

Information of a qubit interacting with a multilevel environment

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Abstract

We consider the interaction of a small quantum system (a qubit) with a structured environment consisting on many levels. The qubit will experience a decoherence process, which implies that its initial information will be transferred to the environment. We investigate how this information is distributed on a given subset of levels as a function of its size, using the mutual information between both entities, in the spirit of the partial information plots studied by Zurek [1]. In this case we observe some differences, which arise from the fact that we are partitioning just one quantum system, and not a collection of them. However some similar features, like a redundancy (in the sense that a given amount of information is shared by many subsets), which increases with the size of the environment, are also found here.

I. INTRODUCTION

Quantum systems are usually subject to interaction with some environment. Such interaction is at the origin of the decoherence experienced by the system [2, 3], a fact that makes difficult the design and performance of quantum computers [4]. In short, decoherence will cause a system S to evolve from a pure state to a mixed one, thus destroying the subtle correlations (entanglement) needed for quantum computation and for many kinds of experiments. During this process, the entropy of the system increases and, as a consequence, the information it contained is transferred to the environment E , where it is inevitably lost.

This is, at least, the usual point of view. Suppose, however, that one can have access to this information. If we consider the combined $S+E$ total system as isolated, then the dynamics of this enlarged system is unitary, which means that the total entropy is conserved. In other words, all the information lost by S must be **necessarily** encoded on E . One can then address the problem of knowing how this information is stored in the environment. There are reasons to justify this research. First, a knowledge of how the interaction of the system with the environment works can be used to design a strategy to protect the system against decoherence. Indeed, procedures to control the effects caused by decoherence have been proposed, and a better understanding of the decoherence mechanism may be used to improve these strategies [5, 6, 7, 8, 9, 10].

A second reason is related to our perception of quantum systems. As pointed out by W. Zurek [1], observers usually perceive those systems by an indirect way, i.e. by interaction with the environment to which the system is coupled. Such interaction leads to a proliferation (redundancy) of the information content of S within E , a fact that can help us to understand why many independent observers can agree about the properties of the system S or, in other words, how these properties can become *objective*. As shown in these references, the above mentioned redundancy comes at the price that only some selected states can give rise to a large offspring. This fact has been referred to as *quantum Darwinism* [1, 11, 12, 13].

In order to investigate these features, Zurek and his coworkers have considered a quantum system (a qubit, for example) coupled to an environment which consists on many identical quantum entities, such as spins or oscillators. One can then choose an arbitrary subset of E , and study the mutual information this particular partition shares with S . From these *Partial Information Plots* (PIPs) one can get insight about how many subsets of the whole

environment share a given amount of information with the smaller system S .

In this paper, we will face a different topic, although it is related to, and motivated by, the above discussion. We calculate the mutual information between a qubit and a multilevel environment (a qudit). We analyze this quantity for a fraction of the total number of levels as a function of the size of the chosen fraction. The purpose of this study is to investigate how many levels of the environment one should “read” in order to obtain information about the qubit, assuming this information is available experimentally. The obtained PIPs differ from the ones mentioned above, because in this case we are discussing the interaction with a subset of a **single** quantum system, instead of a collection of them. Some similarities, however, still exist. We discuss these topics. As we also show, the fraction of levels one should measure in order to gain a substantial information about the system S depends on the total number of levels of the environment.

This paper is organized as follows. In Sect. II we introduce the model used for the system and environment. In Sect. III we present the results that are derived from our calculations. Sect. IV will summarize these results. We work in units such that $\hbar = 1$.

II. MODEL

We use a simple model in order to describe the decoherence of our system (the qubit). It is assumed to interact with an environment consisting on a band of N equally spaced levels. This model can describe relaxation to equilibrium and decoherence effects in a natural way [14], and may be regarded as a simplified version of the two-band model described in [15, 16]. We write the Hamiltonian for the free qubit as

$$H_S = \frac{\Delta E}{2} \sigma_z, \quad (1)$$

where ΔE is the energy gap for our two-level system and σ_z is the third Pauli matrix. The Hamiltonian describing the environment is defined by

$$H_E = \sum_{n=1}^N \frac{\delta \varepsilon}{N} n |n\rangle \langle n| \quad (2)$$

and the interaction between both systems by

$$H_I = \sigma_z C, \quad (3)$$

with

$$C = \lambda \sum_{n_2 > n_1} c(n_1, n_2) |n_1\rangle \langle n_2| + h.c. \quad (4)$$

acting on E and σ_z acting on S . The indices n, n_1 and n_2 label the levels of the energy band. The global strength of the interaction with the qubit is given by λ . The coupling constants $c(n_1, n_2)$ are independent Gaussian random variables. Their averages (denoted by $\langle \rangle$) over the random constants $c(n_1, n_2)$ satisfy:

$$\langle c(n_1, n_2) \rangle = 0, \quad (5)$$

$$\langle c(n_1, n_2) c(n'_1, n'_2) \rangle = 0, \quad (6)$$

$$\langle c(n_1, n_2) c^*(n'_1, n'_2) \rangle = \delta_{n_1, n'_1} \delta_{n_2, n'_2}. \quad (7)$$

III. RESULTS

We now present some results obtained from a simulation of the model introduced in the previous section. The combined $S+E$ total system is considered as isolated, starting from a pure factorizable state $|\Psi(0)\rangle = |\Psi_S(0)\rangle \otimes |\Psi_E(0)\rangle$, and we let the whole system evolve according to the Schrödinger equation, thus obtaining $|\Psi(t)\rangle$ as a function of time from

$$i \frac{d}{dt} |\Psi(t)\rangle = H |\Psi(t)\rangle, \quad (8)$$

with $H = H_S + H_E + H_I$ the total Hamiltonian. As for the initial conditions, we choose $|\Psi_S(0)\rangle = \frac{1}{\sqrt{2}} |+\rangle + \frac{1}{\sqrt{2}} |-\rangle$, where $|\pm\rangle$ are the eigenstates of H_S . For the environment, we take the uniform superposition $|\Psi_E(0)\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^N |n\rangle$. As the total system evolves in time, the qubit becomes entangled with the environment, so that it can not be described as a pure state. We obtain the reduced density matrix of S as

$$\rho_S(t) \equiv Tr_E \{ \rho(t) \}, \quad (9)$$

where $\rho(t) = |\Psi(t)\rangle \langle \Psi(t)|$ is the density matrix corresponding to $S+E$, and Tr_E stands for the partial trace over the environment. From this, we calculate the entropy of system S as $S_S(t) = -\text{Tr} \{ \rho_S(t) \log \rho_S(t) \}$.

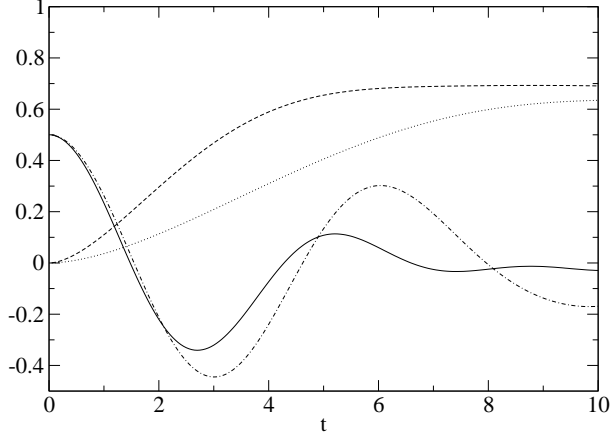


Figure 1: Numerical solution of Eq. (8) for an environment of $N = 10$ or $N = 100$ levels. The dashed-dotted and continuous lines show the evolution of $\rho_S(1,2)$ for $N = 10$ and $N = 100$, respectively. The dotted and dashed lines are plots of the entropy of the qubit for $N = 10$ and $N = 100$, respectively.

Fig. 1 plots the results obtained from a numerical simulation of Eq. (8) when our system interacts with a small environment with $N = 10$ levels. The rest of parameters are $\lambda = 2.5 \times 10^{-2}$, $\Delta E = 1$, $\delta\varepsilon = 0.5$. Although we solve Eq. (8) exactly, this choice warrants that the evolution of $\rho_S(t)$ can be approximated by a master equation of the form [14]

$$\frac{d}{dt}\rho_S(t) = -i[H_S, \rho_S(t)] + \Gamma(\sigma_z\rho_S(t)\sigma_z - \rho_S(t)). \quad (10)$$

with

$$\Gamma = \frac{2\pi\lambda^2 N}{\delta\varepsilon}. \quad (11)$$

It has been showed that Eq. (10) will approximate the evolution of $\rho_S(t)$ when the conditions

$$\begin{aligned} c_1 &\equiv \frac{\lambda N}{\delta\varepsilon} \geq \frac{1}{2} \\ c_2 &\equiv \frac{\lambda^2 N}{\delta\varepsilon^2} \ll 1 \end{aligned} \quad (12)$$

are met [16, 17]. In our case, we find $c_1 = 0.5$, $c_2 = 2.5 \times 10^{-2}$.

We clearly see that the system oscillates, but these oscillations are damped due to the effect of decoherence, which translates into an increase of the entropy. These features can be easily reproduced by studying Eq. (10), and imply that the information initially stored

in the system has degraded. Since the $S+E$ total system is assumed to be isolated, such information **must be present** in the environment E . The question we want now to analyze is how this information is distributed or, in other words, how much of the environment one should scan in order to know this information.

Let us write the time-dependent global state as

$$|\Psi(t)\rangle = \sum_{i=1}^2 \sum_{n=1}^N a_{in}(t) |i\rangle \otimes |n\rangle. \quad (13)$$

In this equation, $\{|i\rangle / i = 1, 2\}$ is a basis of the Hilbert space associated to the qubit, and $a_{in}(t)$ are the coefficients of the expansion in the composite base $\{|i\rangle \otimes |n\rangle\}$. One then has

$$\rho(t) = \sum_{i,j=1}^2 \sum_{n,m=1}^N a_{in}(t) a_{jm}^*(t) |i\rangle \langle j| \otimes |n\rangle \langle m|. \quad (14)$$

Assume one can access a given subset F of n_F levels (not necessarily consecutive) out of the N total number of levels in E . The density matrix ρ_{SF} corresponding to $F+S$ can be obtained from

$$\rho_{SF}(t) = \frac{1}{N_F} \sum_{i,j=1}^2 \sum_{n,m \in F} a_{in}(t) a_{jm}^*(t) |i\rangle \langle j| \otimes |n\rangle \langle m|. \quad (15)$$

where $N_F = \sum_{i=1}^2 \sum_{n \in F} |a_{in}|^2$ is a normalization factor, so that $\text{Tr} \{\rho_{SF}(t)\} = 1$ (we have omitted the dependence on t for brevity). It can be easily checked that this density matrix actually describes a pure state, since

$$\rho_{SF}(t) = |\Psi_{SF}(t)\rangle \langle \Psi_{SF}(t)|, \quad (16)$$

with

$$|\Psi_{SF}(t)\rangle = \sum_{i=1}^2 \sum_{n \in F} a_{in}(t) |i\rangle \otimes |n\rangle. \quad (17)$$

Finally, we obtain the density matrix for F from

$$\rho_F(t) \equiv \text{Tr}_S \{\rho_{SF}(t)\} = \frac{1}{N_F} \sum_{i=1}^2 \sum_{n,m \in F} a_{in}(t) a_{im}^*(t) |n\rangle \langle m|. \quad (18)$$

In order to characterise how much information we can obtain from S by knowing about F , we define the mutual information between F and S

$$I(S : F) = S_S + S_F - S_{SF} = S_S + S_F, \quad (19)$$

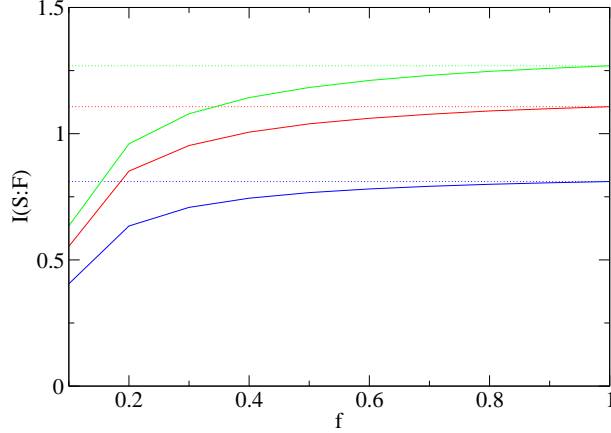


Figure 2: (Color online). Averaged mutual information (solid lines) as a function of the fraction f , corresponding to the model with $N = 10$ levels, for 3 different times: $t = 5, 7, 10$ (from bottom to top). The values of the model are the same as in Fig. (1). For each t , the horizontal dotted line shows the maximum value $2S_S(t)$.

where $S_F = -\text{Tr}\{\rho_F(t) \log \rho_F(t)\}$ is the entropy associated to F , and $S_{SF} = -\text{Tr}\{\rho_{SF}(t) \log \rho_{SF}(t)\}$ is the entropy associated to $F+S$, which vanishes according to the above discussion. The last equality in Eq. (19) immediately follows from this. We now analyze the magnitude $I(S : F)$ as a function of the fraction $f = n_F/N$ of levels involved in F .

Given a value of time, we have obtained, from the numerical simulations, that $I(S : F)$ is not a monotonic function of f , i.e. the information has accumulated in some levels at the expense of the rest. However, if we perform an averaging over all levels participating for a given fraction f , one expects a monotonic increase. This is indeed the case, as we show in Fig. 2. At this point, we observe a similarity with the partial information plots studied in [1, 11, 12, 18]. There is, however, a fundamental difference between both kind of plots, which has to be stressed. In the previous case, the authors consider the interaction of a qubit with an environment composed by **several** quantum systems (like qubits or oscillators). In our case, the environment is just one quantum system, although it consists on many levels. Partitioning these levels is not the same as partitioning several quantum systems into a subset of them. In other words, let \mathcal{H}_N be the Hilbert space associated to E , \mathcal{H}_F the Hilbert space corresponding to F and $\mathcal{H}_{\bar{F}}$ the one associated to its complementary in E . Obviously, \mathcal{H}_N is obtained as the direct sum $\mathcal{H}_N = \mathcal{H}_F \oplus \mathcal{H}_{\bar{F}}$ and **not** as the tensor product

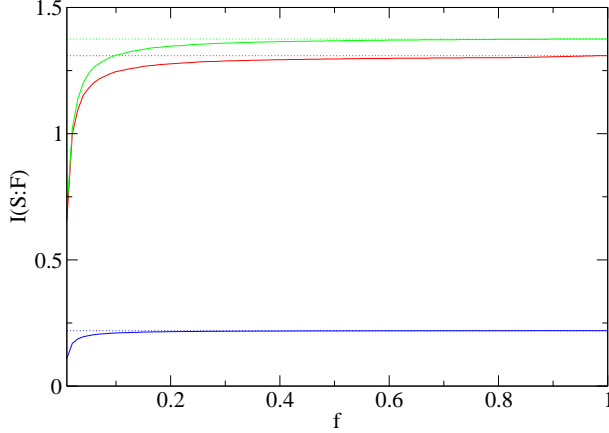


Figure 3: (Color online). Same as Fig. 2 for the model with $N = 100$.

$\mathcal{H}_{\mathcal{F}} \otimes \mathcal{H}_{\bar{\mathcal{F}}}$, as it would appear when the environment is made from several quantum objects, and F represents a subset of them. This has the consequence, for example, that strong subadditivity [4] does not apply to F and its complementary. Another consequence is that the plot of $I(S : F)$ versus f is not symmetric, differently to what is obtained by Zurek and coworkers.

Figure 2 is a plot of $I(S : F)$ as a function of f for the same model considered in Fig. 1 and three different times : $t = 5, 7, 10$. For each value of n_F , we have performed an average over all possible $\binom{N}{n_F}$ combinations. As can be seen, the resulting curves are monotonic functions. The maximum value is attained when $f = 1$ or, equivalently, when $n_F = N$, which amounts to knowing the total information in the environment. In this case, the partition of $F+S$ corresponds to two entangled quantum systems (E and S) sharing the same information $S_S(t)$. Therefore, this maximum value is

$$I(S : E) = 2S_S(t). \quad (20)$$

We have plotted this maximum value for each time t as a horizontal dotted line, corresponding to twice the entropy of the qubit shown in Fig. 1.

In order to explore a more complex environment, we have also performed a simulation when the number of levels is $N = 100$. The results for the time evolution of $\rho_S(1, 2)$ and the entropy $S_S(t)$ are also shown in Fig. 1. In this case we have taken $\lambda = 1.5 \times 10^{-2}$, the rest of parameters being the same as in the previous example, giving $c_1 = 3, c_2 = 9 \times 10^{-2}$.

The results for the quantity $I(S : F)$ are shown in Fig. 3. In this case, it becomes

impossible to perform an average over all possible $\binom{N}{n_F}$ combinations since, for example,

$\binom{100}{50} \approx 10^{29}$. Instead, we have performed an approximated average over a sufficiently

large number of combinations, until we obtain convergence. We observe that the mutual information converges faster towards its maximum value. This means that knowing a low fraction of the entire environment will provide almost complete information about the system. One could also try to interpret this result in the spirit of *redundant information* being stored in the environment [1, 11, 12, 18] (in the sense that many fragments share the same information). It is interesting to observe that a measurement over virtually any small random subset of levels configuring our system E can be used to obtain information about S . There is also a question regarding the relevant time scales for the problem. Fig. 3 suggests that the time scale necessary for information to be distributed throughout the environment may be much shorter than the decoherence time $T_d \sim 1/\Gamma$ (compare this figure with Fig. 1). Clearly, more research is necessary to elucidate this issue.

IV. CONCLUSIONS

In this paper, we studied the interaction of a qubit S with an environment E consisting on N levels. The initial information about the qubit is distributed through E via the well-known process of decoherence. We have investigated how much information about S one can obtain by measuring a subset F of n_F levels, as a function of the fraction $f = n_F/N$. As a measure of the amount of information, we used the mutual information $I(S : F)$. We found some differences with the partial information plots that appear when one considers the mutual information of S with a fraction of a given set of quantum systems that define the environment [1, 11, 12, 18]. The reason is that E is not a bipartite system of F and its complementary. For example, the plots we obtain do not have the symmetry properties that appear in the referenced papers although, when properly averaged over different fragments of the same size, we find that $I(S : F)$ increases monotonically with f .

Our results show that, when N is increases, even a small fraction of E can give information about S . One would be tempted to interpret this result in the spirit of *redundant information* being stored in the environment, in a similar way that Zurek and coworkers suggest for a

multipartite environment, but now applied to a part of a single quantum system E (although possessing a rich internal structure). Our work can be interpreted as a further step in the understanding on how information is distributed throughout the environment. Clearly, more research has to be done in this direction, but such knowledge can be used, in principle, to a better design of quantum systems and quantum computers in the presence of decoherence. Maybe also for a better understanding of how macroscopic observers perceive quantum systems, as suggested by Zurek et. al.

Acknowledgments

I would like to acknowledge the comments made by M.C. Bañuls and I. de Vega during interesting discussions. This work has been supported by the Spanish Ministerio de Educación y Ciencia through Projects AYA2007-67626-C03-01 and FPA2008-03373.

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