD formula provides the leading term in the $1/N_C$ expansion, which effectively sums an infinite number of local ChPT contributions to all orders in momenta (corresponding to the expansion of the ρ propagator in powers of s/M_ρ^2). Assuming that $L_9^r(M_\rho)$ is indeed dominated by the $1/N_C$ result (7), the loop contributions encoded in the functions $A(m_P^2/s, m_P^2/M_\rho^2)$ give the next-to-leading corrections in a combined expansion in powers of momenta and $1/N_C$.

3. Unitarity Constraints

The loop functions $A(m_P^2/s, m_P^2/M_\rho^2)$ contain the logarithmic corrections induced by the final–state interaction of the two pseudoscalars. The strong constraints imposed by analyticity and unitarity allow us to perform a resummation of those contributions.

The pion form factor is an analytic function in the complex s plane, except for a cut along the positive real axis, starting at $s = 4m_\pi^2$, where its imaginary part develops a discontinuity. For real values $s < 4m_\pi^2$, $F(s)$ is real. The imaginary part of $F(s)$, above threshold, corresponds to the contribution of on–shell intermediate states:

$$\mathrm{Im}F(s) = \mathrm{Im}F(s)_{2\pi} + \mathrm{Im}F(s)_{4\pi} + \cdots + \mathrm{Im}F(s)_{K\bar{K}} + \cdots \tag{9}$$

In the elastic region ($s < 16m_\pi^2$), Watson final–state theorem[11] relates the imaginary part of $F(s)$ to the partial wave amplitude T_1^1 for $\pi\pi$ scattering with angular momenta and isospin equal to one:

$$\mathrm{Im}F(s + i\epsilon) = \sigma_\pi T_1^1 F(s)^* = e^{i\delta_1^1}\sin\delta_1^1 F(s)^* = \sin\delta_1^1\,|F(s)| = \tan\delta_1^1\,\mathrm{Re}F(s)\,. \tag{10}$$

Since $\mathrm{Im}F(s)$ is real, the phase of the pion form factor is the same as the phase δ_1^1 of the partial wave amplitude T_1^1. Thus, one can write a (n-subtracted) dispersion relation in the form:

$$F(s) \;=\; \sum_{k=0}^{n-1} \frac{s^k}{k!} \frac{d^k}{ds^k} F(0) + \frac{s^n}{\pi} \int_{4m_\pi^2}^{\infty} \frac{dz}{z^n} \, \frac{\tan\delta_1^1(z)\, \mathrm{Re}F(z)}{z-s-i\epsilon} \, , \tag{11}$$

which has the well–known Omnès[12,13] solution:

$$F(s) \;=\; Q_n(s)\, \exp\left\{ \frac{s^n}{\pi} \int_{4m_\pi^2}^{\infty} \frac{dz}{z^n} \, \frac{\delta_1^1(z)}{z-s-i\epsilon} \right\} , \tag{12}$$

where

$$\ln Q_n(s) \;=\; \sum_{k=0}^{n-1} \frac{s^k}{k!} \, \frac{d^k}{ds^k} \ln F(0) \, . \tag{13}$$

Strictly speaking, this equation is valid only below the inelastic threshold ($s \leq 16 m_\pi^2$). However, the contribution from the higher–mass intermediate states is suppressed by phase space. The production of a larger number of meson pairs is also of higher order in the chiral expansion.

Using the lowest–order ChPT result

$$\sigma T_1^1 = \frac{s\sigma_\pi^3}{96\pi f_\pi^2} \simeq \delta_1^1 \, , \tag{14}$$

the integral in Eq. (12) generates the one–loop function $-sA(m_\pi^2/s, y)/(96\pi^2 f_\pi^2)$, up to a polynomial [in $s/(4m_\pi^2)$] ambiguity, which depends on the number of subtractions applied[a]. Thus, the Omnès formula provides an exponentiation of the chiral logarithmic corrections. The subtraction function $Q_n(s)$ (and the polynomial ambiguity) can be partly determined by matching the Omnès result to the one–loop ChPT result (2), which fixes the first two terms of its Taylor expansion; it remains, however, a polynomial ambiguity at higher orders. This means an indetermination order by order between the non-logarithmic part of the pion form factor which must be in $Q_n(s)$ and the one which must be in the exponential.

The ambiguity can be resolved to a large extent, by matching the Omnès formula to the improved result in Eq. (8), which incorporates the effect of the ρ propagator. One gets then:

$$F(s) = \frac{M_\rho^2}{M_\rho^2 - s} \; \exp\left\{ \frac{-s}{96\pi^2 f^2} A(m_\pi^2/s, m_\pi^2/M_\rho^2) \right\} . \tag{15}$$

This expression for the pion form factor satisfies all previous low–energy constraints and, moreover, has the right phase at one loop.

[a] Obviously, the finite integration does not give rise to any dependence on $y \equiv m_\pi^2/\mu^2$.

Eqs. (15) has obvious shortcomings. We have used an $O(p^2)$ approximation to the phase shift δ_1^1, which is a very poor (and even wrong) description in the higher side of the integration region. Nevertheless, one can always take a sufficient number of subtractions to emphasize numerically the low–energy region. Since our matching has fixed an infinite number of subtractions, the result (15) should give a good approximation for values of s not too large. One could go further and use the $O(p^4)$ calculation of δ_1^1 to correct this result. While this could (and should) be done, it would only improve the low–energy behaviour, where Eq. (15) provides already a rather good description. However, we are more interested in getting an extrapolation to be used at the resonance peak.

4. The Rho Width

The off-shell width of the ρ meson can be easily calculated, using the resonance chiral effective theory.[6,7] One gets:

$$\begin{aligned}\Gamma_\rho(s) &= \frac{M_\rho\, s}{96\pi f_\pi^2}\left\{\theta(s-4m_\pi^2)\,\sigma_\pi^3+\frac{1}{2}\theta(s-4m_K^2)\,\sigma_K^3\right\}\\ &= -\frac{M_\rho\, s}{96\pi^2 f_\pi^2}\,\mathrm{Im}\left[A(m_\pi^2/s,m_\pi^2/M_\rho^2)+\frac{1}{2}A(m_K^2/s,m_K^2/M_\rho^2)\right]. \qquad (16)\end{aligned}$$

At $s=M_\rho^2$, $\Gamma_\rho(M_\rho^2)=144$ MeV, which provides a quite good approximation to the experimental meson width, $\Gamma_\rho^{\mathrm{exp}}=(150.7\pm1.2)$ MeV.

Eq. (16) shows that, below the $K\bar{K}$ threshold, the ρ width is proportional to the $O(p^2)$ $\pi\pi$ phase shift: $\Gamma_\rho(s)=M_\rho\delta_1^1$. Making a Dyson summation of the ρ self-energy corrections amounts to introduce the ρ width into the denominator of the ρ propagator, shifting the pole singularity to $\sqrt{s}=M_\rho-\frac{i}{2}\Gamma_\rho$. However, expanding the resulting propagator in powers of s/M_ρ^2, it becomes obvious that the one–loop imaginary correction generated by the width contribution is exactly the same as the one already contained in the Omnès exponential. This suggests to shift the imaginary part of the loop function $A(m_P^2/s,m_P^2/M_\rho^2)$ from the exponential to the propagator. Including the small contribution from the intermediate state $K\bar{K}$, one has then:

$$\begin{aligned}F(s)=\frac{M_\rho^2}{M_\rho^2-s-iM_\rho\Gamma_\rho(s)}\exp\Bigg\{\frac{-s}{96\pi^2f_\pi^2}\Bigg[&\mathrm{Re}A(m_\pi^2/s,m_\pi^2/M_\rho^2)\\ &+\frac{1}{2}\mathrm{Re}A(m_K^2/s,m_K^2/M_\rho^2)\Bigg]\Bigg\}. \qquad (17)\end{aligned}$$

This change does not modify the result at order p^4, which still coincides with ChPT, but makes the phase shift pass trough $\pi/2$ at the mass of the ρ resonance.

Expanding Eq. (17) in powers of momenta, one can check[14] that it does a quite good job in generating the leading $O(p^6)$ contributions in the chiral expansion.

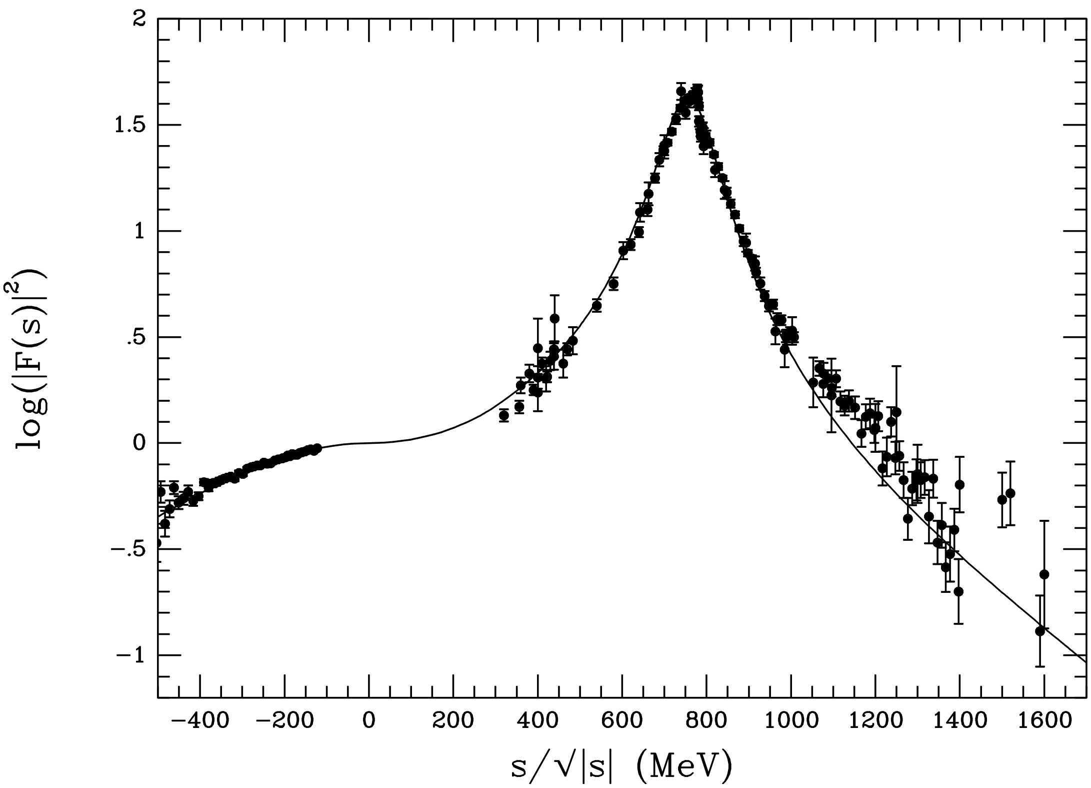

Figure 1: $|F(s)|^2$ (in logarithmic scale) versus $s/\sqrt{s}$. The continuous curve shows the theoretical prediction in Eq. (17).

The known coefficients for linear and quadratic logarithms of the ChPT result,[10] are well reproduced in the chiral limit. While this is also true for Eq. (15), the expansion of Eq. (17) gives a better description[14] of the polynomial part of the $O(p^6)$ ChPT result.

5. Numerical Results

We can see graphically in figure 1 how Eq. (17) provides a very good description of the data[15] up to rather high energies. The only parameters appearing in the pion form factor formula are m_π, m_K, M_ρ and f_π, which have been set to their physical values. Thus, Eq. (17) is in fact a parameter–free prediction. The extremely good agreement with the data is rather surprising.

At low energies the form factor is completely dominated by the polynomial contribution generated by the ρ propagator. Nevertheless, the summation of chiral logarithms turn out to be crucial to get the correct normalization at the ρ peak. The exponential factor in Eq. (17) produces a 17% enhancement of $|F(s)|$ at $s = M_\rho^2$.

The data at $s > 0$ have been taken from $e^+e^- \to \pi^+\pi^-$; so in the vicinity of the ρ peak there is a small isoescalar contamination due to the ω resonance.

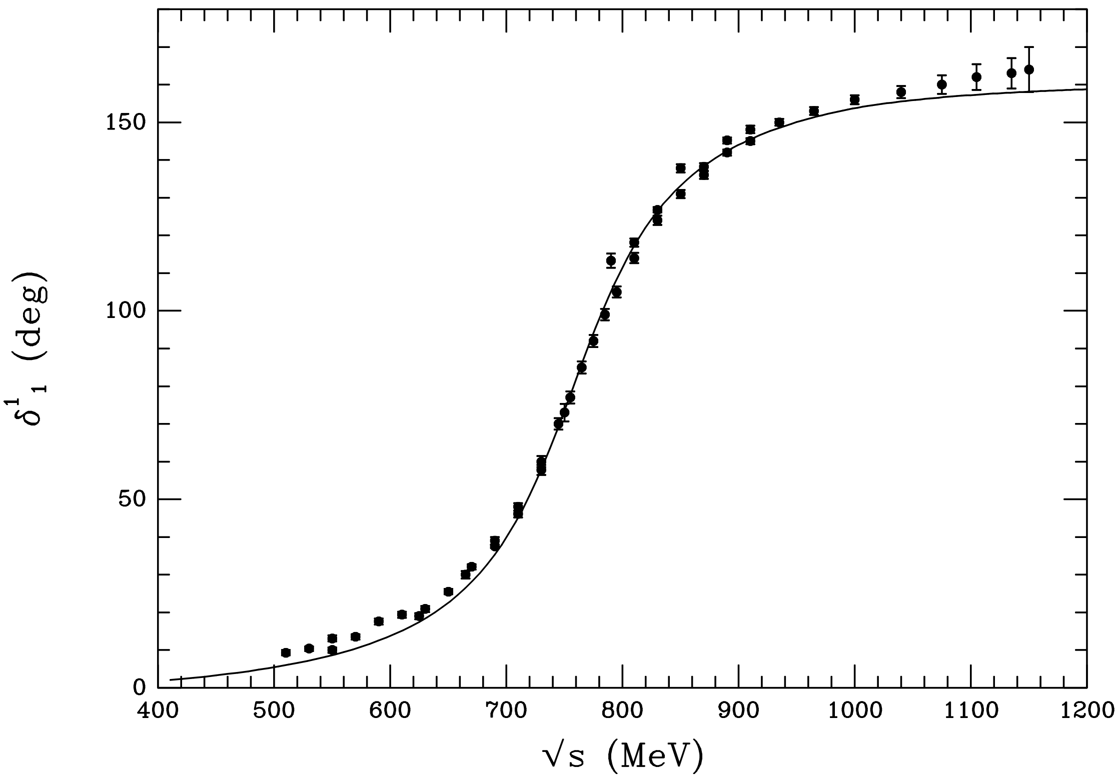

Figure 2: $\delta_1^1(s)$ versus $\sqrt{s}$. The continuous curve shows the theoretical prediction in Eq. (18).

This contribution is well–known and generates a slight distortion of the ρ peak, which can easily be included in the theoretical formula.[16] The effect cannot be appreciated at the scale of the figure.

We also obtain a prediction for the phase shift δ_1^1. From Eq. (17) we get:

$$\delta_1^1(s) \,=\, \arctan\left\{\frac{M_\rho\Gamma_\rho(s)}{M_\rho^2 - s}\right\}. \qquad (18)$$

For $s << M_\rho^2$, this expression reduces to the $O(p^2)$ ChPT result in Eq. (14). As shown in figure 2, the improvement obtained through Eq. (18) provides a quite good description of the experimental data[17,18] over a rather wide energy range. At large energies, the phase shift approaches the asymptotic limit $\delta_1^1(s \to \infty) = \arctan\{-\xi M_\rho^2/(96\pi f_\pi^2)\}$. If only the elastic 2π intermediate state is included, $\xi = 1$ and $\delta_1^1(s \to \infty) = 167^o$; taking into account the $K\bar{K}$ contribution, one gets $\xi = 3/2$, which slightly lowers the asymptotic phase shift to $\delta_1^1(s \to \infty) = 161^o$.

5. Summary

Using our present knowledge on effective hadronic theories, short–distance QCD information, the $1/N_C$ expansion, analyticity and unitarity, we have derived a simple expression for the pion form factor, in terms of m_π, m_K, M_ρ and f_π. The resulting parameter–free prediction gives a surprisingly good description of the experimental data up to energies of the order of 1 GeV.

Our main result, given in Eq. (17), contains two basic components. The ρ propagator provides the leading contribution in the limit of a large number of colours; it sums an infinite number of local terms in the low–energy chiral expansion. Chiral loop corrections, corresponding to the final–state interaction among the two pions, appear at the next order in the $1/N_C$ expansion; the Omnès exponential allows to perform a summation of these unitarity corrections, extending the validity domain of the original ChPT calculation.

Requiring consistency with the Dyson summation of the ρ self-energy, forces us to shift the imaginary phase from the exponential to the ρ propagator. While this change does not modify the rigorous ChPT results at low energies, it does regulate the ρ pole and makes the resulting phase shift pass through $\pi/2$ at the mass of the ρ resonance.

As shown in figures 1 and 2, the experimental pion form factor is well reproduced, both in modulus and phase. Although the ρ contribution is the dominant physical effect, the Omnès summation of chiral logarithms turns out to be crucial to get the correct normalization at the ρ peak.

Many detailed studies of the pion form factor have been already performed previously.[9,10,19,20,21,22,23,24] The different ingredients we have used can in fact be found in the existing literature on the subject. However, it is only when one combines together all those physical informations that such a simple description of $F(s)$ emerges.

Our approach can be extended in different ways (including two–loop ChPT results, ρ' contribution, ...). Moreover, one should investigate whether it can be applied to other current matrix elements where the underlying physics is more involved. We plan to study those questions in future publications.

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