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Next-to-leading order renormalization of the $\Delta B = 2$ operators in the static theory

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Abstract

The renormalization, at the next-to-leading order in α_s, of the $\Delta B = 2$ operators at the lowest order in the heavy quark expansion, namely in the static theory, is computed taking into account previously missed contributions. These operators are relevant for the calculation of the B^0–$\bar{B}^0$ mixing on the lattice.

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1 Introduction

The lack of a precise knowledge of the matrix elements of the operator

$$\langle \bar{B}|\bar{b}\gamma^{\mu}(1-\gamma_5)d\;\bar{b}\gamma_{\mu}(1-\gamma_5)d|B\rangle = \frac{4}{3}f_B^2 B_B M_B, \tag{1}$$

generated by the box diagram with the exchange of virtual top quarks, is the main source of theoretical error in the extraction of the CKM matrix element V_{td} from the B^0–$\bar{B}^0$ mixing parameter x_d. This uncertainty on $f_B^2 B_B$ propagates to other theoretical estimates. In particular the extraction of the CP-violating phase from the combined analysis of x_d and the K^0–$\bar{K}^0$ CP-violating parameter ϵ is affected by a twofold ambiguity that would be eliminated by a precise determination of $f_B^2 B_B$. In turn this would strongly reduce the uncertainty in the prediction of the asymmetry in the decay $B \to J/\psi\; K_S$ [1].

The matrix element in eq. (1) can be evaluated on the lattice [2]. However, since m_b is larger than the current values of the lattice cutoff, the b quark cannot be put on the lattice as a dynamical field. Therefore an effective theory based on the expansion in the heavy quark mass is needed. Such a theory has been built [3], and its discretized version can be used in lattice simulations. In particular the expansion at the lowest order in m_b is used to build the static effective theory both in the continuum and on the lattice. This theory has the usual form

$$\mathcal{H}_{static} = \sum_i C_i Q_i \tag{2}$$

in terms of Wilson coefficients and local four-fermion operators. In order to use lattice results, the effective theory in the continuum must be matched both to its lattice counterpart and to the "full" theory, namely a theory with a dynamical heavy field. In ref. [4] this matching has been computed at $O(\alpha_s)$, using the $\Delta B = 2$ effective Hamiltonian [5] as the "full" theory.

The two matching procedures take place at different scales. In fact the matching to the "full" theory is done at a scale of the order of the ultraviolet cutoff, namely m_b, while the matching to the static theory on the lattice is done at typical current values of the lattice cutoff, $1/a \sim 2$ GeV. A complete determination of the static theory in the continuum requires the calculation of the running of the Wilson coefficients between these two scales. This is usually done by using the renormalization group equations (RGEs). In this way, at the leading order (LO), one resums in the Wilson coefficients terms such as $\alpha_s^n \log^n$, assumed to be of $O(1)$. To be consistent, an $O(\alpha_s)$ matching calls for a next-to-leading (NLO) determination of the Wilson coefficients, which resums terms of the type $\alpha_s^{n+1}\log^n$. Of course, in the case at hand, one may argue that the relevant logs, namely $\log(a^2m_b^2) \sim 1.6$, are not large, so that the running at the leading order can be considered as a pure $O(\alpha_s)$ effect. Anyway the calculation of the anomalous dimension up to the NLO is required to have a regularization-scheme-independent expression of the Wilson coefficients. In the past an effort has been made to calculate this NLO anomalous dimension in the static theory [6]. However, as pointed out in ref. [7], some contributions coming from the operator mixing have been overlooked in ref. [6].

In the next section we calculate these new contributions and present a complete determination of the NLO Wilson coefficients of the static theory.

2 NLO Wilson coefficients in the static theory

In this section we discuss the NLO gluon renormalization of the $\Delta B = 2$ operators at the lowest order in the heavy quark expansion, i.e. in the static limit $m_b \to \infty$. We want to calculate the NLO expression of the Wilson coefficients of the relevant operators. To this end a few steps are required. First of all, the basis of local operators in the effective theory must be identified. Then the effective theory has to be matched against a "full" theory at a scale of the order of the ultraviolet cutoff, fixing in this way the initial conditions of the renormalization group equations. Finally the anomalous dimension matrix in the effective theory must be calculated at the desired order in α_s and the RGE solved to give the Wilson coefficients as functions of the renormalization scale μ^2. The scale-independent physical amplitude is given by the product of the Wilson coefficients $\vec{C}(\mu^2)$ and the matrix elements of the corresponding renormalized operators $\langle\vec{Q}(\mu^2)\rangle$, the latter usually requiring some non-perturbative method to be evaluated.

The $\Delta B = 2$ operator basis in the static limit is given by two dimension-six local operators

$$\vec{Q} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} ,$$

$$Q_1 = 2\bar{h}^{(+)}\gamma_\mu(1-\gamma_5)d\,\bar{h}^{(-)}\gamma^\mu(1-\gamma_5)d\ , \qquad Q_2 = 2\bar{h}^{(+)}(1-\gamma_5)d\,\bar{h}^{(-)}(1-\gamma_5)d\ , \qquad (3)$$

where the field $\bar{h}^{(+)}$ creates a heavy quark and $\bar{h}^{(-)}$ annihilates a heavy antiquark. Our calculation explicitly shows that indeed the basis is closed under renormalization at the NLO.

The next step is the NLO matching. The heavy quark theory has to be matched against the $\Delta B = 2$ effective Hamiltonian by comparing the matrix elements at the scale m_b of the relevant operators in the "full" and the effective theories, up to and including $O(\alpha_s)$ terms. Here the effective Hamiltonian plays the role of the "full" theory, even if it is also an effective theory with the top and the heavy bosons integrated out, which in turn is matched against the Standard Model at the weak scale. The $\Delta B = 2$ effective Hamiltonian has been calculated in ref. [5] and is completely known at the NLO. There exists only one $\Delta B = 2$ operator in this theory, namely

$$Q_{LL} = \bar{b}\gamma_\mu(1-\gamma_5)d\,\bar{b}\gamma^\mu(1-\gamma_5)d\ . \tag{4}$$

The calculation of the matching of this operator onto the operators in eq. (3) requires the expansion of the matrix element $\langle b\bar{d}|Q_{LL}|\bar{b}d\rangle$ in the heavy quark mass. This calculation has been done in refs. [4, 6] and gives the initial condition at the scale m_b

$$\vec{C}(m_b^2) = \begin{pmatrix} 1 + \frac{\alpha_s(m_b^2)}{4\pi}B_1 \\ \frac{\alpha_s(m_b^2)}{4\pi}B_2 \end{pmatrix} C_{LL}(m_b^2),$$
$$B_1 = \begin{cases} -14 & \text{NDR} \\ -11 & \text{DRED} \end{cases}\ , \quad B_2 = -8\ . \tag{5}$$

We have reported the values of B_1 both in the naive dimensional regularization scheme (NDR) and in dimensional reduction (DRED). This scheme dependence cancels out against the corresponding dependence contained in the Wilson coefficient $C_{LL}(m_b^2)$ of the $\Delta B = 2$ effective Hamiltonian[1]. The other initial condition, $C_2(m_b^2)$, is the same both in DRED and NDR. We notice that, starting at $O(\alpha_s)$, it does not pick up the scheme-dependent part inside $C_{LL}(m_b^2)$ at the NLO.

The evolution of the Wilson coefficients between the matching scale m_b^2 and the renormalization scale μ^2 is determined by the renormalization group equation

$$\mu^2\frac{d}{d\mu^2}\vec{C}(\mu^2) = \frac{1}{2}\hat{\gamma}^T\vec{C}(\mu^2)\ . \tag{6}$$

The anomalous dimension matrix $\hat{\gamma}$ is defined as

$$\hat{\gamma} = 2\mu^2\frac{d}{d\mu^2}\hat{Z}\ , \tag{7}$$

[1]As noticed in ref. [6], this implies that the anomalous dimension matrix element $\hat{\gamma}^{(1)}_{11}$, see eq. (11), in the effective theory is the same in DRED and NDR.

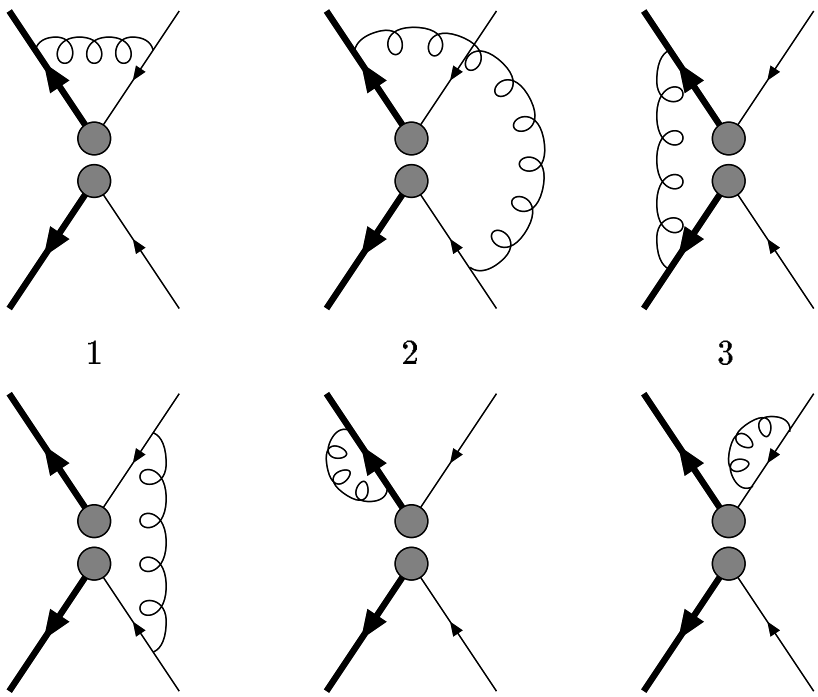

Figure 1: *The Feynman diagrams that contribute to* $\hat{\gamma}^{(0)}$. *Thick (thin) lines represent the heavy (light) quarks. The blobs are the operator insertion point.*

where $\hat{Z}$ is the matrix of the renormalization constants connecting the bare and the renormalized operators

$$\vec{O}_R = \hat{Z}^{-1}\vec{O}_B \ . \tag{8}$$

The formal solution of eq. (6) is

$$\vec{C}(\mu^2) = T_{\alpha_s} e^{\int_{\alpha_s(m_b^2)}^{\alpha_s(\mu^2)} d\alpha_s \frac{\hat{\gamma}(\alpha_s)}{2\beta(\alpha_s)}} \vec{C}(m_b^2) \ , \tag{9}$$

where T_{α_s} is the ordered product with increasing powers of the coupling constant from left to right and $\beta(\alpha_s)$ is the QCD beta function

$$\beta(\alpha_s) = \mu^2 \frac{d\alpha_s}{d\mu^2} \ . \tag{10}$$

In order to calculate the Wilson coefficients at the NLO, the first two terms of the perturbative expansion of $\beta(\alpha_s)$ and $\hat{\gamma}(\alpha_s)$ are needed:

$$\beta(\alpha_s) \ = \ -\frac{\alpha_s^2}{4\pi}\beta_0 - \frac{\alpha_s^3}{(4\pi)^2}\beta_1 + \dots \ ,$$

Diag.	Mult.	$Q_1 \to Q_1$	$Q_1 \to Q_2$	$Q_2 \to Q_1$	$Q_2 \to Q_2$
1	2	$2\xi C_F$	-	-	$2\xi C_F$
2	2	$\xi\left(1-\frac{1}{N}\right)$	-	$-\frac{\xi}{2}$	$-\xi\left(1+\frac{1}{N}\right)$
3	1	$\frac{3-\xi}{2}\left(1-\frac{1}{N}\right)$	-	$-\frac{3-\xi}{4}$	$-\frac{3-\xi}{2}\left(1+\frac{1}{N}\right)$
4	1	$-\frac{3+\xi}{2}\left(1-\frac{1}{N}\right)$	-	$\frac{1+\xi}{4}-\frac{1}{2N}$	$-\frac{1-\xi}{2}\left(1+\frac{1}{N}\right)$
5	1	$(3-\xi)\,C_F$	-	-	$(3-\xi)\,C_F$
6	1	$-\xi C_F$	-	-	$-\xi C_F$

Table 1: *Pole coefficients of the operator insertions into the diagrams of fig. 1, calculated in the linear R_ξ gauge. An overall factor $\alpha_s/4\pi$ is understood. The second column contains the diagram multiplicity factors, which have already been applied to the shown coefficients. To account for the renormalization of the external legs, self-energy diagrams count as* 1/2.

$$\hat{\gamma}(\alpha_s) \;=\; \frac{\alpha_s}{4\pi}\hat{\gamma}^{(0)} + \left(\frac{\alpha_s}{4\pi}\right)^2 \hat{\gamma}^{(1)} + \dots \; . \tag{11}$$

The beta function coefficients are well known:

$$\beta_0 = \frac{11N - 2n_f}{3} \;, \qquad \beta_1 = \frac{34}{3}N^2 - \frac{10}{3}Nn_f - 2C_F n_f \;, \tag{12}$$

where N is the number of colours, n_f is the number of active flavours and $C_F = (N^2-1)/2N$. The anomalous dimension matrices $\hat{\gamma}^{(0)}$ and $\hat{\gamma}^{(1)}$ are calculated by computing the one- and two-loop renormalization of the operators in eq. (3).

The one-loop renormalization matrix is given by the infinite parts of the operator insertion into the diagrams in fig. 1, computed in the static theory. We have used the dimensional regularization[2] to calculate the divergent parts of these diagrams that appear as poles in $\epsilon = (4-D)/2$. The coefficients of the poles are collected in table 1. The calculation is straightforward, the only peculiarity being that few tensor structures can appear in the effective theory because of the equation satisfied by the static fields, $v_\mu \gamma^\mu h^{(\pm)} = \pm h^{(\pm)}$. In particular tensors can be reduced in the following way:

$$\begin{aligned}\bar{h}^{(+)}\sigma_{\mu\nu}(1-\gamma_5)d\;\bar{h}^{(-)}\sigma^{\mu\nu}(1-\gamma_5)d \;&=\; 4\Big[\bar{h}^{(+)}\gamma^\mu(1-\gamma_5)d\;\bar{h}^{(-)}\gamma_\mu(1-\gamma_5)d \\ &\quad + \bar{h}^{(+)}(1-\gamma_5)d\;\bar{h}^{(-)}(1-\gamma_5)d\Big] \; .\end{aligned} \tag{13}$$

The anomalous dimension at the LO is minus twice the pole coefficients in the matrix $\hat{Z}$ of

[2] γ_5 and subtraction prescriptions are immaterial for the anomalous dimension at the LO.

the operator renormalization constants, so that, from table 1, we readily find

$$\hat{\gamma}^{(0)} = \begin{pmatrix} -6C_F & 0 \\ 1+\frac{1}{N} & -6C_F+4+\frac{4}{N} \end{pmatrix} . \tag{14}$$

As expected, the anomalous dimension matrix is independent of the gauge. $\hat{\gamma}^{(0)}_{11}$ agrees with the previous calculations [8], while the other matrix elements were not considered previously. The form of this matrix has some important consequences. On the one hand, Q_1 insertion has vanishing component on Q_2 at the leading order. Since the initial condition of Q_2 is already of order α_s, the one-loop contribution of Q_2 becomes a NLO effect. For the very same reason, we do not need to calculate the two-loop renormalization, i.e. the second row of $\hat{\gamma}^{(1)}$, which would generate a next-to-next-to-leading order term. On the other hand, Q_2 has a non-zero leading component on Q_1, and thus contributes to the Wilson coefficient C_1 at the NLO.

We still have to calculate the first row of $\hat{\gamma}^{(1)}$. This is a hard task, involving the evaluation of the pole parts of several two-loop diagrams. However the renormalization of Q_1 onto itself has already been calculated in ref. [6], while the insertion of Q_1 has vanishing component onto Q_2. This last statement holds to all orders in perturbation theory, provided that one chooses a renormalization scheme that preserves the Fierz symmetry. In fact

$$Q^{(+)} = Q_1 \ , \qquad Q^{(-)} = Q_2 + \frac{1}{4} Q_1 \tag{15}$$

are the eigenvectors of the Fierz transformation with eigenvalues ± 1 [3], therefore they cannot mix under renormalization. This enforces the following relations among the anomalous dimension matrix elements

$$\hat{\gamma}_{12} = 0 \ , \qquad \hat{\gamma}_{21} = \frac{1}{4}\Big(\hat{\gamma}_{22} - \hat{\gamma}_{11}\Big) \ , \tag{16}$$

which indeed are satisfied by eq. (14).

The NLO anomalous dimension matrix is then given by

$$\hat{\gamma}^{(1)} = \begin{pmatrix} -\frac{N-1}{12N}\Big[127N^2+143N+63-\frac{57}{N}+8\pi^2\Big(N^2-2N+\frac{4}{N}\Big)-n_f\,(28N+44)\Big] & 0 \\ \frac{1}{4}\Big(X-\hat{\gamma}^{(1)}_{11}\Big) & X \end{pmatrix} , \tag{17}$$

where, as already noticed, the entry marked with X is not needed at the NLO.

[3]This is a consequence of the properties of the static fields, $v_\mu \gamma^\mu h^{(\pm)} = \pm h^{(\pm)}$.

We are now ready to write down the solution of the RGE, eq. (6), at the NLO. Using eqs. (12), (14) and (17), we obtain

$$\begin{aligned} C_1(\mu^2) &= \left(\frac{\alpha_s(m_b^2)}{\alpha_s(\mu^2)}\right)^{d_1}\left(1+\frac{\alpha_s(\mu^2)-\alpha_s(m_b^2)}{4\pi}J\right)C_1(m_b^2) \\ &+\left[\left(\frac{\alpha_s(m_b^2)}{\alpha_s(\mu^2)}\right)^{d_2}-\left(\frac{\alpha_s(m_b^2)}{\alpha_s(\mu^2)}\right)^{d_1}\right]\frac{\hat{\gamma}_{21}^{(0)}}{\hat{\gamma}_{22}^{(0)}-\hat{\gamma}_{11}^{(0)}}C_2(m_b^2) \\ C_2(\mu^2) &= \left(\frac{\alpha_s(m_b^2)}{\alpha_s(\mu^2)}\right)^{d_2}C_2(m_b^2)\ , \end{aligned} \tag{18}$$

where

$$d_i=\frac{\hat{\gamma}_{ii}^{(0)}}{2\beta_0}\ ,\qquad J=\beta_1\frac{d_1}{\beta_0}-\frac{\hat{\gamma}_{11}^{(1)}}{2\beta_0}\ . \tag{19}$$

The new contribution to the NLO running of C_1 is the term proportional to $\hat{\gamma}_{21}^{(0)}$, while C_2 is renormalized multiplicatively.

Numerically the new term contributes to the running between m_b^2 and a typical lattice scale $\mu^2=4$ GeV^2 by increasing C_1 of a few per cent, roughly doubling the already known NLO enhancement coming from $\hat{\gamma}_{11}^{(1)}$. Moreover the operator Q_2 also contributes at the NLO and should be included in lattice calculations of $f_B^2 B_B$ in the static limit.

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