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ALGEBRAS DE FUNCIONES ANALÍTICAS ACOTADAS. INTERPOLACIÓN

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Álgebras de Funciones Analíticas Acotadas. Interpolación

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A mis padres

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Este trabajo ha sido realizado durante el periodo de disfrute de una beca predoctoral con referencia AP2002-0121 dentro del Programa Nacional de Formación de Profesorado Universitario del Ministerio de Educación y Ciencia. En el marco de este programa, se han efectuado dos estancias de investigación en el *Department of Mathematical Analysis* de *Kent State University* (Ohio, U.S.A.) durante los periodos octubre-diciembre de 2004 y agosto-enero de 2006. La investigación realizada también ha sido financiada parcialmente por los proyectos *Diferenciabilidad y funciones analíticas en espacios de Banach* de la Universidad de Valencia de referencia *AE 2003-0392* (2003), por el proyecto *Diferenciabilidad y funciones analíticas en espacios de Banach* otorgado por el Ministerio de Ciencia y Tecnología de referencia *BFM 2003-07540* (2003-2006) y el proyecto *Diferenciabilidad y funciones analíticas en espacios de Banach* de referencia *MTM 2007-64521* del Ministerio de Educación y Ciencia (2007-2009).

Contents

Resumen y Conclusiones en Español

Este trabajo resume, de forma parcial, la investigación realizada durante mi periodo predoctoral. Esta investigación pertenece, de forma general, a la teoría de álgebras de Banach conmutativas y álgebras uniformes y, en particular, se desarrolla principalmente en el ámbito de las álgebras de funciones analíticas acotadas en dominios de espacios de Banach finito e infinito dimensionales.

La teoría de álgebras de Banach conmutativas fue ampliamente desarrollada por I. M. Gelfand en los años 40. Las álgebras uniformes son una rama de esta teoría y están profundamente relacionadas con la teoría de funciones analíticas. La teoría de álgebras uniformes fue inicialmente desarrollada por G. E. Shilov a principios de los años 50. Algunas álgebras concretas de funciones analíticas nos proporcionan modelos para el estudio de álgebras uniformes y, recíprocamente, el estudio de las álgebras uniformes nos conduce a analizar propiedades de funciones analíticas. Pueden encontrarse referencias sobre álgebras uniformes en [Gam84], [Rud73], [Dal00] y [GK01], sobre la teoría general de funciones analíticas en [Din99] y [Muj86] y sobre álgebras de funciones analíticas en [AG89], [AG88], [Glo79] y [CG86].

Las líneas centrales de este trabajo son las siguientes:

- Sucesiones de Interpolación para Álgebras Uniformes
- Operadores de Composición
- Propiedades Topológicas de Álgebras de Funciones Analíticas

La investigación realizada sobre sucesiones de interpolación para álgebras uniformes se puede dividir en dos partes: una genérica en la que se proporcionan algunos resultados de carácter general sobre sucesiones de interpolación para álgebras uniformes, y una parte más específica, en que se tratan sucesiones de interpolación para algunas álgebras de funciones analíticas acotadas. Estos puntos se tratan en los Capítulos 2 y 3. El estudio de operadores de composición, principalmente sobre $H^\infty(B_E)$, centra el contenido del Capítulo 4. En este capítulo estudiaremos una descripción del espectro de estos operadores y los llamados operadores de composición de Radon-Nikodým. Con respecto a la tercera línea que hemos citado, estudiaremos los llamados operadores de tipo Hankel en el capítulo 5. Éstos nos permitirán tratar el concepto de álgebra tight y las álgebras de Bourgain de un subespacio de $C(K)$, que están estrechamente relacionadas con la propiedad de Dunford-Pettis.

A continuación presentamos un resumen de cada capítulo del trabajo:

1. Preliminares

En este primer capítulo a modo de preliminares, introducimos gran parte de la notación (estándar y no estándar) que seguiremos durante todo el trabajo y presentamos varios resultados de diversa índole que también serán necesarios para desarrollarlo.

Para ello, comenzamos con algunos resultados de topología y geometría de espacios de Banach y continuamos introduciendo la teoría de álgebras de Banach y uniformes.

A continuación, aportamos algunos resultados sobre análisis complejo: Introducimos el álgebra del disco $A(\mathbf{D})$, que es el conjunto de funciones analíticas definidas en $\mathbf{D}$ que se extienden contínuamente a $\overline{\mathbf{D}}$ dotado con la norma del supremo y H^∞, el álgebra de funciones analíticas acotadas definidas en $\mathbf{D}$, que se dota con la misma norma. También recordamos las llamadas *álgebras del polidisco* y *álgebras de la bola* para después dar paso a la introducción de la teoría de funciones analíticas en dominios de espacios de Banach en términos de series polinomiales (ver [Din99] y [Muj86]). En este punto, definimos las llamadas topologías polinomiales y definimos las álgebras de funciones analíticas que serán tratadas durante el resto del trabajo. La primera de ellas es $H^\infty(B_E)$, una extensión de H^∞ cuando el dominio de las funciones es la bola unidad abierta B_E de un espacio de Banach complejo E. También trataremos con dos extensiones del álgebra del disco $A(\mathbf{D})$: Por una parte $A_\infty(B_E)$, la extensión del álgebra del disco según hemos definido antes tomando como dominio la bola B_E y $A_u(B_E)$, el espacio de funciones analíticas definidas en B_E que son uniformemente contínuas. Es fácil ver que estas extensiones coinciden en caso de que E sea finito dimensional pero, sin embargo, $A_u(B_E)$ está estrictamente contenida en $A_\infty(B_E)$ si E es infinito dimensional [ACG91]. Para un estudio más amplio de estas álgebras, consultar los textos citados en la introducción de este resumen relativos a álgebras de funciones analíticas.

En el marco de estudio de éstas álgebras, tratamos con la extensión de Davie-Gamelin, que surge del trabajo de A. M. Davie y T. W. Gamelin [DG89], en que prueban que las funciones analíticas acotadas definidas en B_E pueden extenderse a funciones analíticas acotadas en $B_{E^{**}}$ mediante la *extensión de Aron-Berner* para polinomios. Este resultado permitirá extender varios resultados en el tercer capítulo de este trabajo gracias a la inclusión de $B_{E^{**}}$ en el espectro M_A en caso de que A sea una de las álgebras de funciones analíticas que hemos definido.

También recordamos la conocida representación de $H^\infty(B_E)$ dada por J. Mujica en [Muj91a] como espacio dual de

$$G^\infty(B_E) = \{u \in H^\infty(B_E)^* : u|_B \text{ es } \tau_c\text{−contínuo }\},$$

donde B denota la bola unidad de $H^\infty(B_E)$ y damos algunos resultados relacionados. Esto será de gran utilidad en el cuarto capítulo, cuando tratemos con los operadores de Radon-Nikodým.

Por otra parte, recordamos la *distancia pseudohiperbólica* para $z, w \in \mathbf{D}$, que está definida por

$$\rho(z,w) = \left| \frac{z-w}{1-z\bar{w}} \right|.$$

Esta distancia se extiende a álgebras uniformes mediante la fórmula

$$\rho_A(x,y) = \sup\{\rho(f(x), f(y)) : f \in A, \|f\| < 1\}.$$

La distancia pseudohiperbólica es clave en el estudio de sucesiones de interpolación para algunas álgebras uniformes como veremos en el capítulo 3.

Finalmente, introducimos los llamados operadores de composición entre espacios de funciones y damos algunos resultados relacionados. Los operadores de composición son esenciales en el cuarto capítulo y los resultados citados nos servirán para encontrar un nuevo enfoque a un resultado de A. M. Davie en el capítulo 2.

2. Interpolación en Álgebras Uniformes

El segundo capítulo está dedicado al estudio de sucesiones de interpolación para álgebras uniformes. Dada un álgebra uniforme A, decimos que una sucesión $(x_n) \subset M_A$ es *de interpolación (o interpolante)* para A si, dada cualquier sucesión $(\alpha_n) \in \ell_\infty$, existe $f \in A$ tal que $\widehat{f}(x_n) = \alpha_n$ para todo $n \in \mathbb{N}$, donde $\widehat{f}$ denota la transformada de Gelfand de $f \in A$. Consideremos la aplicación $R : A \longrightarrow \ell_\infty$ definida por $R(f) = (\widehat{f}(x_n))$. Se tiene que una sucesión $(x_n) \subset M_A$ es de interpolación para A si y sólo si existe una aplicación $T : \ell_\infty \longrightarrow A$ tal que $R \circ T = id_{\ell_\infty}$. Se dice que una sucesión es de *interpolación lineal* si T es un operador lineal. Decimos que la sucesión es de c_0*−interpolación (lineal)* si las definiciones anteriores se satisfacen para c_0 en lugar de ℓ_∞.

Con el objetivo de controlar las funciones interpolantes, se introduce el concepto de constante de interpolación. Dada $\alpha = (\alpha_j) \in \ell_\infty$, consideramos el conjunto $M_\alpha = \inf\left\{\|f\|_\infty : \widehat{f}(x_j) = \alpha_j,\ j \in \mathbb{N}, f \in A\right\}$. La *constante de interpolación* para la sucesión interpolante (x_n) está definida de la siguiente manera:

$$M = \sup\{M_\alpha : \alpha \in \ell_\infty, \|\alpha\|_\infty \leq 1\}.$$

Análogamente, se obtiene la constante de interpolación para el caso de sucesiones de c_0-interpolación sustituyendo ℓ_∞ por c_0.

Dada $(x_n) \subset M_A$, decimos que una sucesión $(f_k) \subset A$ es una sucesión de *funciones de Beurling* para (x_n) si, para cualesquiera $k, j \in \mathbb{N}$, se tiene que $\widehat{f_k}(x_j) = \delta_{k,j}$ y existe una constante $M > 0$ tal que $\sum_{j=1}^\infty |\widehat{f_j}(x)| \leq M$ para todo $x \in M_A$.

El punto de partida de nuestro estudio sobre interpolación es un resultado de P. Beurling (consultar [Gar81] y [Car62]) que prueba que, dada cualquier sucesión de interpolación $(z_n) \subset \mathbf{D}$ para H^∞, existe una sucesión de funciones de Beurling para (z_n). En este caso, la sucesión (z_n) también será de interpolación lineal para H^∞.

N. Th. Varopoulos (consultar [Var71] y [Gar81]) presentó algunos resultados sobre sucesiones de interpolación finitas para álgebras uniformes y P. Galindo, T. W. Gamelin y M. Lindström (consultar [GGL04]) mejoraron estos resultados extendiéndolos a sucesiones de interpolación finitas o infinitas cualesquiera para álgebras uniformes.

En nuestro estudio sobre interpolación, tratamos en primer lugar la relación entre sucesiones de interpolación y de interpolación lineal. A. M. Davie demostró que las sucesiones de c_0-interpolación para $A(\mathbf{D})$ son de c_0-interpolación lineal [Dav72]. En la Proposición 2.2.2, probamos lo siguiente:

Sea A un álgebra uniforme y $(x_n) \subset M_A$. Consideremos las siguientes afirmaciones:

a) (x_n) es una sucesión de interpolación para A.

b) Existe una sucesión de funciones de Beurling $(f_n) \subset A$ para (x_n).

c) (x_n) es una sucesión de c_0-interpolación lineal para A.

Entonces, se tiene que (b) y (c) son equivalentes y (a) implica ambas, (b) y (c).

Además, demostraremos que las sucesiones de c_0-interpolación lineal para A son de interpolación lineal para A^{**} en la Proposición 2.2.3.

Continuamos con el estudio de álgebras uniformes duales $A = X^*$. Nuestro objetivo es probar que, en este contexto, las sucesiones de c_0-interpolación son sucesiones de interpolación lineal. Para ello, comenzamos demostrando que, si $A = X^*$, las sucesiones de c_0-interpolación lineal son de interpolación lineal en la

Proposición 2.3.1. En segundo lugar, veremos que las sucesiones de c_0-interpolación son siempre de c_0-interpolación lineal cuando seguimos en el contexto de álgebras duales, obteniendo el resultado mencionado (ver Teorema 2.3.3):

Sea $A = X^*$ un álgebra uniforme dual y consideremos una sucesión $(x_n) \subset M_A \cap X$. Las siguientes afirmaciones son equivalentes:

i) Todo subconjunto finito de (x_n) es interpolante para A y existe una constante de interpolación independiente del número de términos de la sucesión interpolados.

ii) (x_n) es una sucesión c_0-interpolante para A.

iii) (x_n) es una sucesión interpolante A.

La última parte de esta sección demuestra que, bajo ciertas hipótesis, el álgebra resulta ser dual y el predual X puede escogerse para que satisfaga $(x_n) \subset X$:

Sea A una subálgebra cerrada de $\ell_\infty(Y)$ tal que los puntos del conjunto Y están separados por A. Supongamos que el límite de cualquier red de funciones de A que converge puntualmente en Y también pertenece a A. Si (x_n) es una sucesión c_0-interpolante para A, entonces (x_n) es interpolante para A.

A. M. Davie proporciona [Dav72] un ejemplo de sucesión de c_0-interpolación para $A_u(B_{c_0})$ que no es de c_0-interpolación lineal. Nosotros presentamos un enfoque distinto a este resultado por medio de algunos resultados conocidos de la teoría de operadores de composición.

3. Interpolación en $H^\infty(B_E)$. Separabilidad en $A_\infty(B_E)$ y $A_u(B_E)$

Una vez estudiados algunos resultados generales sobre interpolación para álgebras uniformes en general, continuamos con el estudio de sucesiones de interpolación para las álgebras de funciones analíticas $H^\infty(B_E)$ y $A_\infty(B_E)$ en el tercer capítulo. Damos algunas condiciones suficientes para que una sucesión $(x_n) \subset B_{E^{**}}$ sea interpolante para $H^\infty(B_E)$ y completamos el estudio realizado por J. Globevnik sobre la existencia de sucesiones de interpolación para $A_\infty(B_E)$. En consecuencia, probamos que $A_\infty(B_E)$ es separable sólo en caso de que E sea finito dimensional.

El estudio de sucesiones de interpolación para H^∞ parte de los resultados de L. Carleson [Car58], W. K. Hayman [Hay58] y D. J. Newman [New59]. En particular, de los trabajos de Hayman y Newman, se obtiene el siguiente resultado:

Sea $(z_n) \subset \mathbf{D}$ y supongamos que la sucesión $(|z_n|)$ crece exponencialmente a 1*, es decir, existe* $0 < c < 1$ *tal que la siguiente condición se satisface,*

$$\frac{1-|z_{n+1}|}{1-|z_n|} < c.$$

Entonces, (z_n) es interpolante para $H^\infty(B_E)$.

El principal resultado de interpolación para H^∞ fue probado por L. Carleson en [Car58], quien dio una condición necesaria y suficiente sobre la sucesión (z_n) para ser interpolante para H^∞ en términos de la distancia pseudohiperbólica ρ en $\mathbf{D}$. Esta condición expresa que existe una constante $\delta > 0$ tal que

$$\prod_{k \neq j} \rho(z_k, z_j) \geq \delta \qquad \text{para todo } j \in \mathbb{N}.$$

Nos referiremos a esta condición como la *condición de Carleson*. Como la noción de distancia pseudohiperbólica puede ser extendida al ámbito de las álgebras uniformes, se puede generalizar la condición de Carleson a éstas. En este caso, dada una sucesión $(x_n) \subset M_A$, diremos que (x_n) satisface la *condición de Carleson generalizada*.

En este contexto, B. Berndtsson demostró [Ber85] que la condición de Carleson generalizada es suficiente para que una sucesión $(x_n) \subset B_H$ sea interpolante para $H^\infty(B_H)$ cuando tratamos con espacios de Hilbert finito dimensionales H. B. Berndtsson, S-Y. A. Chang y K-C. Lin [BCL87] extendieron el resultado al espacio finito dimensional $(\mathbb{C}^n, \|\cdot\|_\infty)$ y probaron en este caso que la condición de Carleson generalizada no es una condición necesaria para que la sucesión (x_n) sea interpolante. P. Galindo, T. W. Gamelin y M. Lindström remarcaron [GGL08] que los resultados dados por B. Berndtsson pueden extenderse a cualquier espacio de Hilbert infinito dimensional. Este caso será tratado en el Teorema 3.2.10, donde daremos una construcción explícita de las funciones de interpolación bajo el supuesto de que se cumpla la condición de Carleson generalizada. Para esto, adaptaremos algunos de los resultados dados por B. Berndtsson y estudiaremos los automorfismos de B_H para un espacio de Hilbert H. La fórmula explícita de la distancia pseudohiperbólica para espacios de Hilbert dada por

$$1-\rho(x,y)^2 = \frac{(1-\|x\|^2)(1-\|y\|^2)}{|1-<x,y>|^2}$$

facilitará en gran medida este estudio. Denotando por M_a el automorfismo de B_H estudiado que transforma 0 en a, en primer lugar simplificamos la expresión de $<M_{-y}(x), M_{-y}(z)>$ en términos de productos escalares de los elementos $x, y, z \in B_H$

en el Lema 3.2.4. Este resultado nos permitirá, junto con algunos lemas técnicos, enunciar el siguiente resultado de carácter general en espacios de Hilbert (ver Teorema 3.2.10):

Sea H un espacio de Hilbert complejo y (x_n) una sucesión en B_H. Supongamos que existe $\delta > 0$ tal que (x_n) satisface la condición de Carleson generalizada para δ. Entonces, existe una sucesión de funciones de Beurling (F_k) para (x_n). En particular, la sucesión (x_n) es interpolante para $H^\infty(B_H)$ y la constante de interpolación está acotada por

$$\frac{2048}{e\delta}(1+2\log\frac{1}{\delta})^2.$$

Por otra parte, cuando tratamos con espacios de Banach en general, no se conocen fórmulas explícitas para la distancia pseudohiperbólica en la mayor parte de los casos. Esto nos conducirá, cuando tratemos con espacios de Banach, a plantear condiciones más restrictivas para asegurar que una sucesión sea de interpolación para $H^\infty(B_E)$. Así, probamos que la condición de crecimiento exponencial a 1 dada en el resultado de Hayman-Newman aplicada a la sucesión de normas $(\|x_n\|)$ es suficiente para asegurar que la sucesión $(x_n) \subset B_{E^{**}}$ es interpolante para $H^\infty(B_E)$ si E es cualquier espacio de Banach finito o infinito dimensional (ver Corolario 3.2.16). Esto es consecuencia de un resultado más fuerte (ver Teorema 3.2.14):

Sea (x_n) una sucesión en $B_{E^{**}}$ y supongamos que la sucesión $(\|x_n\|)$ satisface la condición de Carleson. Entonces, para todo $0 < s < 1$, existe una sucesión de funciones de Beurling $(F_j) \subset H^\infty(B_E)$ para (x_n). En particular, la sucesión (x_n) es interpolante para $H^\infty(B_E)$ y la constante de interpolación puede escogerse de forma que esté acotada por

$$\frac{4e^2}{(1-s)\delta}\left(1+2\log\frac{1}{(1-s)\delta}\right).$$

En particular, es obvio que la constante de interpolación para (x_n) estará acotada por

$$\frac{4e^2}{\delta}\left(1+2\log\frac{1}{\delta}\right).$$

Este resultado nos permite concluir varios corolarios. Por una parte, como ya hemos mencionado, obtenemos que la condición de Hayman-Newman para la sucesión de normas $(\|x_n\|)$ permite concluir que $(x_n) \subset B_{E^{**}}$ es interpolante para $H^\infty(B_E)$. A su vez, una consecuencia clara de este resultado, es un teorema debido a R. M. Aron, B. Cole y T. Gamelin [ACG91]: Sea $(x_n) \subset B_{E^{**}}$ una sucesión tal que $\lim_{n\to\infty} \|x_n\| = 1$. Entonces, existe una subsucesión interpolante para $H^\infty(B_E)$.

Una pregunta que surge de modo natural a partir de nuestro estudio sobre interpolación en $H^\infty(B_E)$ es si la condición de Carleson generalizada para una sucesión $(x_n) \subset B_{E^{**}}$ es siempre suficiente para afirmar que (x_n) es interpolante para $H^\infty(B_E)$ o, en general, qué ocurre en el caso de una sucesión $(x_n) \subset M_A$ que cumple la condición de Carleson para un álgebra determinada A. En el último parágrafo de esta sección, reducimos esta discusión al caso de $E = c_0$, probando el siguiente resultado (ver Teorema 3.2.18) basado en el trabajo de Berndtsson, Chang y Li (ver [BCL87]):

Supongamos que cualquier sucesión $(x_n) \subset B_{c_0}$ satisfaciendo la condición de Carleson generalizada es interpolante para $H^\infty(B_{c_0})$ con una constante de interpolación que depende solamente de δ. Entonces, dada cualquier álgebra dual $A = X^*$, todas las sucesiones $(x_n) \subset X$ satisfaciendo la condición de Carleson generalizada son sucesiones de interpolación lineal para A cuya constante de interpolación depende sólo de δ.

Si se satisface la hipótesis de este resultado, entonces tendremos, en particular, que para todo espacio de Banach E, la condición de Carleson generalizada es suficiente para que una sucesión sea interpolante para $H^\infty(B_E)$ ya que éstas son álgebras duales.

En último lugar, en esta sección estudiamos algunas condiciones necesarias para que una sucesión sea interpolante para $H^\infty(B_E)$. En particular, probamos que si los polinomios sobre E son débil contínuos sobre conjuntos acotados, entonces las sucesiones de interpolación (x_n) para $H^\infty(B_E)$ cumplen que $\|x_n\|$ converge a 1.

En la segunda sección del capítulo tratamos la separabilidad de las álgebras $A_\infty(B_E)$ y $A_u(B_E)$. La existencia de sucesiones de interpolación para un álgebra implica que ésta no es separable. Cuando tratamos con $A = A_\infty(B_E)$, la existencia de sucesiones de interpolación para A fue probada por J. Globevnik [Glo78] para una amplia clase de espacios de Banach infinito dimensionales. De hecho, Globevnik probó este resultado para los espacios de Banach cumpliendo que exista una sucesión $(x_n) \subset S_E$ de puntos fuertemente expuestos sin puntos de acumulación. Esta hipótesis la cumplen, entre otros, los espacios de Banach reflexivos. Nosotros extendemos este resultado, probando la existencia de sucesiones de interpolación para $A_\infty(B_E)$ para cualquier espacio de Banach infinito dimensional E, caracterizando por tanto la separabilidad de $A_\infty(B_E)$ en términos de que E tenga dimensión finita:

Sea E un espacio de Banach complejo. Las siguientes afirmaciones son equivalentes:

i) $A_\infty(B_E)$ no es separable.
ii) E es infinito dimensional.
iii) Existen sucesiones de interpolación para $A_\infty(B_E)$.

Por último, estudiamos la separabilidad del álgebra $A_u(B_E)$. En caso de que E tenga dimensión finita, se tiene que $A_\infty(B_E) = A_u(B_E)$ y, por tanto, $A_u(B_E)$ es separable. Sin embargo, el caso infinito dimensional presenta algunas diferencias con $A_\infty(B_E)$. En este caso, hay ejemplos de espacios de Banach E para los cuales $A_u(B_E)$ es separable y otros tantos para los que no lo es. Una forma de probar que $A_u(B_E)$ no es separable es, como ya hemos mencionado, demostrar la existencia de sucesiones de interpolación para el álgebra, como ocurre para el caso $E = \ell_p$ para $1 \leq p \leq \infty$. Entre los casos en que $A_u(B_E)$ es separable, encontramos $E = c_0$ o el espacio de Tsirelson T^*.

Es bien conocido que un álgebra A es separable si y sólo si su espectro M_A es metrizable para la topología de Gelfand, es decir, para la restricción de la $w(A^*,A)-$topología. Nosotros afinamos este resultado, probando que un álgebra es separable si y sólo si su espectro M_A es $\tau(A^*,P(E))$-metrizable. Esta condición nos conduce al estudio de la metrizabilidad para la topología polinomial de subconjuntos acotados de M_A, en particular de aquellos de B_E. En este contexto, en primer lugar presentamos un ejemplo de un espacio de Banach complejo y un subconjunto acotado $L \subset B_E$ que es metrizable para la topología polinomial pero $(P(^2E), \|\cdot\|_L)$ no es separable, obteniendo por tanto que $A_u(B_E)$ no es separable ya que el espacio de polinomios $P(E)$ es denso en $A_u(B_E)$. Sin embargo, obtenemos el siguiente resultado:

Sea E un espacio de Banach real o complejo y L un subconjunto separable absolutamente convexo, acotado y cerrado de E que es $\tau(E,P(E))$-metrizable. Entonces, $(E^*, \|\cdot\|_L)$ es separable.

Por último, damos un ejemplo de un espacio de Banach cuya bola es metrizable para la topología polinomial que prueba que la separabilidad de L en el resultado anterior es necesaria para que sea válido.

4. Operadores de Composición sobre $H^\infty(B_E)$

En el cuarto capítulo estudiamos los llamados operadores de composición de $H^\infty(B_F)$ en $H^\infty(B_E)$. Estos son aplicaciones $C_\phi : H^\infty(B_F) \longrightarrow H^\infty(B_E)$ definidas por $C_\phi(f) = f \circ \phi$, donde ϕ es una función analítica de B_E en B_F denominada *símbolo*. Dividimos este estudio en dos partes. En la primera, damos una descripción del espectro de algunos operadores de composición de $H^\infty(B_E)$ en sí mismo, extendiendo así un teorema de L. Zheng para el caso H^∞ y, en segundo lugar, estudiamos la clase de operadores de Radon-Nikodým de $H^\infty(B_F)$ en $H^\infty(B_E)$ por medio de los conjuntos de Asplund, íntimamente relacionados con estos operadores.

Previamente, presentamos un lema (ver Lema 4.1.1) de carácter general sobre operadores de composición en el que aplicamos algunos resultados de interpolación. Este lema se aplicará al estudio de los operadores de Radon-Nikodým en la Proposición 4.3.21:

Sea $\phi : B_E \longrightarrow B_F$ una aplicación analítica y supongamos que no existe ningún $0 < r < 1$ tal que $\phi(B_E) \subset rB_F$. Entonces, existen operadores lineales $T : H^\infty(B_E) \longrightarrow \ell^\infty$ y $S : \ell^\infty \longrightarrow H^\infty(B_F)$ tales que

$$T \circ C_\phi \circ S = Id_{\ell^\infty}.$$

Como ya hemos mencionado, L. Zheng [Zhe02] describió el espectro de algunos operadores de composición de H^∞ en sí mismo (ver Teorema 4.2.2):

Sea $\phi : \mathbf{D} \longrightarrow \mathbf{D}$ una aplicación analítica no constante que no es un automorfismo y supongamos que existe $a \in \mathbf{D}$ tal que $\phi(a) = a$. Entonces, o bien el espectro $\sigma(C_\phi)$ es $\overline{\mathbf{D}}$ si C_ϕ no tiene potencia compacta o $\sigma(C_\phi) = \left\{\phi'(a)^k : k \in \mathbb{N}\right\} \cup \{0,1\}$ si C_ϕ tiene potencia compacta.

Estos resultados fueron extendidos por P. Galindo, T. W. Gamelin y M. Lindström a $H^\infty(B_E)$ en [GGL08] para el caso en que C_ϕ tenga potencia compacta, donde E denota cualquier espacio de Banach complejo. En este trabajo, los autores también tratan el caso en que C_ϕ no tenga potencia compacta, extendiendo también en este caso el resultado de L. Zheng a $H^\infty(B_H)$ cuando H es un espacio de Hilbert.

Parece ser que H. Kamowitz fue el primero en utilizar sucesiones de interpolación para analizar el espectro de algunos operadores de composición. Actualmente, la existencia de tales sucesiones de interpolación se deriva de la siguiente estimación:

$$\text{Dado } 0 < r < 1, \text{ existe } \varepsilon > 0 \text{ tal que } \frac{1 - |\phi(z)|}{1 - |z|} > 1 + \varepsilon \quad \text{si } |z| > r$$

donde ϕ es una aplicación analítica de $\mathbf{D}$ en sí mismo cumpliendo $\phi(0) = 0$ y $|\phi'(0)| < 1$. Esta estimación (consultar Lemma 7.33 en [CM95]) se obtiene normalmente usando un lema de Julia (consultar Lemma 2.41 en [CM95]) y derivadas angulares.

En su estudio [GGL08] del espectro de operadores de composición C_ϕ sobre $H^\infty(B_H)$ para un espacio de Hilbert H, Galindo, Gamelin y Lindström obtuvieron una estimación análoga a la que hemos citado y a la que denominaron *estimación de tipo Julia*. Por tanto, nosotros seguiremos denominando así a esta desigualdad cuando tratemos con un espacio de Banach cualquiera E y sustituyamos el valor absoluto por la norma de E.

Una estimación de tipo Julia nos permite, bajo algunos supuestos más, concluir el siguiente resultado (ver Teorema 4.2.12) en el que utilizamos el resultado de interpolación 3.2.16. Éste resultado es una extensión a espacios de Banach del resultado probado en [GGL08] para el caso de un espacio de Hilbert H:

Sea $\phi : B_E \longrightarrow B_E$ una aplicación analítica tal que $\phi(0) = 0$, $\|\phi'(0)\| < 1$ y supongamos que $\phi(B_E)$ es relativamente compacto en B_E. Supongamos que C_ϕ no tiene potencia compacta y que ϕ cumple la siguiente estimación de tipo Julia: Para todo $0 < \delta < 1$, existe $\varepsilon > 0$ tal que

$$\frac{1 - \|\phi(x)\|}{1 - \|x\|} \geq 1 + \varepsilon, \quad \text{para todo } x \in \phi(B_E) \text{ tal que } \|x\| \geq \delta.$$

Entonces, el espectro de C_ϕ coincide con el disco unidad cerrado $\overline{\mathbf{D}}$.

Diremos que un subconjunto $W \subset B_E$ *se aproxima a S_E de forma compacta* si cualquier sucesión $(x_n) \subset W$ tal que $\|x_n\| \to 1$, tiene una subsucesión convergente.

La descripción del espectro de operadores de composición C_ϕ que no tienen potencia compacta para el caso del espacio $\mathbb{C}^n$ dotado con la norma supremo no se había producido ya que no se conocía una estimación de tipo Julia para este espacio. En el presente capítulo, probamos que existe tal estimación para cualquier espacio de tipo $E = C_0(X)$:

Sea $E = C_0(X)$ y consideremos una aplicación analítica $\phi : B_E \longrightarrow B_E$ tal que $\phi(0) = 0$ y $\|\phi'(0)\| < 1$. Supongamos que W se aproxima a S_E de forma compacta. Entonces, se tiene la siguiente estimación de tipo Julia: para cualquier $\delta > 0$, existe $\varepsilon > 0$ tal que

$$\frac{1 - \|\varphi(x)\|}{1 - \|x\|} \geq 1 + \varepsilon \quad \text{para todo } x \in W \text{ tal que } \|x\| \geq \delta.$$

Para describir el espectro de los operadores de composición C_ϕ para el caso $E = C_0(X)$, basta tomar $W = \phi(B_E)$ y aplicar el resultado que acabamos de mencionar al Teorema 4.2.12 citado antes. En particular, obtendremos la descripción de $\sigma(C_\phi)$ en el caso $\mathbb{C}^n$.

En la segunda parte de este capítulo, estudiamos la clase de operadores de Radon-Nikodým de $H^\infty(B_E)$ en $H^\infty(B_F)$. En primer lugar, recordamos algunos resultados sobre la propiedad de Radon-Nikodým e introducimos el concepto de operador de Radon-Nikodým. Un operador $T : E \longrightarrow F$ se dice que es un *operador de Radon-Nikodým* si $T(B_E)$ es un conjunto de Radon-Nikodým. Estos operadores están estrechamente relacionados con la propiedad de Asplund, así que continuamos con el estudio de esta propiedad para conjuntos acotados de espacios de Banach. Debido al enfoque de la propiedad de Asplund a los operadores de Radon-Nikodým, introducimos tal propiedad según la definición dada por S. Fitzpatrick [Fit80]. El autor demostró que esta definición es equivalente a la dada anteriormente relacionada con la diferenciabilidad de la norma de E. Dado $D \subset E$, se dice que D es un *conjunto de Asplund* si el espacio $(E^*, \|\cdot\|_A)$ es separable para cualquier conjunto numerable $A \subset D$. Un espacio de Banach E se dice que es un *espacio de Asplund* si B_E tiene la propiedad de Asplund. Un operador se dice que es un *operador de Asplund* si transforma la bola unidad en un conjunto de Asplund. La relación entre los conjuntos de Asplund y los operadores de Radon-Nikodým surge en el siguiente resultado (ver [Bgi83]): *Un operador lineal T es de Asplund si y sólo si T^* es de Radon-Nikodým*. Este resultado nos lleva al siguiente corolario:

Un operador de composición $C_\phi : H^\infty(B_F) \longrightarrow H^\infty(B_E)$ es de Radon-Nikodým si y sólo si el operador $C^\phi : G^\infty(B_E) \longrightarrow G^\infty(B_F)$ dado por la restricción de C_ϕ^* al predual $G^\infty(B_E)$ es de Asplund. A su vez, el operador C^ϕ es de Asplund si y sólo si el conjunto $\{\delta_{\phi(x)} : x \in B_E\}$ es de Asplund en G^∞.

Es bien conocido que la propiedad de Asplund se conserva por transformaciones lineales. El anterior corolario nos muestra el interés en extender este resultado a transformaciones analíticas entre espacios de Banach. En particular, probamos el siguiente resultado en la Proposición 4.3.13:

Sean E y F espacios de Banach y $D \subset E$ un conjunto de Asplund.

a) *Supongamos que $P(^kE) = \overline{P_f(^kE)}$ para algún $k \in \mathbb{N}$. Si $P : E \longrightarrow F$ es un polinomio k-homogéneo, entonces $P(D)$ es un conjunto de Asplund.*

b) *Supongamos que $P(^kE) = \overline{P_f(^kE)}$ para cualquier $k \in \mathbb{N}$. Si $f : U \subset E \longrightarrow F$ es una función analítica de tipo acotado y D es U- acotado, entonces $f(D)$ es un conjunto de Asplund.*

El principal resultado sobre operadores de composición de Radon-Nikodým es el siguiente (ver Teorema 4.3.21):

El operador de composición $C_\phi : H^\infty(B_F) \longrightarrow H^\infty(B_E)$ es de Radon-Nikodým si y sólo si existe $0 < r < 1$ tal que $\phi(B_E) \subset rB_F$ y $(P(F), \|\cdot\|_A)$ es separable para cualquier conjunto numerable $A \subset \phi(B_E)$.

Además, presentamos algunas condiciones suficientes para que C_ϕ sea un operador de composición de Radon-Nikodým:

Sea $\phi : B_E \longrightarrow B_F$ una aplicación analítica.

a) Supongamos que $P(^kF) = \overline{P_f(^kF)}$ para cualquier $k \in \mathbb{N}$. Si $\phi(B_E)$ es un conjunto de Asplund y existe $0 < r < 1$ tal que $\phi(B_E) \subset rB_F$, entonces el operador de composición $C_\phi : H^\infty(B_F) \longrightarrow H^\infty(B_E)$ es de Radon-Nikodým.

b) Supongamos que $A_u(B_F)$ es separable. Si $\phi(B_E)$ es un cojunto de Asplund y existe $0 < r < 1$ tal que $\phi(B_E) \subset rB_F$, entonces el operador de composición $C_\phi : H^\infty(B_F) \longrightarrow H^\infty(B_E)$ es de Radon-Nikodým.

5. Operadores de tipo Hankel

En este capítulo, estudiamos algunas propiedades relacionadas con los llamados operadores de tipo Hankel sobre álgebras uniformes. Estos operadores son extensiones de los clásicos operadores de Hankel definidos en el espacio de Hardy H^2. Referencias sobre los operadores de Hankel en H^2 pueden encontrarse en [Pow82], [Zhu90] y [Pel98]. Dada un álgebra uniforme A de $C(K)$ y $g \in C(K)$, el operador de tipo Hankel $S_g : A \longrightarrow C(K)/A$ está definido por $S_g(f) = gf + A$.

El estudio de los operadores de Hankel permite, por una parte, estudiar la propiedad de Dunford-Pettis mediante las llamadas álgebras de Bourgain y, por otra, introducir el concepto de álgebra tight que caracteriza a las subálgebras uniformes A de $C(K)$ tales que $A^{**} + C(K)$ sigue siendo una subálgebra cerrada de $C(K)^{**}$.

La propiedad de Dunford-Pettis surge del resultado dado por N. Dunford y B. J. Pettis en [DP40] que fue posteriormente mejorado por R. Phillips en [Phi40]:

Sea F un espacio de Banach y μ una medida. Supongamos que el operador lineal $T : L^1(\mu) \longrightarrow F$ es débil contínuo. Entonces, T es completamente contínuo.

A. Grothendieck [Gro53] probó, algunos años más tarde, que el resultado dado por N. Dunford y B. J. Pettis sigue siendo válido para el caso de espacios de tipo $C(K)$ y se refiere a esta propiedad como propiedad de Dunford-Pettis: Dado un espacio de Banach F y un operador débil compacto $T : C(K) \longrightarrow F$, se tiene que T es completamente contínuo.

Diremos que un espacio de Banach E tiene la *propiedad de Dunford-Pettis* si dado cualquier espacio de Banach F y un operador débil compacto $T : E \longrightarrow F$, se tiene que T es completamente contínuo. Es bien conocido [Die80] que E tiene la propiedad de Dunford-Pettis si y sólo si $\lim_n x_n^*(x_n) = 0$ para cualesquiera sucesiones $x_n \xrightarrow{w} 0$ en E y $x_n^* \xrightarrow{w} 0$ en E^*. Por tanto, es claro que E tiene la propiedad de

Dunford-Pettis si E^* también la tiene. El recíproco es falso en general, como probó C. Stegall en [Ste72]. La propiedad de Dunford-Pettis es hereditaria para subespacios complementados pero no para subespacios cerrados en general [PS65]. También es bien conocido que no existen espacios de Banach reflexivos de dimensión infinita con la propiedad. Más resultados sobre la propiedad pueden encontrarse en [Die80].

En el contexto de las álgebras de funciones analíticas, encontramos diversos resultados relacionados con la propiedad de Dunford-Pettis: J. Chaumat probó que el álgebra del disco $A(\mathbf{D})$ tiene la propiedad [Cha74] y J. Bourgain lo probó para H^∞ [Bou84b] y también para las álgebras de la bola $A(B_n)$ y las álgebras del polidisco $A(\mathbf{D}^n)$ [Bou84a].

Como decíamos, los operadores de tipo Hankel están estrechamente relacionados con la propiedad de Dunford-Pettis a través de las llamadas álgebras de Bourgain introducidas por J. A. Cima y R. M. Timoney en [CT87]. Una función $g \in C(K)$ pertenece al álgebra de Bourgain A_b de A (resp. A_B) si el operador de tipo Hankel S_g es completamente contínuo (resp. S_g^{**} es completamente contínuo). El trabajo de J. Bourgain [Bou84a] fue reformulado en [CT87], así que, en este contexto, una condición suficiente para que una subálgebra A de $C(K)$ tenga la propiedad de Dunford-Pettis es que $A_b = C(K)$. Si $A_B = C(K)$, entonces el espacio dual A^* tiene la propiedad de Dunford-Pettis y, por tanto, también la tiene A.

Desde este punto de vista, los resultados dados por J. Bourgain también pueden ser enfocados utilizando tal reformulación y obtenemos que las álgebras de Bourgain de $A(B_n)$ son iguales a $C(\overline{B}_n)$ y, por tanto, $A(B_n)$ y su dual tienen la propiedad de Dunford-Pettis. En relación a esto, en el mismo trabajo, una prueba por inducción le permitió a J. Bourgain concluir que $A(\mathbf{D}^n)$ y su dual tienen ambos la propiedad de Dunford-Pettis. Sin embargo, la línea de esta prueba no nos permite concluir si las álgebras de Bourgain de $A(\mathbf{D}^n)$ coinciden con todo el espacio $C(\overline{\mathbf{D}}^n)$. En contra de lo que ocurre en el caso del álgebra de la bola $A(B_n)$, en la sección 5.2 probamos el siguiente resultado:

Sea A el álgebra $A(\mathbf{D}^n)$ considerada como subespacio de $C(\bar{\mathbf{D}}^n)$ para $n \geq 2$. Entonces,

$$A_B = A_b = A.$$

La última sección de este capítulo trata el concepto de álgebra tight. Este concepto fue introducido por B. J. Cole y T. W. Gamelin en [CG82]. Un álgebra uniforme A de $C(K)$ se dice que es un *álgebra tight* sobre K si los operadores de tipo Hankel S_g son débilmente compactos para todo $g \in C(K)$. Como ya hemos mencionado, la propiedad de ser tight caracteriza a las subálgebras uniformes

A de $C(K)$ tales que $A^{**} + C(K)$ es una subálgebra cerrada de $C(K)^{**}$. Además, esta propiedad es, como se menciona en [CG82], una aproximación a la versión abstracta de la resolubilidad de un determinado problema $\bar{\partial}$.

En [CG82], los autores probaron que $A(\mathbf{D}^n)$ no es tight sobre su espectro para $n \geq 2$. En este trabajo presentamos un nuevo enfoque de este resultado, extendiéndolo a álgebras $A_u(B_E)$ para espacios de Banach $E = \mathbb{C} \times F$ dotados con la norma del supremo:

Sea E un espacio de Banach y $F = \mathbb{C} \times E$ dotado con la norma del supremo $\|(z,x)\|_F = \sup\{|z|, \|x\|_E\}$. Entonces $A = A_u(B_F)$ no es tight en su espectro.

En esta línea, también estudiamos la propiedad de ser tight en el caso de H^∞. Un lema de tipo técnico (ver Proposición 5.3.5) y algunos resultados que recopilamos en el Lema 5.3.6 nos conducen al siguiente resultado:

Sea E un espacio de Banach complejo. Entonces, $H^\infty(B_E)$ no es tight sobre su espectro.

Resumen en Español

Introduction

This thesis partially summarizes the research done during my predoctoral period. This research belongs in a general way to the Theory of uniform algebras. In particular, our work is mainly developed in the area of spaces of analytic functions on domains of finite and infinite dimensional Banach spaces. The lines treated in this thesis are the following:

- Interpolating Sequences for Uniform Algebras
- Composition Operators
- Topological Properties in Algebras of Analytic Functions

The research done on interpolating sequences for uniform algebras can be split into two parts: a general one, where some general results about interpolating sequences for uniform algebras have been proved and, a more specific one for some particular algebras of bounded analytic functions. This topic is discussed in Chapters 2 and 3. The study of composition operators, mainly those on $H^\infty(B_E)$, is the central content of Chapter 4. With regard to the third item stated above, the so-called Hankel-type operators have been studied in Chapter 5. These operators are useful in order to study several topological properties of the algebras.

Therefore, in the first chapter we give some notation and background on several areas which will be necessary in the sequel. It begins with several results on topology and geometry of Banach spaces and continue with background on Banach and uniform algebras since these are the basis of our research. Further results on Banach and uniform algebras can be found in [Gam84].

Next, we give some background on $A(\mathbf{D})$, the algebra of complex analytic functions defined on $\mathbf{D}$ which extend continuously to $\overline{\mathbf{D}}$ and H^∞, the algebra of complex bounded analytic functions defined on $\mathbf{D}$. In addition, we introduce the *polydisk* and *ball algebras*. Then, we introduce the theory of analytic functions on domains of Banach spaces in terms of polynomial series (see [Din99] and [Muj86]). At this point, we introduce algebras of analytic functions which will be treated during the rest of the thesis. The first one is $H^\infty(B_E)$, an extension of H^∞ when the domain is the open unit ball B_E of a complex Banach space E. We will also deal with two extensions of the disk algebra $A(\mathbf{D})$: $A_\infty(B_E)$, the extension of the disk algebra on B_E such as we defined it above and $A_u(B_E)$, the space of analytic functions on B_E which are uniformly continuous. For further results related to these algebras, see also [AG89], [AG88], [Glo79] and [CG86].

We continue with the study of the pseudohyperbolic distance ρ on $\mathbf{D}$ and the spectrum M_A of a uniform algebra.

Finally, we introduce the so-called composition operators between spaces of functions and give some related results.

The second chapter is devoted to the study of *interpolating sequences* on uniform algebras A. A sequence (x_n) in the spectrum M_A is *interpolating* for A if for any sequence $(\alpha_n) \in \ell_\infty$, there exists a function $f \in A$ such that $\widehat{f}(x_n) = \alpha_n$ for all $n \in \mathbb{N}$. Given an interpolating sequence (x_n) for A, let $T : \ell_\infty \longrightarrow A$ be a function which maps any $(\alpha_n) \in \ell_\infty$ into its interpolating function $f \in A$. If T is a linear operator, then the sequence (x_n) is said to be *linear interpolating*. The sequence (x_n) is said to be c_0-*(linear) interpolating* if the definition is satisfied by c_0 instead of ℓ_∞.

Given a sequence $(x_n) \subset M_A$, we say that a sequence $(f_k) \subset A$ is a sequence of *Beurling functions* for (x_n) if for any $k, j \in \mathbb{N}$, we have that $\widehat{f_k}(x_j) = \delta_{kj}$ and there exists a constant $M > 0$ such that $\sum_{j=1}^{\infty} |\widehat{f_j}(x)| \leq M$ for any $x \in M_A$.

The starting point for our research on interpolation is a result of P. Beurling (see [Gar81] and [Car62]), which states that for any interpolating sequence $(z_n) \subset \mathbf{D}$ for H^∞, there exists a sequence of Beurling functions for (z_n). Then, it is clear that the sequence (z_n) is linear interpolating for $H^\infty(B_E)$.

N. Th. Varopoulos (see [Var71] and [Gar81]) gave some results on finite interpolating sequences for uniform algebras and P. Galindo, T. W. Gamelin and M. Lindström (see [GGL04]) improved these results by extending them to any finite or infinite interpolating sequence for uniform algebras.

After this background, we first deal with the connection between interpolating sequences and linear interpolating sequences. A. M. Davie proved that c_0-interpolating sequences for $A(\mathbf{D})$ are c_0-linear interpolating. In Proposition 2.2.2 we prove that (x_n) is c_0-linear interpolating for A if and only if there exists a sequence of Beurling functions $(f_n) \subset A$ for (x_n). Moreover, we will show that c_0-linear interpolating sequences for A are linear interpolating for A^{**} in Proposition 2.2.3.

Next, we deal with dual uniform algebras $A = X^*$. In this context, we prove first that c_0-linear interpolating sequences are linear interpolating in Proposition 2.3.1. Then, we show that c_0-interpolating sequences are, indeed, c_0-linear interpolating, obtaining that c_0-interpolating sequences $(x_n) \subset M_A \cap X$ become linear interpolating. The last part of this section shows that, for $A \subset \ell_\infty(Y)$, under the assumption that limits of any bounded net of functions in A that converges pointwise on Y also belongs to A, then the algebra becomes a dual algebra and the predual X can be chosen to satisfy $(x_n) \subset X$.

An example of a c_0-interpolating sequence for $A_u(B_{c_0})$ which is not c_0-linear interpolating was given by A. M. Davie in [Dav72]. We provide a different approach to his result via composition operators.

Once we have studied some general results on interpolation for general uniform algebras, we continue with the study of interpolating sequences for the particular algebras of analytic functions $H^\infty(B_E)$ and $A_\infty(B_E)$ in the third chapter. We find some sufficient conditions for a sequence $(x_n) \subset B_{E^{**}}$ to be interpolating for $H^\infty(B_E)$ and complete the study done by J. Globevnik about the existence of interpolating sequences for $A_\infty(B_E)$. It turns out that $A_\infty(B_E)$ is separable only if E is finite dimensional.

The study of interpolating sequences for H^∞ arises from the results of L. Carleson [Car58], W. K. Hayman [Hay58] and D. J. Newman [New59]. In particular, from [Hay58] and [New59], the next result follows,

Let $(z_n) \subset \mathbf{D}$ and suppose that $(|z_n|)$ increases exponentially to 1, that is, there exists $0 < c < 1$ such that the following condition holds,

$$\frac{1-|z_{n+1}|}{1-|z_n|} < c.$$

Then, (z_n) is interpolating for $H^\infty(B_E)$.

The main result on interpolation for H^∞ was given by L. Carleson [Car58], who gave a necessary and sufficient condition on the sequence (z_n) to be interpolating for H^∞. This condition involves the pseudohyperbolic distance ρ in the unit disk and states that there is a constant $\delta > 0$ such that

$$\prod_{k\neq j} \rho(z_k, z_j) \geq \delta \qquad \text{for any } j \in \mathbb{N}.$$

We will refer to this condition as the *Carleson condition*. One can try to generalize it to uniform algebras since the notion of pseudohyperbolic distance can be carried over to a uniform algebra. Then, given $(x_n) \subset M_A$, we will say that the sequence (x_n) satisfies the *generalized Carleson condition*.

B. Berndtsson proved [Ber85] that this condition is sufficient for a sequence $(x_n) \subset B_H$ to be interpolating for $H^\infty(B_H)$ when we deal with finite dimensional Hilbert spaces H. Then, B. Berndtsson, S-Y. A. Chang and K-C. Lin [BCL87] extended the result to the finite dimensional space $(\mathbb{C}^n, \|\cdot\|_\infty)$ and proved that the condition is not necessary in this case. P. Galindo, T. W. Gamelin and M. Lindström [GGL08] noticed that the results given by B. Berndtsson could be extended to any infinite Hilbert space.

When we deal with functions defined on the unit ball of a Hilbert space, we provide explicitly the interpolating functions under the assumption of the generalized Carleson condition in Theorem 3.2.10. For this, we adapt some results given by B. Berndtsson and study the automorphisms on B_H for a Hilbert space H. The explicit formula of the pseudohyperbolic distance for Hilbert spaces given by

$$1-\rho(x,y)^2 = \frac{(1-\|x\|^2)(1-\|y\|^2)}{|1-<x,y>|^2}$$

makes this study easier.

On the other hand, when we deal with general Banach spaces, there are no explicit formulas for the pseudohyperbolic distance in general. We prove that the Hayman-Newman condition for the sequence of norms is also sufficient for a sequence $(x_n) \subset B_{E^{**}}$ to be interpolating for $H^\infty(B_E)$ if E is any finite or infinite dimensional Banach space (see Corollary 3.2.16). This is a consequence of a stronger result (see Theorem 3.2.14):

The Carleson condition for the sequence $(\|x_n\|) \subset \mathbf{D}$ is sufficient for (x_n) to be interpolating for $H^\infty(B_E)$.

Actually, the result holds for sequences in $B_{E^{**}}$ thanks to the Davie-Gamelin extension.

In our dealing with the generalized Carleson condition, we reduce the discussion for uniform dual algebras to the case of $H^\infty(B_{c_0})$ in a similar way to what Berndtsson, Chang and Lin did in [BCL87] (see Theorem 3.2.18):

Suppose that any sequence $(x_n) \subset B_{c_0}$ satisfying the generalized Carleson condition is interpolating for $H^\infty(B_{c_0})$ with interpolation constant depending only on δ. Then, for any dual uniform algebra $A = X^*$, all sequences $(x_n) \subset X$ satisfying the generalized Carleson condition are linear interpolating sequences for A with constant of interpolation depending only on δ.

If the assumption of this result is satisfied, then it is clear that the generalized Carleson condition is sufficient for a sequence to be interpolating for $H^\infty(B_E)$ regardless the Banach space E.

When we deal with the algebra $A = A_\infty(B_E)$, the existence of interpolating sequences for A was proved by J. Globevnik [Glo78] for a wide class of infinite-dimensional Banach spaces. Indeed, he proved this result if there exists a sequence $(x_n) \subset S_E$ of strongly exposed points with no cluster points. We show the existence of interpolating sequences for $A_\infty(B_E)$, for any infinite-dimensional Banach space E, characterizing the separability of $A_\infty(B_E)$ in terms of the finite dimension of E.

In connection with the existence of interpolating sequences, we study the separability of $A_u(B_E)$. This algebra is always separable when E is finite dimensional and there are plenty of infinite dimensional spaces such that $A_u(B_E)$ is non separable. One way used to prove this is to show the existence of interpolating sequences for the algebra. Nevertheless, there are also cases such that $A_u(B_E)$ is separable, as $E = c_0$ or the Tsirelson space T^*. In our approach to the separability of $A_u(B_E)$, we show that the algebra is separable if and only if the spectrum M_A is $\tau(A^*, P(E))$-metrizable.

This condition leads us to study the metrizability of bounded subsets of M_A, in particular those of B_E. We provide an example of a complex Banach space E and a bounded subset $L \subset B_E$ which is metrizable for the polynomial topology but $(P(^2E), \|\cdot\|_L)$ is not separable. However, we prove that, under the assumption that L is a closed bounded absolutely convex separable set in E which is metrizable for the polynomial topology, then at least $(E^*, \|\cdot\|_L)$ is separable. We also give an example that the separability of L cannot be removed in this result if the conclusion is to hold.

In chapter 4 we deal with composition operators on $H^\infty(B_E)$. These are transformations C_ϕ defined according to $C_\phi(f) = f \circ \phi$, where ϕ is an analytic self map of B_E. We give a description of the spectra of some of these operators and study the class of Radon-Nikodým composition operators on $H^\infty(B_E)$.

L. Zheng [Zhe02] described the spectrum of some composition operators on H^∞ (see Theorem 4.2.2). Indeed, suppose that $\phi : \mathbf{D} \longrightarrow \mathbf{D}$ is a non constant, analytic self-map, not an automorphism and there exists $a \in \mathbf{D}$ such that $\phi(a) = a$. Then, she proved that either the spectrum $\sigma(C_\phi)$ is $\overline{\mathbf{D}}$ if C_ϕ is non power compact or $\sigma(C_\phi) = \{\phi'(a)^k : k \in \mathbb{N}\} \cup \{0,1\}$ if C_ϕ is power compact. Her results where extended to $H^\infty(B_E)$, E any complex Banach space, for the power compact case in [GGL08]. In this work, the authors also deal with the non power compact case: they extend the result given by L. Zheng to $H^\infty(B_E)$ for the non power compact case when E is a Hilbert space.

It was H. Kamowitz (see [Kam73] and [Kam75]) who seems to be the first who used interpolating sequences to analyze the spectrum of some composition operators. The existence of such suitable interpolating sequences is nowadays usually derived from the following estimate:

$$\text{Given } 0 < r < 1, \text{ there is } \varepsilon > 0 \text{ such that } \frac{1 - |\phi(z)|}{1 - |z|} > 1 + \varepsilon \quad \text{if } |z| > r$$

where ϕ is an analytic self map of $\mathbf{D}$ satisfying $\phi(0) = 0$ and $|\phi'(0)| < 1$. This estimate (see Lemma 7.33 in [CM95]) is typically obtained by using Julia's lemma (see Lemma 2.41 in [CM95]) and angular derivatives.

In their study [GGL08] of the spectrum of composition operators C_ϕ on $H^\infty(B_H)$ for a Hilbert space H, P. Galindo, T. W. Gamelin and M. Lindström obtained a similar estimate to the above one which they called *Julia-type estimate*. Thus we will continue to call Julia-type estimate to the above inequality with the absolute value replaced by the norm. Inspired by [GGL08] and using our interpolation result 3.2.16, we prove the following result for general Banach spaces,

Let $\phi : B_E \longrightarrow B_E$ be an analytic map such that $\phi(0) = 0$, $\|\phi'(0)\| < 1$ and suppose that $\phi(B_E)$ is relatively compact in B_E. If ϕ satisfies a Julia-type estimate and C_ϕ is non power compact, then the spectrum of C_ϕ coincides with the closed unit disk $\overline{\mathbf{D}}$.

The description of the spectrum of non power compact composition operators C_ϕ for the $n-$fold product space $\mathbb{C}^n$ was not yet done because a Julia-type estimate was not yet available. In the present chapter we prove that it holds whenever $E = C_0(X)$. As a consequence, we complete the description of $\sigma(C_\phi)$ in the case of $\mathbb{C}^n$.

Next, we study the class of Radon-Nikodým composition operators from $H^\infty(B_E)$ to $H^\infty(B_F)$. These operators are closely related to the Asplund property, so we study this property for bounded sets of Banach spaces. A subset $D \subset E$ is said to be an *Asplund set* if the space $(E^*, \|\cdot\|_A)$ is separable for any countable set $A \subset D$. The Asplund property is preserved by linear operators and we are interested in extending this result to analytic mappings between Banach spaces. In particular, we prove in Proposition 4.3.13 that, under the assumption that $P_f(^kE)$ is dense in $P(^kE)$, then k-homogeneous polynomials $P : E \longrightarrow F$ preserve Asplund sets and obtain similar results for some analytic mappings.

A linear operator $T : E \longrightarrow F$ is a Radon-Nikodým operator if $T(B_E)$ is a Radon-Nikodým set. It is known (see [Bgi83]) that the linear operator T is Asplund if and only if T^* is Radon-Nikodým. Thus there is a connection between the previous study of Asplund sets and Radon-Nikodým composition operators. Our main result is,

The composition operator $C_\phi : H^\infty(B_F) \longrightarrow H^\infty(B_E)$ is Radon-Nikodým if and only if there exists $0 < r < 1$ such that $\phi(B_E) \subset rB_F$ and $(P(F), \|\cdot\|_A)$ is separable for any countable set $A \subset \phi(B_E)$.

In addition, we show some sufficient conditions for C_ϕ to be Radon-Nikodým.

Chapter 5 deals with properties related to Hankel-type operators. These operators are extensions of the classical Hankel operators on the Hardy space H^2. Let A be a uniform algebra on K. Given $g \in C(K)$, the Hankel-type operator $S_g : A \longrightarrow C(K)/A$ is defined by $S_g(f) = gf + A$.

The concept of tight algebra was introduced by B. Cole and T. Gamelin in [CG82]. An algebra $A \subset C(K)$ is said to be a *tight algebra* on K if the operator S_g is weakly compact for all $g \in C(K)$. Tightness characterizes uniform subalgebras A of $C(K)$ such that $A^{**} + C(K)$ is a closed subalgebra of $C(K)^{**}$. In addition, this property is, roughly speaking, the abstract analogue of the solvability of a certain abstract $\bar{\partial}$-problem with a small gain in smoothness.

In [CG82], the authors proved that $A(\mathbf{D}^n)$ is not tight on its spectrum for $n \geq 2$. We present a new approach to this result extending it to algebras $A_u(B_E)$ for Banach spaces $E = \mathbb{C} \times F$ endowed with the supremum norm.

In addition, we show that $H^\infty(B_E)$ is never tight on its spectrum regardless the Banach space E. For this, we first prove the result for H^∞ using some properties of H^∞ as a closed subalgebra of L^∞ and then we prove the general case $H^\infty(B_E)$ by means of Proposition 5.3.5.

Hankel-type operators are also closely related to the *Dunford-Pettis property* through the so-called Bourgain algebras introduced by J. A. Cima and R. M. Timoney in [CT87]. A function $g \in C(K)$ belongs to the Bourgain algebra A_b of A (resp. A_B) if the Hankel-type operator S_g is completely continuous (resp. S_g^{**} is completely continuous). Bourgain's work [Bou84a] was reformulated in [CT87] by proving that a sufficient condition for a subalgebra A of $C(K)$ to enjoy the Dunford-Pettis property is that $A_b = C(K)$. If $A_B = C(K)$, then the dual space A^* enjoys the Dunford-Pettis property and, hence, so does A itself.

What J. Bourgain did can be restated, using such reformulation, by saying that the Bourgain algebras of $A(B_n)$ are $C(\overline{B}_n)$ and, therefore, $A(B_n)$ and its dual enjoy the Dunford-Pettis property. A proof by induction allowed J. Bourgain to conclude that $A(\mathbf{D}^n)$ and its dual have the Dunford-Pettis property. However, the line of this proof does not allow us to conclude whether the Bourgain algebras of the polydisk algebras are the whole $C(\overline{\mathbf{D}}^n)$. Contrary to the results obtained for $A(B_n)$, we prove that the Bourgain algebras of $A(\mathbf{D}^n)$ as a subspace of $C(\bar{D}^n)$ are themselves.

Chapter 1

Preliminaries

The aim of this chapter is to collect concepts and notation which will be used along this work. We will also include some classical theorems related to Banach spaces and theory of analytic functions.

1.1 Topology and Banach spaces

Let E be a *Banach space*. The Banach spaces considered during this work will be complex Banach spaces unless it is otherwise stated. Its *open unit ball* will be denoted by

$$B_E = \{x \in E : \|x\| < 1\}.$$

Its *unit sphere* and its *closed unit ball* will be denoted by

$$S_E = \{x \in E : \|x\| = 1\} \quad \text{and} \quad \overline{B}_E = \{x \in E : \|x\| \leq 1\}$$

respectively.

Given E and F Banach spaces, a *linear operator* from E into F means a linear and continuous map from E into F. We denote by $L(E,F)$ the Banach space of all the operators from E into F endowed with the norm

$$\|T\| = \sup\{\|T(x)\| : x \in B_E\};$$

when $Y = \mathbb{C}$, we write $E^* = L(E,C)$ to denote the topological dual of E. We will use sometimes the notation of dual pair $< x, x^* >$ to denote $x^*(x)$ for $x \in E$ and $x \in E^*$.

The linear mapping $i : E \longrightarrow E^{**}$, defined by $i(x)(x^*) = x^*(x)$, is an isometry and, therefore, E becomes a closed subspace of E^{**} via i. The Banach space E is said to be *reflexive* if i is also surjective, hence an isometric isomorphism. We will use both, E and $i(E)$, to denote the corresponding subspace of E^{**}.

The *weak topology* on E, denoted by $w(E,E^*)$ (or simply w), is the coarsest topology for which all the elements of E^* are continuous. The *weak* topology* on E^*, that we denote by $w(E^*,E)$ (or simply w^*), is the coarsest topology for which all the elements of $i(E)$ are continuous. Recall that a linear mapping $T : E \longrightarrow F$ is continuous if and only if $T : (E,w) \longrightarrow (F,w)$ is continuous.

Given a topological space (X,τ), the closure of a subset $A \subset X$ is denoted by $\overline{A}^{\tau}$. For a Banach space E and $A \subset E$, $\overline{A}$ will denote the norm closure unless it is otherwise stated.

Mazur's Theorem can be stated as follows.

Theorem 1.1.1 (Mazur)**.** *For any convex set A of a Banach space E, the norm closure $\overline{A}$ equals its $w(E,E^*)-$closure $\overline{A}^{w}$.*

The *absolutely convex hull* of a subset $A \subset E$ is defined by

$$\Gamma(A) = \left\{ \sum_{n=1}^{\infty} t_n x_n : \ x_n \in A, \sum_{n=1}^{\infty} |t_n| \leq 1 \right\}.$$

The *polar* of a set $A \subset E$ is given by

$$A^{\circ} = \{L \in E^* : |L(x)| \leq 1 \text{ for any } x \in A\}.$$

The polar of a set $B \subset E^*$ is given by

$$B^{\circ} = \{x \in E : |L(x)| \leq 1 \text{ for any } L \in B\}.$$

Recall that the *Bipolar Theorem* states that, for $A \subset E$, we have that

$$A^{\circ\circ} = \overline{\Gamma(A)}. \tag{1.1}$$

Given a closed subspace Y of the Banach space X, the quotient space X/Y is endowed with the norm $\|[x]\| = \inf\{\|x-y\| : y \in Y\}$ for $[x] \in X/Y$ and then X/Y becomes a Banach space. The orthogonal $Y^{\perp}$ is the closed subspace of X^* defined by

$$Y^{\perp} = \{x^* \in X^* : x^*(y) = 0 \text{ for any } y \in Y\}.$$

It is well-known that $Y^{\perp}$ is isomorphic to X/Y.

A linear operator $T : E \longrightarrow F$ is *compact* if $T(B_E)$ is relatively compact. We say that $T \in L(E)$ is *power compact* if there exists $n \in \mathbb{N}$ such that the linear operator

$T^n = T \circ \overset{n}{\ldots} \circ T$ is compact. The linear operator T is said to be *weakly compact* if $T(B_E)$ is relatively weakly compact. In case that T transforms weakly null sequences into norm null sequences, it is said that T is *completely continuous*. If $T(E)$ is finite-dimensional, T is said to be a *finite rank operator*. The identity operator on E will be denoted by Id_E.

It is clear that any operator $T : E \longrightarrow E$ which is limit of finite rank operators is compact. Conversely, a Banach space E has the *approximation property* (AP) if for any $\varepsilon > 0$ and any compact subset K of E, there exists a finite rank operator $T : E \longrightarrow E$ such that $\|T(x) - x\| < \varepsilon$ for any $x \in K$.

The following result is well-known:

Theorem 1.1.2. *Let E be a Banach space. The dual space E^* has the approximation property if and only if for every Banach space F, every compact linear mapping T from E into F and every $\varepsilon > 0$ there exists a finite rank operator $T_0 \in L(E,F)$ such that $\| T - T_0 \| \leq \varepsilon$.*

Recall that the *adjoint* of a linear operator $T : E \longrightarrow F$ is the linear operator $T^* : F^* \longrightarrow E^*$ defined by

$$< x, T^*(y^*) > = < T(x), y^* > \qquad \text{for any } x \in E,\ y^* \in F^*.$$

It is well-known that T^* is continous if and only if T is continuous. In addition, $\|T^*\| = \|T\|$. We also have that $T^* : E^* \longrightarrow F^*$ is continuous if and only if $T^* : (E^*, w^*) \longrightarrow (F^*, w^*)$ is continuous.

Recall *Bartle-Graves Selection Theorem* [BG52], which is stated as follows.

Theorem 1.1.3 (Bartle-Graves)**.** *If E and F are Banach spaces and $T : E \longrightarrow F$ is a surjective linear operator, then there exists a continuous function $g : F \longrightarrow E$ such that $T \circ g$ is the identity map on F.*

The well-known *Fredholm Alternative* states the following,

Theorem 1.1.4 (Fredholm Alternative)**.** *Let E be a Banach space, $T : E \longrightarrow E$ a compact operator and $\lambda \neq 0$. Then either $T - \lambda Id$ is both one-to-one and surjective, or it is neither one-to-one nor surjective.*

Compact sets will be assumed to be Hausdorff unless it is otherwise stated and will be denoted by K. The Banach space $C(K)$ is the space of complex valued continuous functions defined on K endowed with the norm of uniform convergence $\|f\| = \sup_{x \in K} |f(x)|$.

For locally compact Hausdorff spaces X, we let $C_0(X)$ denote the space of complex valued continuous functions on X which vanish at infinity, endowed with the supremum norm.

We will denote the space of complex-valued bounded functions on a set Y by $\ell_\infty(Y)$. This space is also endowed with the supremum norm and $(\ell_\infty(Y), \|\cdot\|_\infty)$ becomes a Banach space.

1.2 Geometry of Banach spaces

In this section, we recall some concepts and theorems about geometry of Banach spaces.

Given a Banach space E, an element $x \in S_E$ is said to be an *exposed point* if there exists a continuous linear functional $L \in E^*$ such that $L(x) = \|x\| = 1$ and x is the only point in the unit ball that is mapped to 1. There is a refinement of the concept of exposedness: we call $x \in S_E$ *strongly exposed* if there exists a functional $L \in E^*$ with the properties $L(x) = \|L\| = 1$ and for any sequence (x_n) in X such that $\lim_n L(x_n) = 1$, then $\lim_n x_n = x$ in X. Clearly, a strongly exposed point is exposed.

The weak and weak-star topologies on an infinite dimensional Banach space are never metrizable. It is possible, however, to show that under certain conditions these topologies are metrizable when restricted to bounded sets.

Theorem 1.2.1. *Let E be a Banach space. Then*

a) E is separable if and only if $(B_{E^*}, w(E^*, E))$ is metrizable.
b) If E^* is separable, then $(B_E, w(E, E^*))$ is metrizable.

Now, we recall the *James' reflexivity Theorem*, which characterizes the reflexivity of real and complex Banach spaces E in terms of the attainment of the norm of its functionals.

Theorem 1.2.2 (James). *Let E be a Banach space. Then E is reflexive if and only if any $L \in E^*$ attains its norm on $\overline{B}_E$.*

A Banach space E is said to have the *Dunford-Pettis property* (DPP) if for sequences $(x_n) \subset E$ and $(x_n^*) \subset E^*$, such that $x_n \xrightarrow{w} 0$ and $x_n^* \xrightarrow{w} 0$, we have $x_n^*(x_n) \to 0$. We will study this property in paragraph 5.1.2.

A Banach space E is said to have the *Schur property* if weakly convergent sequences in E are norm convergent. The typical example of a Schur space is the classical sequence space ℓ_1.

1.3 Banach and uniform algebras

The theory of commutative Banach algebras was widely developed by I. M. Gelfand in the 40's. Uniform algebras form a branch of the theory of commutative Banach algebras and they are deeply related to theory of analytic functions. This theory was firstly developed by G. E. Shilov in the early 1950's. Concrete algebras of analytic functions will provide models for the study of uniform algebras and, conversely, the study of uniform algebras will lead us to some properties of analytic functions. In this section we will introduce the concepts of Banach and uniform algebra and will give some background related to them. Further references may be found in [Gam84], [Rud73], [Dal00] and [GK01].

1.3.1 Banach algebras

Recall that a *Banach algebra* $(A, \|.\|)$ is a complex Banach space A which is also an associative algebra, such that the multiplication and the norm are related by the inequality

$$\|f.g\| \leq \|f\|.\|g\| \quad \text{for any } f, g \in A.$$

We will assume, unless it is otherwise stated, that all the Banach algebras we will consider are commutative and have an identity denoted by 1 that satisfies $\|1\| = 1$.

An element $f \in A$ is *invertible* if there exists an element $f^{-1} \in A$ such that $f.f^{-1} = 1$. We denote by $res(f)$ the set of λ such that $\lambda 1 - f$ is invertible. The *spectrum* of $f \in A$ is the set $\sigma(f) = \{\lambda \in \mathbb{C} : \lambda 1 - f \text{ is not invertible}\}$ and the *spectral radius* of f is defined by $\rho(f) = \sup\{|\lambda| : \lambda \in \sigma(f)\}$. It is well-known that for any $f \in A$, the spectrum $\sigma(f)$ is non-void and it is a compact set of $\mathbb{C}$. Moreover, if $\lambda \in \sigma(f)$, then $|\lambda| \leq \|f\|$.

It is also well-known that the maximal ideals of a Banach algebra A are closed. The set of maximal ideals of A is called the *maximal ideal space* or spectrum of A, and it is denoted by M_A. If J is a maximal ideal of A, then A/J is isometrically isomorphic to the field $\mathbb{C}$.

Consider the set

$$Hom(A) = \{\phi : A \to \mathbb{C} : \phi \text{ is a non-zero homomorphism of algebras }\}$$

and define $A_\phi = ker\phi$. The mapping $\mathscr{T} : Hom(A) \longrightarrow M_A$ defined by $\mathscr{T}(\phi) = A_\phi$ is a bijective correspondence, so we will identify each maximal ideal in M_A with the complex-valued homomorphism that it determines. To define a topology for M_A, recall that $\|\phi\| = 1 = \phi(1)$ for any $\phi \in Hom(A)$. Therefore, M_A is a subset of the unit sphere S_{A^*} and can be endowed with the $w(A^*,A)$-topology that inherits from A^*. This topology will be called the *Gelfand topology* of M_A and M_A becomes a compact Hausdorff space when endowed with it.

The *Gelfand transform* of $f \in A$ is the complex-valued function $\widehat{f}$ on M_A defined by

$$\begin{array}{rccc} \widehat{f}: & M_A & \longrightarrow & \mathbb{C} \\ & \phi & \mapsto & \phi(f). \end{array}$$

The function $\widehat{f}$ is the restriction to M_A of the complex-valued operator δ_f on A^* defined by $\delta_f(\phi) = \phi(f)$, so δ_f is $w(A^*,A)$-continuous and hence $\widehat{f} \in C(M_A)$. We will also use the term *Gelfand transform* to refer to the mapping $f \mapsto \widehat{f}$ from A into $\widehat{A} = \{\hat{x} : x \in A\}$. This is a homomorphism of algebras which is continuous since $\|\widehat{x}\|_\infty \leq \|x\|$ and $\widehat{A}$ can be considered as a subalgebra of $C(M_A)$. The Gelfand topology turns out to be the coarsest topology which makes continuous all the functions in $\widehat{A}$.

A Banach space E is said to be a *dual space* if there exists a Banach space X such that $X^* = E$. A *dual algebra* A is a dual space which is a Banach algebra.

1.3.2 Uniform algebras

Now we introduce some concepts of *uniform algebras* and discuss the Gelfand transform in this case.

Recall that a set of functions B defined on a set S is said to separate points of S if for any $x \neq y$ in S we have that there exists $f \in B$ such that $f(x) \neq f(y)$.

Definition 1.3.1. *A Banach algebra A is said to be a uniform algebra if $\|x^2\| = \|x\|^2$ for any $x \in A$.*

The classical examples of uniform algebras are the closed subalgebras A of $C(K)$ which separates points of K and contain the constant function 1. Then, K is homeomorphic to a subset of M_A. Conversely, the following result holds,

Proposition 1.3.2. *A Banach algebra A is uniform if and only if its Gelfand transform is an isometry, that is, $\|\widehat{x}\|_\infty = \|x\|$ for any $x \in A$.*

We also have the following corollary,

Corollary 1.3.3. *A Banach algebra A is uniform if and only if A is isometrically isomorphic to a closed Banach subalgebra of a $C(K)$ space which separates points of K and contains the constant function* 1.

Recall that a subset E of M_A is a *boundary* for A if for any function $\widehat{f} \in \widehat{A}$ we have that

$$\|f\| = \sup_{\phi \in E} |\phi(f)|.$$

It is known that the intersection of all closed boundaries for A is a boundary for A. This intersection is called the *Shilov boundary* of A and it is denoted by ∂_A.

The following result is also well-known (see [Bas77] or [CCG89] for a proof).

Theorem 1.3.4. *Let A be an infinite-dimensional uniform algebra. Then A contains a copy of c_0.*

Recall that the bidual A^{**} of a uniform algebra A is also a uniform algebra endowed with the Arens product (see [Are51b]). The evaluation functionals at points of M_A extend uniquely to be weak-star continuous multiplicative functionals on A^{**}, so we can regard M_A as a subset of $M_{A^{**}}$. Further results can be found in [Gam73] and [DH79].

Finally, we recall the concept of peak point. Let K be a metrizable compact space and A a uniform algebra on K. A point $x \in K$ is a *peak point* for A if there exists a function $f \in A$ such that $f(x) = 1$ while $|f(x)| < 1$ for $y \in X$, $y \neq x$. A point $x \in K$ is said to be a *strong peak point* for A if there is a function $f \in A$ such that $f(x) = 1$ while for any $r > 0$ there exists $\varepsilon > 0$ such that $|f(y)| < 1 - \varepsilon$ for $d(x,y) > r$.

1.4 Complex Analysis

1.4.1 Algebras of analytic functions on D

Recall that, for an open set U of the complex plane $\mathbb{C}$, the set $H(U)$ denotes the set of analytic complex-valued functions defined on U.

We will denote by $\mathbf{D}$ the *open unit disk* of the complex plane. Its closure will be denoted by $\overline{\mathbf{D}}$ and the *unit circle* will be denoted by $\partial\mathbf{D}$. Recall that H^∞ is the space of all bounded analytic functions on $\mathbf{D}$ and the *disk algebra* is defined by

$$A(\mathbf{D}) = \left\{ f : \overline{D} \longrightarrow \mathbb{C} : f \in H(\mathbf{D}) \text{ and continuous on } \overline{D} \right\}.$$

It is clear that these sets of functions become Banach algebras endowed with the supremum norm on the unit disk. Further references about these algebras can be found in [Hof62].

Recall the *Rudin-Carleson Theorem* [Gam84],

Theorem 1.4.1 (Rudin-Carleson)**.** *Let K be a closed set of Lebesgue measure zero on the unit circle $\partial\mathbf{D}$, and let $f \in C(K)$. Then, there exists a function $g \in A(\mathbf{D})$ whose restriction to K is f, that is, $g|_K = f$ and $\|g\|_{\mathbf{D}} = \|f\|_K$.*

Now we recall some basic results about *Blaschke products*.

Definition 1.4.2. *Let (α_n) be a sequence of non-zero complex numbers in $\mathbf{D}$ such that the product $\prod_{n=1}^{\infty} |\alpha_n|^{p_n}$ is convergent. A Blaschke product is a function $B : \mathbf{D} \longrightarrow \mathbb{C}$ defined by*

$$B(z) = z^p \prod_{n=1}^{\infty} \left[\frac{\bar{\alpha}_n}{|\alpha_n|} \frac{\alpha_n - z}{1 - \bar{\alpha}_n z} \right]^{p_n}$$

where $p, p_1, p_2, \dots$ are non-negative integers.

Blaschke products satisfy $B \in H^\infty$ and $|B(z)| \leq 1$ for any $z \in \mathbf{D}$. Moreover, $z = 0$ is a zero of order p and, for $z \neq 0$, we have

$$B(z) = 0 \text{ if and only if } z = \alpha_n.$$

It is also well-known that the formal product which defines the Blaschke product converges for $z \in \partial\mathbf{D}$ if and only if z is a non cluster point of the sequence (α_n).

It is clear that $A(\mathbf{D}) \subset H^\infty$. To show that the converse is false, it is sufficient to consider any Blaschke product. For instance, let B be the Blaschke product defined by the sequence $(\alpha_n) = (1 - 1/n^2)_{n=1}^{\infty}$. The function B belongs to H^∞ but it is easy to show that it cannot be extended continuously to $z = 1$.

Recall that the *Hardy space H^2* is defined by

$$H^2 = \left\{ f \in H(\mathbf{D}) : \sup_{0<r<1} \left(\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \right) < \infty \right\}.$$

This set becomes a Banach space endowed with the norm given by

$$\|f\|_{H_2} = \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \right)^{\frac{1}{2}}.$$

The *Cauchy projection* $\mathscr{C} : L^2 \longrightarrow H^2$ is given by

$$\mathscr{C}(f)(z) = \int_{\partial \mathbf{D}} \frac{f(\xi)}{(1 - z\bar{\xi})} d\xi \text{ for any } z \in \mathbf{D}. \tag{1.2}$$

Further results about H^2 and the Cauchy projection can be found in [Woj91].

1.4.2 The polydisk and ball algebras

The study of analytic functions on the unit disk can be carried over to an analogous situation in several variables, namely to *polydisks* or finite-dimensional balls. These are the unit ball of the Banach space $(\mathbb{C}^n, \|\cdot\|_\infty)$ and the Hilbert space $(\mathbb{C}^n, \|\cdot\|_2)$ respectively. It turns out that there are some analogies with classical complex analysis but also many differences.

We will denote the n-finite dimensional polydisk by $\mathbf{D}^n$ and the n-dimensional ball by $\mathbf{B}_n$. Therefore, this will allow us to define two extensions of the classical disk algebra.

Let $z = (z_1, \ldots, z_n)$ denote the variables of $z \in \mathbb{C}^n$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ a multi-index in $\mathbb{Z}_+^n = \mathbb{Z}_+ \times \overset{n}{\ldots} \times \mathbb{Z}_+$. The expression z^α denotes the monomial $z_1^{\alpha_1} \ldots z_n^{\alpha_n}$ of degree $|\alpha| := \sum_{j=1}^n |\alpha_j|$. Let $\mathbf{B}$ the n-finite dimensional polydisk or ball. A function $f : \mathbf{B} \longrightarrow \mathbb{C}$ is said to be *analytic* if it is the sum of the *multiple power series*

$$f(z) = \sum_{\alpha \in \mathbb{Z}_+^n} a_\alpha z^\alpha \text{ for any } z \in \mathbf{B},$$

where the series converges uniformly on compact sets of $\mathbf{B}$.

The n-dimensional *polydisk algebra* $A(\mathbf{D}^n)$ is defined by

$$A(\mathbf{D}^n) = \left\{ f : \mathbf{D}^n \longrightarrow \mathbb{C} : f \text{ is analytic and extends continuously to } \bar{\mathbf{D}}^n \right\}.$$

Given $f \in A(\mathbf{D}^n)$, the function $|f|$ attains its maximum on $\overline{\mathbf{D}}^n$ since f is continuous on the compact set $\overline{\mathbf{D}}^n$. We endow this algebra with the norm given by

$$\|f\|_\infty := \sup_{z \in \bar{\mathbf{D}}^n} |f(z)|,$$

and then $(A(\mathbf{D}^n), \|.\|_\infty))$ becomes a uniform algebra with spectrum $\overline{\mathbf{D}}^n$. It is clear, by the Maximum Modulus Theorem, that $\|f\| = \sup_{\|z\|=1} |f(z)|$.

The n-dimensional *ball algebra* is defined by

$$A(\mathbf{B}_n) = \left\{ f : \mathbf{B}_n \longrightarrow \mathbb{C} : f \text{ is analytic and extends continuously to } \bar{\mathbf{B}}_n \right\}.$$

As above, $(A(\mathbf{B}_n), \|.\|_\infty)$ also becomes a uniform algebra.

1.5 Polynomials on Banach spaces

This section is devoted to polynomials on Banach spaces; multilinear mappings, tensor products, restrictions to finite dimensional spaces and differential calculus may all be used to define polynomials over infinite dimensional spaces. We adopt an approach to them using multilinear mappings. Polynomials are used to eventually define analytic functions by means of series expansions analogously to the one dimensional case. We will also recall the polynomial topologies $\tau(E,P(^nE))$ and $\tau(E,P(E))$, which will be used in Chapter 3. References about polynomials on Banach spaces can be found in [Din99], [Muj86] y [Gam94].

1.5.1 Polynomials on Banach spaces

Let E and F be complex Banach spaces and $n \in \mathbb{N}$. An *$n-$homogeneous (continuous) polynomial* $P : E \longrightarrow F$ is the restriction of a (continuous) $n-$linear map $L : E \times \overset{n)}{\cdots} \times E \longrightarrow F$ to its diagonal, that is,

$$P(x) = L(x, \overset{n)}{\ldots}, x) \qquad \text{for any } x \in E.$$

The space of n-homogeneous continuous polynomials from E into F is denoted by $P(^nE,F)$, which becomes a Banach space endowed with the norm

$$\|P\| := \sup\{\|P(x)\| : \|x\| \leq 1\}.$$

By $P_0(E,F)$ we denote the set of F-valued constant functions defined on E. If $F = \mathbb{C}$, we will denote the space of complex valued $n-$homogeneous continuous polynomials on E by $P(^nE)$ and the set of complex constant functions on E by $P_0(E)$.

It is well-known that a polynomial P is continuous if and only if P is locally bounded at some point of E (see [Din99]). From now on, all polynomials will be supposed to be continuous.

A polynomial $P \in P(^nE,F)$ is said to be a *finite type polynomial* if there exist finite sequences $(\phi_j)_{j=1}^p \subset E^*$ and $(y_j)_{j=1}^p \subset F$ such that

$$P(x) = \sum_{j=1}^{p} \phi_j^n(x) y_j \quad \text{for any } x \in E.$$

We denote the set of finite type $n-$homogeneous polynomials by $P_f(^nE,F)$. If $F = \mathbb{C}$, the finite type polynomials are simply linear combinations of $n-$th powers of functionals on E. We denote this set by $P_f(^nE)$. This set contains the products $\phi_1 \cdot \ldots \cdot \phi_n$, for $\phi_j \in E^*$ and $j = 1,\ldots,n$.

The closure of $P_f(^nE,F)$ in $P(^nE,F)$ is called the set of *approximable polynomials* and will be denoted by $P_A(^nE,F)$ and $P_A(^nE)$ respectively.

The sets of n-homogeneous polynomials which are $w(E,E^*)-$continuous on bounded sets are denoted by $P_w(^nE,F)$ and $P_w(^nE)$ respectively. The following results are well-known [Din99],

Proposition 1.5.1. *Let E be a complex Banach space. Then,*

a) We have that $P_A(^nE,F) = P_w(^nE,F)$ if and only if E^* has the approximation property.
b) If E has the Dunford-Pettis property, then any polynomial $P \in P(E)$ is weakly sequentially continuous.
c) The space ℓ_1 is not contained in E if and only if all the weakly sequentially continuous polynomials are weakly continuous on bounded sets.

1.5.2 The polynomial topologies

Now we define the *polynomial topology* $\tau(E,P(^nE))$ for a Banach space E. This is the coarsest topology on E which makes continuous all the polynomials in $P(^nE)$. The $\tau(E,P(E))$-topology is the coarsest topology on E which makes continuous all the polynomials in $P(E)$.

A basis of $\tau(E,P(E))$-neighbourhoods of 0 is given by the open sets

$$\{x \in E : |P(x)| < 1,\ P \in F\},$$

where F is a finite set of polynomials in $P(E)$.

1.6 Algebras of analytic functions on Banach spaces

We give here some background related to infinite dimensional holomorphy, which is the study of analytic functions on complex Banach spaces.

1.6.1 Analytic functions on Banach spaces

Analytic functions on Banach spaces can be introduced by means of Gâteaux and Fréchet derivatives or polynomial expansions. We have decided to introduce them using polynomial expansions. Further results on infinite dimensional holomorphy can be found in [Din99] and [Muj86].

Definition 1.6.1. *Let E and F be complex Banach spaces and U an open set of E. A mapping $f : U \longrightarrow F$ is said to be* analytic *on U if for any $a \in U$ there exists a sequence $(P_n)_{n\geq 0}$ of $n-$homogeneous polynomials and a neighbourhood V of a such that*

$$f(x) = \sum_{n=0}^{\infty} P_n(x-a) \text{ for any } x \in V$$

where the series converges uniformly on V.

We denote by $H(U,F)$ the set of analytic F-valued functions on U. The function f is said to be *entire* if $U = E$. If $F = \mathbb{C}$, the set of analytic functions on U is denoted by $H(U)$. The polynomials of $P(E,F)$ are entire functions, that is, $P(E,F) \subset H(E,F)$.

The τ_c-topology for $H(U,F)$ is the topology given by the convergence of functions in $H(U,F)$ on compact sets. We recall now *Montel's theorem* for analytic functions on Banach spaces,

Theorem 1.6.2. *Let U be an open set of the complex Banach space E. A subset $\mathscr{B}$ of $H(U)$ is τ_c-relatively compact if, and only if, $\mathscr{B}$ is τ_c-bounded.*

1.6.2 Algebras of analytic functions on Banach spaces

There are many algebras of analytic functions defined on open subsets of Banach spaces. We will be mainly interested in extensions of H^∞ and the disk algebra $A(\mathbf{D})$. These extensions have been studied in [ACG91], [AG89], [AG88], [Glo79] and [CG86] among other references.

Definition 1.6.3. *Let E be a complex Banach space and F a complex Banach algebra. We define the following set,*

$$H^\infty(B_E,F) = \{f : B_E \longrightarrow F : f \text{ analytic and bounded } \}$$

Whenever E is a finite dimensional Banach space, analytic functions on B_E which are continuously extendible to S_E are also uniformly continuous since $\overline{B}_E$ is compact. Nevertheless, if E is infinite dimensional, then $\overline{B}_E$ is non compact and the continuity on $\overline{B}_E$ is not sufficient to guarantee the uniform continuity. Therefore, the disk algebra $A(\mathbf{D})$ can be generalized in two ways. We assume that F is a uniform algebra. The first generalization is given by the algebra $A_\infty(B_E,F)$, defined by

$$A_\infty(B_E,F) = \{f \in H^\infty(B_E,F) : f \text{ extends continuously to } \overline{B}_E\}.$$

The other way to extend the disk algebra is $A_u(B_E,F)$, defined by

$$A_u(B_E,F) = \{f : B_E \longrightarrow F : f \text{ analytic and uniformly continuous }\}.$$

We have that $H^\infty(B_E,F)$, $A_\infty(B_E,F)$ and $A_u(B_E,F)$ are uniform algebras endowed with the supremum norm

$$\|f\|_\infty = \sup_{x\in B_E} \|f(x)\|_F.$$

In particular, if $F = \mathbb{C}$, we denote these algebras by $H^\infty(B_E)$, $A_\infty(B_E)$ and $A_u(B_E)$ respectively.

It is easy to prove that if $f \in A_u(B_E)$, then f is bounded and it extends to a uniformly continuous function $f : \overline{B}_E \longrightarrow \mathbb{C}$. In consequence, $A_u(B_E) \subset A_\infty(B_E)$. On the other hand, we have that $A_\infty(B_E) \subsetneq H^\infty(B_E)$ as in the one-dimensional case. Therefore, we obtain

$$A_u(B_E) \subset A_\infty(B_E) \subsetneq H^\infty(B_E).$$

If E is finite dimensional, the algebras $A_u(B_E)$ and $A_\infty(B_E)$ are the same. If E is infinite dimensional, we have that $A_u(B_E) \subsetneq A_\infty(B_E)$ by Theorem 12.2 in [ACG91], so the three of them are different.

The following result will be used in the next chapters.

Theorem 1.6.4. *Let E be a Banach space. Then the set of polynomials $P(E)$ is a dense set in $A_u(B_E)$.*

By Proposition 1.5.1, one concludes that if E has the Dunford-Pettis property and ℓ_1 is not contained in E, then $P_f(E)$ is dense in $A_u(B_E)$ if and only if E^* has the approximation property.

1.6.3 The Davie-Gamelin extension

The *Davie-Gamelin extension* allows us to extend analytic functions on B_E to $B_{E^{**}}$ when we deal with functions in $H^\infty(B_E)$ and $A_u(B_E)$. R. Arens [Are51a] extended bilinear maps $A : X \times Y \longrightarrow Z$ to bilinear maps $\widetilde{A} : X^{**} \times Y^{**} \longrightarrow Z^{**}$ preserving its norm using Goldstine's Theorem. This technique can be used to extend 2-homogeneous polynomials $P \in P(^2E)$ to $\widetilde{P} \in P(^2E^{**})$. R. M. Aron and P. Berner generalized this result [AB78] proving that for any $P \in P(^nE)$, there exists an extension $\widetilde{P} \in P(^nE^{**})$ such that $\|P\| = \|\widetilde{P}\|$. A. M. Davie and T. W. Gamelin sharpened this result [DG89] proving the following theorem,

Theorem 1.6.5. *Let E be a complex Banach space and $f \in H^\infty(B_E)$. There exists an extension $\widetilde{f} \in H^\infty(B_{E^{**}})$ such that $\|f\|_{B_E} = \|\widetilde{f}\|_{B_{E^{**}}}$. Furthermore, the operator $f \to \widetilde{f}$ is linear, continuous and multiplicative, that is, $\widetilde{fg} = \tilde{f}\tilde{g}$ for all $f, g \in H^\infty(B_E)$.*

We have that the Davie-Gamelin extension of a function in $A_u(B_E)$ belongs to $A_u(B_{E^{**}})$, extending the results given in Theorem 1.6.5,

Corollary 1.6.6. *Let f be a function in $A_u(B_E)$. Then the Davie-Gamelin extension $\widetilde{f}$ belongs to $A_u(B_{E^{**}})$.*

Proof. Consider the operator $T : H^\infty(B_E) \longrightarrow H^\infty(B_{E^{**}})$ such that $T(f)$ is the Davie-Gamelin extension of f. Then we have that

$$T(A_u(B_E)) = T(\overline{P(E)}) \subseteq \overline{T(P(E))} \subseteq \overline{P(E^{**})} = A_u(B_{E^{**}})$$

and we obtain the result. □

1.6.4 Spectrum of algebras of analytic functions

We present in this section some results about the spectrum of the algebras $H^\infty(B_E)$, $A_\infty(B_E)$ and $A_u(B_E)$. Let A be one of these algebras and $x^{**} \in B_{E^{**}}$. The homomorphism of algebras $\delta_{x^{**}} : A \longrightarrow \mathbb{C}$ given by

$$\delta_{x^{**}}(f) = \widetilde{f}(x^{**})$$

is clearly well-defined and continuous. In addition, we also have that the mapping $\delta : B_{E^{**}} \longrightarrow M_A$ is injective since, for $x^{**} \neq y^{**}$ in E^{**}, there exists $x^* \in E^*$ such that $< x^*, x^{**} > \neq < x^*, y^{**} >$ and then, $\delta_{x^{**}}(x^*) \neq \delta_{y^{**}}(x^*)$. Therefore, identifying elements x^{**} of $B_{E^{**}}$ with the corresponding homomorphism $\delta_{x^{**}}$, we obtain the following result,

Proposition 1.6.7. *Let E be a complex Banach space and A one of the algebras $H^\infty(B_E)$, $A_\infty(B_E)$ or $A_u(B_E)$. Then,*

$$B_{E^{**}} \subset M_A.$$

In addition, if $A = A_u(B_E)$, then $\overline{B}_{E^{**}} \subset M_A$.

We present now a proposition which characterizes the algebra $A_u(B_E)$ in terms of its spectrum for Banach spaces E which enjoy the approximation property. This is based on the following lemma, which proves the existence of homomorphisms $\phi \in M_A$ which are not given by evaluations $\delta_{x^{**}}$ for any $x^{**} \in B_{E^{**}}$. The necessary and sufficient condition for E is the existence of a continuous polynomial which is not $w(E, E^*)$-continuous. It is an adaptation of Proposition 1.5. in [AGGM96].

Lemma 1.6.8. *Let E be a complex Banach space. Suppose that there exists an $n-$homogeneous polynomial $P \in P(^nE)$ which is not $w(E,E^*)$-continuous in bounded sets. Then there exists a homomorphism of algebras $\phi : A_u(B_E) \longrightarrow \mathbb{C}$ which does not belong to $\overline{B}_{E^{**}}$.*

From Lemma 1.6.8, we obtain the following proposition [GL01]

Proposition 1.6.9. *Suppose that E^* has the approximation property and let M_A be the spectrum of the algebra $A = A_u(B_E)$. Then,*

$$M_A = \overline{B}_{E^{**}}$$

if and only if the set of finite type polynomials $P_f(E)$ is dense in $A_u(B_E)$.

1.6.5 $H^\infty(U)$ as a dual space

Let U be an open set in a Banach space E. We will denote by $H^\infty(U)$ the algebra of bounded complex-valued analytic functions on U. In this paragraph we recall that $H^\infty(U)$ is a dual space and give some results related.

Consider $H^\infty(U)$ endowed with the τ_c- topology, that is, the topology of uniform convergence on compact sets of U. It is well-known that the closed unit ball of $H^\infty(U)$, which we will denote by $\overline{B}_U$, is τ_c- relatively compact by Montel's Theorem 1.6.2. Define the set

$$G^\infty(U) = \left\{ u \in H^\infty(U)^* : u|_{\overline{B}_U} \text{ is } \tau_c - \text{ continuous } \right\}.$$

J. Mujica proved in [Muj91a] the following result,

Proposition 1.6.10. *The set $G^\infty(U)$ is a closed subspace of $H^\infty(U)^*$ and the linear operator $T : H^\infty(U) \longrightarrow G^\infty(U)^*$ given by $T(f)(u) = u(f)$, is an isometric isomorphism.*

In consequence, we have that $G^\infty(U)^* = H^\infty(U)$, so $H^\infty(U)$ is a dual algebra. In particular, $H^\infty(B_E)$ is a dual algebra for any Banach space E.

Denote by B_G the unit ball of $G^\infty(B_E)$. In [Muj91a], the following result is also noticed:

$$\overline{B}_G = \overline{\Gamma}\{\delta_x : x \in B_E\}. \tag{1.3}$$

1.7 The pseudohyperbolic metric

Let E be a complex Banach space and $\mathscr{D}$ a domain of E, that is, an open and connected subset of E. We say that $f : \mathscr{D} \longrightarrow \mathscr{D}$ is an automorphism of $\mathscr{D}$ if f

is bijective, analytic and f^{-1} is analytic. It is well-known [GR84] that the last condition is redundant if E is finite dimensional. When we deal with infinite-dimensional Banach spaces, this is an open question.

Definition 1.7.1. *Let $\rho : \mathbf{D}\times\mathbf{D} \longrightarrow \mathbb{R}$ be the map defined by*

$$\rho(z,w) = \left|\frac{z-w}{1-z\overline{w}}\right|.$$

Then $(\mathbf{D},\rho)$ is a complete metric space which induces the usual topology of $\mathbf{D}$. The distance ρ will be called the pseudohyperbolic distance.

It is clear, by the Schwarz-Pick Lemma, that analytic mappings $f : \mathbf{D} \longrightarrow \mathbf{D}$ are contractive for the pseudohyperbolic distance, that is,

$$\rho(f(z),f(w)) \leq \rho(z,w) \quad \text{for any } z,w \in \mathbf{D}. \tag{1.4}$$

Moreover, the equality is satisfied if and only if f belongs to $Aut(\mathbf{D})$.

The following lemma summarizes well-known results on the pseudohyperbolic distance. They are simple calculations.

Lemma 1.7.2. *We have the following statements,*

$$a) \qquad \rho(|z|,|w|) \leq \rho(z,w) \quad \textit{for all } z,w \in \mathbf{D}, \tag{1.5}$$

$$b) \qquad 1-\left|\frac{z-w}{1-\bar{z}w}\right|^2 = \frac{(1-|z|^2)(1-|w|^2)}{|1-\bar{z}w|^2} \quad \textit{for all } z,w \in \mathbf{D}. \tag{1.6}$$

From b) we conclude

$$1-\rho(z,w)^2 = \frac{(1-|z|^2)(1-|w|^2)}{|1-\bar{z}w|^2} \quad \text{for all } z,w \in \mathbf{D}.$$

Since this metric is an isometry for $f \in Aut(\mathbf{D})$, we have

$$\rho(z,w) = \sup\{\rho(f(z),f(w)) : f \in H^\infty, \|f\| \leq 1\}. \tag{1.7}$$

Then, the notion of pseudohyperbolic distance can be carried over to uniform algebras as follows,

Definition 1.7.3. *Let A be a uniform algebra, M_A its spectrum and $x,y \in M_A$. The pseudohyperbolic distance $\rho_A(x,y)$ is defined by*

$$\rho_A(x,y) = \sup\{\rho(f(x),f(y)) : f \in A, \|f\| \leq 1\}.$$

From now on, we will not specify the uniform algebra A in $\rho_A(x,y)$ unless it is necessary.

If we deal with $E = C_0(X)$ and $A = H^\infty(B_E)$, in [AGL03] we find an explicit expression for the pseudohyperbolic distance given by

$$\rho(x,y) = \sup_{t\in X} \left| \frac{x(t)-y(t)}{1-x(t)\overline{y(t)}} \right| \qquad \text{for all } x,y \in B_E. \tag{1.8}$$

1.8 Composition Operators

Now, we introduce some background related to composition operators. Further results can be found in [CM95] and [Sha93].

Let E and F be complex Banach spaces and U and V open sets in E and F respectively.

Definition 1.8.1. *Let $\phi : U \longrightarrow V$ be an analytic function and let $\mathscr{A}(U)$, $\mathscr{A}(V)$ be spaces of analytic functions defined on U and V respectively. The* composition operator $C_\phi : \mathscr{A}(V) \longrightarrow \mathscr{A}(U)$ *of* symbol ϕ *is defined by*

$$C_\phi(f) = f \circ \varphi$$

whenever $f \circ \phi \in \mathscr{A}(U)$ for any $f \in \mathscr{A}(V)$.

It is easy to check that any symbol $\phi : B_E \longrightarrow B_F$ gives rise to a composition operator $C_\phi : H^\infty(B_F) \longrightarrow H^\infty(B_E)$. On the other hand, if we deal with $A_u(B_E)$ and $A_u(B_F)$, the corresponding composition operator with symbol $\phi : B_E \longrightarrow B_F$ will not be well-defined in general as the following example [AGL97] shows,

Example 1.8.2. *Let $\phi : B_{\ell_2} \longrightarrow B_{\ell_2}$ be the analytic map given by $\phi(x) = (x_n^n)_{n=1}^\infty$. The composition operator $C_\phi : A_u(B_{\ell_2}) \longrightarrow A_u(B_{\ell_2})$ of symbol ϕ is not well-defined.*

Proof. Consider $\varepsilon = 1/4$. For any $\delta > 0$, consider $r = 1-\delta/2$ and $s_n = 1 - \frac{1}{n}$ for any $n \in \mathbb{N}$. Then, there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ we have $\|re_n - s_ne_n\|_2 = |r - s_n| < \delta$ but

$$\|\varphi(re_n) - \phi(s_ne_n)\|_2 = |r^n - s_n^n| \longrightarrow \frac{1}{e} \text{ for } n \to \infty,$$

so the function ϕ cannot be uniformly continuous. Now, taking the function $f \in A_u(B_{\ell_2})$ given by $f(x) = \sum_{n=1}^\infty x_n^2$, we have that $f \circ \phi \notin A_u(B_{\ell_2})$.

This situation can be overcome by considering the symbol $\varphi : B_E \longrightarrow B_F$ to be uniformly continuous. Then, the composition operator $C_\phi : A_u(B_F) \longrightarrow A_u(B_E)$ is clearly well-defined. Nevertheless, when we deal with $E = c_0$, this assumption is unnecessary since composition operators are always well-defined (see also [AGL97]).

Now we recall some results which characterize some classes of composition operators. The characterization of compact composition operators can be found in [AGL97]. The other ones are in [GLR99]. Recall that a set $A \subset E$ is said to be a *Dunford-Pettis set* if for any sequences $(x_n^*) \subset E^*$ such that $x_n^* \xrightarrow{w} 0$, and $(x_n) \subset A$, we have that $x_n^*(x_n) \to 0$ when $n \to \infty$.

Theorem 1.8.3. *Let $C_\varphi : H^\infty(B_F) \longrightarrow H^\infty(B_E)$ be a composition operator of symbol ϕ. Then,*

- *a) C_ϕ is compact if and only if $\phi(B_E)$ is a relatively compact set in B_F and there exists $0 < r < 1$ such that $\phi(B_E) \subset rB_F$, that is, $\phi(B_E)$ is* strictly inside *B_F.*
- *b) If there exists $0 < r < 1$ such that $\phi(B_E) \subset rB_F$ and $\phi(B_E)$ is a $\sigma(F, P(F))$-relatively compact set, then C_ϕ is weakly compact. The converse is satisfied if F has the approximation property .*
- *c) C_ϕ is completely continuous if and only if $\phi(B_E)$ is a Dunford-Pettis set in F and there exists $0 < r < 1$ such that $\phi(B_E) \subset rB_F$.*

Chapter 2

Interpolation

Interpolation theory has been developed since the earliest times to the present date. There are many connections between the results obtained in different ages, thereby putting the techniques currently used in many fields of science. In this chapter, we will study interpolating sequences for uniform algebras and their connection with linear interpolation when we deal with dual uniform algebras. In addition, we give a new approach to prove the existence of interpolating sequences which are not linear interpolating by using results on composition operators. Some results will allow us to study linear interpolating sequences for the particular algebras of analytic functions $H^\infty(B_E)$ and $A_\infty(B_E)$ in Chapter 3.

2.1 Background

We begin this chapter by giving some background related to interpolating sequences for uniform algebras. In this section we give the concepts of interpolating sequence and linear interpolating sequence for a uniform algebra. We also introduce c_0-interpolating sequences and the constant of interpolation. In addition, some results related to interpolating sequences are quoted.

2.1.1 Interpolating sequences

Let A be a uniform algebra and (x_n) a sequence of elements in M_A. Recall that the Gelfand transform $\widehat{f}$ is continuous on the spectrum M_A and this is a compact space, so then the sequence $(\widehat{f}(x_n))$ is bounded. Consider the restriction map $R : A \longrightarrow \ell_\infty$ defined by

$$R(f) = (\widehat{f}(x_n)).$$

It is clear that R is well-defined. Moreover, R is linear and continuous since $\|R(f)\| = \sup_{n=1}^{\infty} |(\widehat{f}(x_n))| \leq \|\widehat{f}\|_\infty$.

Definition 2.1.1. *Let A be a uniform algebra and (x_n) a sequence in M_A. If there is a map $T : \ell_\infty \to A$ such that $R \circ T = id_{\ell_\infty}$, then (x_n) is called an* interpolating sequence *for A. If the map $T : \ell_\infty \to A$ is a linear operator, then (x_n) is called a* linear interpolating sequence *for A.*

If there exists an interpolating sequence for A, then A is not separable since the map R is onto. It is also clear that a sequence $(x_n) \subset M_A$ is interpolating for A if and only if for any bounded sequence $(\alpha_n)_{n=1}^{\infty} \subset \mathbb{C}$, there exists $f \in A$ such that $\widehat{f}(x_n) = \alpha_n$. If we deal with a finite sequence $\{x_1, \dots, x_N\}$, we say that the sequence is interpolating for A if for any $(\alpha_1, \dots, \alpha_N) \in \mathbb{C}^N$, there exists $f \in A$ such that $\widehat{f}(x_n) = \alpha_n$ for any $n = 1, \dots, N$.

In order to control the interpolating functions, the constant of interpolation is introduced.

Definition 2.1.2. *Let (x_n) be an interpolating sequence for A. For any $\alpha = (\alpha_j) \in \ell_\infty$ consider the set $M_\alpha = \inf\left\{\|f\|_\infty : \widehat{f}(x_j) = \alpha_j,\ j \in \mathbb{N}, f \in A\right\}$. The* constant of interpolation *for (x_n) is defined by*

$$M = \sup\{M_\alpha : \alpha \in \ell_\infty, \|\alpha\|_\infty \leq 1\}.$$

Analogously, the notion of *c_0-(linear) interpolating sequence* is defined by simply replacing ℓ_∞ by c_0 in Definition 2.1.1. The constant of interpolation for c_0-interpolating sequences is defined by

$$M = \sup\{M_\alpha : \alpha \in c_0, \|\alpha\|_\infty \leq 1\}.$$

We will say that M is a constant of interpolation for (x_n) if it is an upper bound for the constant of interpolation of (x_n).

As we stated in paragraph 1.3.2, the bidual of a uniform algebra A is also a uniform algebra and we can regard M_A as a subset of $M_{A^{**}}$. We say that a sequence $(x_n) \subset M_A$ is interpolating for A^{**} if (x_n) is interpolating for A^{**} as a subset of $M_{A^{**}}$. See also [GGL04].

2.1.2 Results on uniform algebras

We recall several results about interpolating sequences for uniform algebras. They will pave our way to study linear interpolating sequences, in particular when we deal with dual uniform algebras.

Our starting point for this research is a result of P. Beurling (see [Gar81] and [Car62]):

Theorem 2.1.3. *Let $(z_j) \subset \mathbf{D}$ be an interpolating sequence for H^∞ and let M be its constant of interpolation. Then, there exists a sequence of functions $(f_j) \subset H^\infty$ such that*

$$f_k(z_j) = \delta_{kj} \quad \text{for } k, j \in \mathbb{N} \quad \text{and} \quad \sum_{j=1}^{\infty} |f_j(z)| \leq M \quad \text{for any } z \in \mathbf{D}.$$

In order to extend the results of this theorem, we introduce the following definition,

Definition 2.1.4. *Let A be a uniform algebra, a sequence $(x_j) \subset M_A$ and a sequence of functions $(f_k) \subset A$. We say that (f_k) is a* sequence of Beurling functions for (x_n) *if there is $M > 0$ such that*

$$\widehat{f}_k(x_j) = \delta_{kj} \quad \text{for any } k, j \in \mathbb{N} \quad \text{and} \quad \sum_{j=1}^{\infty} |\widehat{f}_j(x)| \leq M \quad \text{for any } x \in M_A.$$

N. Th. Varopoulos proved (see [Var71] and [Gar81]) a general result on uniform algebras replacing the constant of interpolation M by a worse one.

Theorem 2.1.5. *Let A be a uniform algebra on a compact set K. Let $\{x_1, x_2, \ldots, x_n\}$ be a finite sequence in K and let M be the constant of interpolation of this sequence.*
For any $\varepsilon > 0$, there exist functions $f_1, f_2, \ldots, f_n$ in A such that $\widehat{f}_k(x_j) = \delta_{kj}$ for any $k, j \in \mathbb{N}$ and such that

$$\sup_{x \in K} \sum_{j=1}^{n} |\widehat{f}_j(x)| \leq M^2 + \varepsilon.$$

P. Galindo, T.W. Gamelin and M. Lindström improved this result in [GGL04],

Theorem 2.1.6. *Let A be a uniform algebra and $(x_n) \subset M_A$. Let $M \geq 1$ such that for each finite collection $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of complex numbers in $\partial\mathbf{D}$, there exists $f \in A$ satisfying $\widehat{f}(x_j) = \alpha_j$ for any $1 \leq j \leq n$ and $\|f\| \leq M$. Then, there is a sequence of functions $(f_n)_{n=1}^{\infty} \subset A^{**}$ such that*

$$\widehat{f}_k(x_j) = \delta_{kj} \quad \text{for any } k, j \in \mathbb{N} \quad \text{and} \quad \sum_{j=1}^{\infty} |\widehat{f}_j(x)| \leq M^2 \quad \text{for any } x \in M_{A^{**}}.$$

2.2 Linear interpolation

In this section we will study the connection between interpolating sequences, linear interpolating sequences and c_0-(linear) interpolating sequences.

First, we recall a result which shows the connection between c_0-interpolating sequences and c_0-linear interpolating sequences when we deal with the disk algebra $A(\mathbf{D})$. It follows from Theorem 3.1 in [Dav72],

Theorem 2.2.1. *Let (z_n) be a c_0-interpolating sequence for the disk algebra $A(\mathbf{D})$. Then, (z_n) is a c_0-linear interpolating sequence for $A(\mathbf{D})$.*

The following result shows the equivalence for a sequence $(x_n) \subset M_A$ to be c_0-linear interpolating and the existence of Beurling functions in A.

Proposition 2.2.2. *Let A be a uniform algebra and $(x_n) \subset M_A$. Consider the following statements:*

a) (x_n) is a linear interpolating sequence for A.
b) There exists a sequence $(f_n) \subset A$ of Beurling functions for (x_n).
c) (x_n) is a c_0-linear interpolating sequence for A.

Then b) and c) are equivalent and a) implies both b) and c).

Proof. It is clear that $a) \Rightarrow c)$ by considering the restriction mapping $T|_{c_0} : c_0 \longrightarrow A$.

$c) \Rightarrow b)$ Set $f_n := T(e_n) \in A$ so that $\widehat{f_n}(x_k) = \delta_{nk}$. Now we consider the adjoint $T^* : A^* \to c_0^*$, which satisfies that

$$T^*(\delta_x)(e_n) = \delta_x(T(e_n)) = \delta_x(f_n) = \widehat{f_n}(x).$$

Since T^* is continuous, we have that $M = \|T^*\|$ is finite. Therefore, by the duality given by $c_0^* = \ell_1$, we have that

$$\|T^*(\delta_x)\| = \sum_{n=1}^{\infty} |T^*(\delta_x)(e_n)| = \sum_{n=1}^{\infty} |\widehat{f_n}(x)|$$

and we obtain

$$M = \|T^*\| = \sup_{x \in S_{A^*}} \|T^*(\delta_x)\|_{c_0^*} \geq \sup_{x \in M_A} \|T^*(\delta_x)\|_{c_0^*} =$$

$$\sup_{x \in M_A} \sum_{n=1}^{\infty} |T^*(\delta_x)(e_n)| = \sup_{x \in M_A} \sum_{n=1}^{\infty} |\widehat{f_n}(x)|$$

so there exists $M > 0$ such that $\sup_{x \in M_A} \sum_{n=1}^{\infty} |\widehat{f_n}(x)| \leq M$ and condition b) is satisfied.

$b) \Rightarrow c)$ Set $\alpha = (\alpha_n) \in c_0$. Since $\sup_{x \in M_A} \sum_{n=1}^{\infty} |\widehat{f_n}(x)| \leq M$ for any $x \in M_A$, we have that $\sum_{n=1}^{\infty} \alpha_n \widehat{f_n}(x)$ is also defined for any $x \in M_A$. Moreover, $\sum_{n=1}^{k} \alpha_n f_n \in A$ converges uniformly on M_A to the function $\sum_{n=1}^{\infty} \alpha_n f_n$ since

$$\| \sum_{n=1}^{\infty} \alpha_n f_n - \sum_{n=1}^{k} \alpha_n f_n \|_\infty = \sup_{x \in M_A} | \sum_{n=k+1}^{\infty} \alpha_n \widehat{f_n}(x) | \leq$$

$$\sup_{n\geq k+1}|\alpha_n|\sup_{x\in M_A}\sum_{n=k+1}^{\infty}|\widehat{f_n}(x)|\leq M\sup_{n\geq k+1}|\alpha_n|\to 0.$$

Therefore, $\sum_{n=1}^{\infty}\alpha_n f_n$ belongs to A since A is a Banach space. Let $R : A \longrightarrow c_0$ be the restriction map. We define its right inverse $T : c_0 \longrightarrow A$ by

$$T((\alpha_n)) := \sum_{n=1}^{\infty}\alpha_n f_n.$$

This is a well-defined linear operator and we have that

$$R\circ T(\alpha) = R(\sum_{n=1}^{\infty}\alpha_n f_n) = \left(\sum_{n=1}^{\infty}\alpha_n \widehat{f_n}(x_j)\right)_{j=1}^{\infty} = (\alpha_n) = \alpha$$

for all $\alpha \in c_0$. Therefore, (x_n) is a c_0-interpolating sequence for A. □

It is not difficult to prove that, in general, (c) does not imply (a). A counterexample can be found just considering the disk algebra $A(\mathbf{D})$ and a convergent sequence in the unit circle. By the Rudin-Carleson Theorem 1.4.1, this is is a c_0-interpolating sequence for $A(\mathbf{D})$ and further linear interpolating by Theorem 2.2.1. It is clear that this sequence is not interpolating for $A(\mathbf{D})$ since $A(\mathbf{D})$ is separable. But, though there are c_0-interpolating sequences which are not interpolating sequences, c_0-linear interpolating sequences are always linear interpolating when we deal with the bidual A^{**}. Indeed, we have

Proposition 2.2.3. *Let A be a uniform algebra and $(x_n) \subset M_A$. If (x_n) is a c_0-linear interpolating sequence for A, then (x_n) is linear interpolating for A^{**}.*

Proof. Since (x_n) is c_0-linear interpolating, then there exists a linear operator $T : c_0 \to A$ such that $R\circ T = Id_{c_0}$. Then, for $\alpha = (\alpha_k) \in c_0$, we have that

$$T(\alpha)(x_n) = \alpha_n \quad \text{for any n } \in \mathbb{N}.$$

Consider the second adjoint $T^{**} : \ell_\infty \longrightarrow A^{**}$ and fix $\alpha \in \ell_\infty$. We have that $T^*(x_n) \in \ell_1$ and, considering the sections $\alpha^k = (\alpha_1,\ldots,\alpha_k,0,\ldots) \in c_0$, we have that the sequence (α^k) $w(\ell_\infty,\ell_1)-$converges to α. Therefore, it follows that

$$< T^{**}(\alpha),x_n >=< \alpha,T^*(x_n) >=< \lim_k(\alpha_1,\ldots,\alpha_k,0,\ldots),T^*(x_n) >=$$

$$\lim_k < (\alpha_1,\ldots,\alpha_k,0,\ldots),T^*(x_n) >= \lim_k < T((\alpha_1,\ldots,\alpha_k,0,\ldots)),x_n >= \alpha_n.$$

Hence, for any $\alpha \in \ell_\infty$, we have that $T^{**}(\alpha)(x_n) = \alpha_n$ for any $n \in \mathbb{N}$. □

2.3 Results for dual uniform algebras

In this section, we deal with dual uniform algebras $A = X^*$. We begin by proving that c_0-linear interpolating sequences for A are also linear interpolating when we deal with sequences $(x_n) \subset M_A \cap X$. Then we prove that the result remains true if we consider c_0-interpolating sequences.

Proposition 2.3.1. *Let A be a dual uniform algebra $A = X^*$ for some Banach space X, and consider $(x_n) \subset M_A \cap X$ a c_0-linear interpolating sequence for A. Then, (x_n) is also linear interpolating for A.*

Proof. Since (x_n) is c_0-linear interpolating, by Proposition 2.2.2 we can choose a sequence of Beurling functions $(f_n) \subset A$ for (x_n), that is, $\widehat{f_n}(x_k) = \delta_{nk}$ and $\sup_{x\in M_A} \sum_{n=1}^{\infty} |\widehat{f_n}(x)| \leq M$ for a constant $M > 0$.

Fix a sequence $\alpha = (\alpha_j) \in \ell_\infty$. We have, for any $x \in M_A$,

$$\sum_{n=1}^{\infty} |\alpha_n \widehat{f_n}(x)| \leq \|\alpha\|_\infty \sum_{n=1}^{\infty} |\widehat{f_n}(x)| \leq M\|\alpha\|_\infty,$$

so the series converges for any $x \in M_A$. Moreover, for $u \in A^*$ we have that

$$u(\sum_{n=1}^{k} \alpha_n f_n) \leq \|u\| \| \sum_{n=1}^{k} \alpha_n f_n\|_\infty = \|u\| \sup_{x\in M_A} |\sum_{n=1}^{k} \alpha_n \widehat{f_n}(x)| \leq$$

$$\|u\| \|\alpha\|_\infty \sup_{x\in M_A} \sum_{n=1}^{k} |\widehat{f_n}(x)| \leq M\|u\| \|\alpha\|_\infty.$$

This shows that $\sum_{n=1}^{\infty} \alpha_n f_n$ is a $w(A, A^*)-$Cauchy series in A and, then, a $w(A, X)-$Cauchy series, hence convergent by the $w(A, X)-$compactness of the ball in A of radius $M\|\alpha\|_\infty$. In addition, we have that

$$\| \sum_{n=1}^{\infty} \alpha_n f_n\| \leq M\|\alpha\|_\infty.$$

Define the map $T : \ell_\infty \longrightarrow A$ by

$$T((\alpha_j)) := \sum_{j=1}^{\infty} \alpha_j f_j.$$

The map T is clearly a linear operator and, since each x_k belongs to X, we obtain that

$$\left(\widehat{\sum_{j=1}^{\infty} \alpha_j f_j}\right)(x_k) = \sum_{j=1}^{\infty} \alpha_j \widehat{f_j}(x_k) = \alpha_k.$$

Thus (x_n) is linear interpolating for A. □

From Proposition 2.3.1 we conclude the following,

Corollary 2.3.2. *Let A be a dual uniform algebra $A = X^*$ and $(x_n) \subset M_A \cap X$. If there exists a sequence $(f_k) \subset A$ of Beurling functions for (x_n), then the sequence (x_n) is linear interpolating for A.*

Theorem 2.3.3. *Let A be a dual uniform algebra $A = X^*$ and consider a sequence $(x_n) \subset M_A \cap X$. The following statements are equivalent:*

i) Every finite subset of (x_n) is interpolating for A and there exists a constant of interpolation independent of the number of interpolated terms.

ii) (x_n) is c_0-interpolating for A.

iii) (x_n) is linear interpolating for A.

Proof. It is clear that *iii*) $\Rightarrow$ *ii*).

ii) $\Rightarrow$ *i*) Let (x_n) be a c_0-interpolating sequence. Then, there is a map $T : c_0 \to A$ such that $R \circ T = id_{c_0}$. Consider the set $B = R^{-1}(c_0)$. Since R is continuous, we have that B is a closed subspace of A and, therefore, it is a Banach subspace of A. Moreover, the mapping $R_{|_B} : B \longrightarrow c_0$ is surjective since for all $\alpha \in c_0$ we have that $T(\alpha) \in B$ and $R(T(\alpha)) = \alpha$. In consequence, $R_{|_B}$ is an open mapping and the quotient mapping

$$\widetilde{R_{|_B}} : B/\ker\left(R_{|_B}\right) \longrightarrow c_0$$

has a continuous linear inverse. Consider $M = \|\widetilde{R_{|_B}}^{-1}\| > 0$. We have that

$$\sup_{\|\alpha\|_\infty \leq 1} \{\inf\{\|f\|_\infty : f \in A, (\widehat{f}(x_n))_n = \alpha\}\} \leq$$

$$\sup_{\|\alpha\|_\infty \leq 1} \{\inf\{\|f\|_\infty : f \in B, (\widehat{f}(x_n))_n = \alpha\}\} = M.$$

Now, for every finite subset $\{x_1, \ldots, x_n\}$, it is clear that the constant of interpolation

$$M_n := \sup_{\|(\eta_j)\|_\infty \leq 1} \{\inf\{\|f\|_\infty : f \in A, \widehat{f}(x_j) = \eta_j,\ j = 1, 2, \ldots, n\}\}$$

satisfies the inequality $M_n \leq M$.

i) $\Rightarrow$ *iii*) We will use Theorem 2.1.5 and a normal families reasoning. In addition, we will use some ideas from Theorem 2.1.6. Let (ε_n) be a null sequence of positive numbers. By Theorem 2.1.5, we have that for each ε_n there are functions $f_1^n, \ldots, f_n^n$ in A such that $\widehat{f_j^n}(x_k) = \delta_{jk}$ for $j,k = 1,\ldots,n$ and such that

$$\sup_{x\in M_A} \sum_{j=1}^{n} |\widehat{f_j^n}(x)| \leq M^2 + \varepsilon_n.$$

Since $A = X^*$, its unit ball is a $w(A,X)-$relatively compact set, thus from the sequence $(f_j^n)_{n\geq j}$ we may obtain a net f_j^α $w(A,X)-$convergent to an element $F_j \in A$. Since $x_k \in X$, it follows that

$$\widehat{F_j}(x_k) = \lim_\alpha \widehat{f_j^\alpha}(x_k) = \delta_{jk} \quad \text{for any } j,k \in \mathbb{N}.$$

Fix $m \geq 1$ and let $a_1,\ldots,a_m$ be complex numbers of unit modulus. For any $n \geq m$ and all $x \in M_A$ we have

$$|\sum_{j=1}^{m} a_j \widehat{f_j^n}(x)| \leq \sum_{j=1}^{m} |\widehat{f_j^n}(x)| \leq \sum_{j=1}^{n} |\widehat{f_j^n}(x)| \leq M^2 + \varepsilon_n.$$

From this, we have for any $n \geq m$, $\|\sum_{j=1}^m a_j f_j^n\|_\infty \leq M^2 + \varepsilon_n$. Thus for all $u \in B_X$, we have $| < u, \sum_{j=1}^m a_j f_j^n > | \leq M^2 + \varepsilon_n$, and by passing to the $w(A,X)-$limit, we obtain $| < u, \sum_{j=1}^m a_j F_j > | \leq M^2$, hence $\|\sum_{j=1}^m a_j F_j\|_\infty \leq M^2$. Therefore, we obtain

$$|\sum_{j=1}^{m} a_j \widehat{F_j}(x)| \leq M^2 \quad \text{for any } x \in M_A.$$

This is true in particular for $a_j = |\widehat{F_j}(x)|/\widehat{F_j}(x)$ if $F_j(x) \neq 0$ and, therefore, we obtain

$$\sum_{j=1}^{m} |\widehat{F_j}(x)| \leq M^2 \quad \text{for all } x \in M_A.$$

By Proposition 2.2.2, we obtain that (x_n) is a c_0-linear interpolating sequence for A and the proof is completed by applying Proposition 2.3.1. □

The following corollary proves that, under some assumptions, uniform algebras A are dual algebras $A = X^*$ and we can choose the predual X to contain the interpolating sequence (x_n).

Recall that the set of complex-valued bounded functions on Y is denoted by $\ell_\infty(Y)$, which is the dual space of $\ell_1(Y)$. If A is a subalgebra of $\ell_\infty(Y)$, we consider

$$A^\perp = \{x \in \ell_\infty(Y)^* : < f, x >= 0 \ \text{ for all } f \in A\},$$

so

$$\ell_1(Y) \cap A^\perp = \{x \in \ell_1(Y) :< x, f >= 0 \text{ for all } f \in A\}.$$

We define X to be the quotient space

$$X = \frac{\ell_1(Y)}{\ell_1(Y) \cap A^\perp}.$$

If A is a weak* closed subspace of $\ell_\infty(Y)$, we have that the space X^* is isomorphic to $\left(\ell_1(Y_1) \cap A^\perp\right)^\perp$ since the dual of a quotient space E/F is isomorphic to the orthogonal $F^\perp$ if F is a weak* closed subspace of E. Moreover,

$$\left(\ell_1(Y) \cap A^\perp\right)^\perp = \left\{f \in \ell_\infty(Y) : f(x) = 0 \text{ for any } x \in \ell_1(Y) \cap A^\perp\right\} =$$

$$\ell_\infty(Y) \cap A^{\perp\perp}.$$

Since A is a weak* closed subspace of $\ell_\infty(Y)$, we have that $A^{\perp\perp} = A$ and, therefore, $A = X^*$.

Corollary 2.3.4. *Let A be a closed subalgebra of $\ell_\infty(Y)$ for some set Y whose points are separated by A. Suppose that the limit of any bounded net of functions in A that converges pointwise on Y also belongs to A. If (x_n) is a c_0-interpolating sequence for A, then it is linear interpolating for A.*

Proof. Set $Y_1 = Y \cup \{x_n : n \in \mathbb{N}\}$. Clearly, $Y \subset M_A$, so we have that

$$Y \subset Y_1 \subset M_A$$

and A is also a closed subalgebra of $\ell_\infty(Y_1)$. Since Y_1 satisfies the same assumptions as Y, the condition on pointwise bounded limits guarantees that A is a weak* closed subspace of $\ell_\infty(Y_1)$. Thus, as we have mentioned above, such an algebra A is the dual of the Banach space

$$X := \frac{\ell_1(Y_1)}{\ell_1(Y_1) \cap A^\perp}$$

and every $y \in Y_1$ is identified with the characteristic function δ_y. Therefore, $(x_n) \subset X$. Now, it suffices to apply Theorem 2.3.3. □

J. Mujica proved that if $(x_n) \subset U$ is an interpolating sequence for $H^\infty(U)$, then it is also linear interpolating [Muj91b]; this is a particular case of Theorem 2.3.3 since $(x_n) \subset G^\infty(U)$, the predual space of $H^\infty(U)$ found by J. Mujica in [Muj91a].

This result is extended by Corollary 2.3.4 to any c_0-interpolating sequence since $H^\infty(U)$ is a closed subalgebra of $\ell_\infty(U)$ fulfilling the assumptions by Montel's theorem 1.6.2.

On the other hand, there are examples of c_0-interpolating sequences which are not c_0-linear interpolating. Indeed, A. M. Davie proved [Dav72] that, when we deal with the algebra $A_u(2B_{c_0})$ on its spectrum $M_A = 2\overline{B}_{\ell_\infty}$, there exists an example of a c_0-interpolating sequence which is not c_0-linear interpolating. We provide a somehow different proof of this result as an application of results on composition operators.

Theorem 2.3.5. *Let A be the algebra $A_u(2B_{c_0})$. There exists a c_0-interpolating sequence for A which does not admit linear interpolating subsequences.*

Proof. Let $\{f^j\}$ be a dense sequence in the unit ball of c_0, chosen in c_{00}, that is, $f^j(n) = 0$ for n large enough depending on j. Define $x^i \in B_{\ell_\infty}$ by $x^i(j) = f^j(i)$. We know, by Theorem 1.6.9, that the spectrum of $A_u(2B_{c_0})$ is given by

$$M_A = 2\overline{B}_{\ell_\infty} = \{(z_n) : |z_n| \leq 2\}.$$

It is clear that each x^i belongs to the spectrum and the sequence (x^i) converges to 0 there since the Gelfand topology coincides with the pointwise topology and $x^i(j) = f^j(i) \to 0$ when $i \to \infty$.

Consider the restriction map $R : A_u(2B_{c_0}) \longrightarrow c$ defined by $R(f) = (\widehat{f}(x^i))_{i=1}^\infty$. For any $j \in \mathbb{N}$, let $z_j \in A_u(2B_{c_0})$ be the coordinate functions defined by $z_j(x) = x_j$. We have that $z_j(x^i) = x^i(j) = f^j(i)$, so then R maps the unit ball of $A_u(2B_{c_0})$ onto a dense set of B_{c_0}. In consequence, the mapping $R : A_u(2B_{c_0}) \to c$ is onto by the open mapping Theorem and therefore, by the Bartle-Graves Theorem 1.1.3, there exists a map $T : c \longrightarrow A_u(2B_{c_0})$ such that $R \circ T = Id_c$. By taking the restriction map $T : c_0 \to A_u(2B_{c_0})$, we obtain that (x^i) is a c_0-interpolating sequence.

We show that (x^i) has no linear interpolating subsequences. Consider the natural embedding $\imath : B_{c_0} \longrightarrow 2B_{c_0}$ and the composition operator

$$C_\imath : H^\infty(2B_{c_0}) \longrightarrow H^\infty(B_{c_0})$$

defined by $C_\imath(f) = f|_{B_{c_0}}$. This operator is completely continuous according to Theorem 1.8.3. Observe that the restriction $C_\imath : A_u(2B_{c_0}) \longrightarrow A_u(B_{c_0})$ is still completely continuous and the adjoint $C_\imath{}^*$ restricted to the spectrum $\overline{B}_{\ell_\infty}$ is the canonical embedding

$$\overline{B}_{\ell_\infty} \longrightarrow 2\overline{B}_{\ell_\infty}.$$

Indeed, if $\delta_x \in \overline{B}_{\ell_\infty}$, the homomorphism $C_\iota^*(\delta_x)$ coincides with δ_x since both coincide on the linear functionals on c_0 (i.e., on ℓ_1) and consequently on the dense subspace of finite type polynomials $P_f(c_0)$.

Suppose that (x^i) has a c_0-linear interpolating subsequence. Without loss of generality, we can assume that (x^i) itself is c_0-linear interpolating. Then there exist a linear operator $T : c_0 \to A_u(2B_{c_0})$ and a sequence $(F_k) \subset A_u(2B_{c_0})$ such that $\widehat{F_k}(x^i) = \delta_{ki}$ and further,

$$\sum_{j=1}^{\infty} |\widehat{F_j}(x)| \leq M \quad \text{for all } x \in \overline{B}_{\ell_\infty}.$$

That means that the series $\sum_{j=1}^{\infty} F_j$ is weakly Cauchy, so the series $\sum_{j=1}^{\infty} C_\iota(F_j) = \sum_{j=1}^{\infty} F_j|_{B_{c_0}}$ is a Cauchy series in $A_u(B_{c_0})$. Therefore, $(F_k|_{B_{c_0}})_k$ is a null sequence there. However, this is not possible since

$$\|F_k|_{B_{c_0}}\| = \|C_\iota(F_k)\| \geq |< x^k, C_\iota(F_k) >| = |< C_\iota^*(x^k), F_k >| = |\widehat{F_k}(x^k)| = 1.$$

□

Chapter 3

Interpolation for $H^\infty(B_E)$. Separability of $A_\infty(B_E)$ and $A_u(B_E)$

This chapter is concerned with the study of interpolating sequences for the algebras of analytic functions defined in 1.6.3. First, we aim to study sufficient conditions on a sequence (x_n) to be interpolating for $H^\infty(B_E)$. At this point, we will study the extension of the classical Carleson and Hayman-Newman Theorems on interpolating sequences for H^∞. In particular, we prove the sufficiency of the Hayman-Newman condition on a sequence (x_n) to be interpolating for $H^\infty(B_E)$ and the sufficiency of the Carleson condition on $(\|x_n\|)$ for the sequence (x_n) to be interpolating for $H^\infty(B_E)$. In addition, when we deal with Hilbert spaces, we provide explicitly the interpolating functions.

The existence of interpolating sequences for $A_\infty(B_E)$ was proved by J. Globevnik [Glo78] for a big class of infinite-dimensional Banach spaces. We prove that this fact can be extended to any infinite-dimensional Banach space, solving open questions asked in [Glo78] and characterizing the separability of $A_\infty(B_E)$ in terms of the finite dimension of E.

Finally, we study some conditions related to the separability of the algebra $A_u(B_E)$.

3.1 Background

We begin with some background which includes some results related to the pseudohyperbolic distance and the classical theorems of interpolation for $H^\infty(B_E)$.

3.1.1 The pseudohyperbolic metric

In this paragraph, we present some results related to the pseudohyperbolic metric. The following lemma summarizes several calculations which will be very useful.

Lemma 3.1.1. *We have the following statements:*

$$a) \qquad \rho(a,c) \geq \rho(b,c) \quad \text{for real numbers } 0 \leq a \leq b \leq c < 1. \tag{3.1}$$

$$b) \qquad 1 - x \leq -\log x \quad \text{for } 0 < x \leq 1. \tag{3.2}$$

$$c) \qquad \Re e \left[\frac{1+\alpha z}{1-\alpha z}\right] = \frac{1-|\alpha|^2|z|^2}{|1-\alpha z|^2} \quad \text{for any } \alpha \in \overline{\mathbf{D}}, z \in \mathbf{D}. \tag{3.3}$$

Proof. a) Let $0 \leq a \leq b \leq c < 1$ be real numbers. We have that

$$\rho(a,c) = \frac{c-a}{1-ac} \geq \frac{c-b}{1-bc} = \rho(b,c)$$

if and only if $(c-a)(1-bc) \geq (c-b)(1-ac)$, which is equivalent to inequality $c-a-bc^2+abc \geq c-b-ac^2+abc$. This happens if and only if $b(1-c^2) \geq a(1-c^2)$ and this condition is equivalent to $b \geq a$ since $1-c^2 > 0$.

b) Consider the function $f(x) = 1-x+\log x$ defined for $x \in (0,1]$. Then, the derivative is given by $f'(x) = -1+1/x$ and, therefore, $f'(x) \geq 0$ for any $x \in (0,1]$. Hence, the function f is increasing in $(0,1]$ and, since $f(1) = 0$, we conclude that $f(x) \leq 0$ for $x \in (0,1]$ and we obtain the inequality.

c) Let $\alpha \in \overline{\mathbf{D}}$ and $z \in \mathbf{D}$. Since $\Re e\, w = (w+\overline{w})/2$ for any $w \in \mathbb{C}$, we have that

$$\Re e \left[\frac{1+\alpha z}{1-\alpha z}\right] = \frac{1}{2}\left[\frac{1+\alpha z}{1-\alpha z} + \overline{\frac{1+\alpha z}{1-\alpha z}}\right] =$$

$$\frac{1}{2}\left[\frac{1-\overline{\alpha z}+\alpha z-\alpha^2|z|^2+1+\overline{\alpha z}-\alpha z-|\alpha|^2|z|^2}{|1-\alpha z|^2}\right] = \frac{1-|\alpha|^2|z|^2}{|1-\alpha z|^2}$$

□

The following result will prove that the norm is a contractive function when we deal with the pseudohyperbolic distance on $A = H^\infty(B_E)$. Recall (see paragraph 1.6.4) that elements $x, y \in B_E$ can be also seen as elements of M_A. Therefore, when we deal with $\rho(\|x\|,\|y\|)$, we mean the pseudohyperbolic distance on $\mathbf{D}$ and the norm is calculated for x and y as elements of the Banach space E. However, the expression $\rho(x,y)$ denotes the pseudohyperbolic distance on $A = H^\infty(B_E)$, that is, x and y are considered as elements of M_A.

Proposition 3.1.2. *Let E be a complex Banach space, $A = H^\infty(B_E)$ and $x^{**}, y^{**} \in B_{E^{**}}$. Then,*

$$\rho(\|x^{**}\|, \|y^{**}\|) \leq \rho(x^{**}, y^{**}).$$

Proof. If $\rho(\|x^{**}\|, \|y^{**}\|) = 0$, the result is clear. So then we can suppose that $\|y^{**}\| < \|x^{**}\|$ without loss of generality. Since $x^{**} \in B_{E^{**}}$, there exists a sequence $(f_n) \subset E^* \subset A$, such that $\|f_n\| = 1$ and $\lim_n |f_n(x^{**})| = \|x^{**}\|$. It is clear that

$$\rho(f_n(x^{**}), f_n(y^{**})) \leq \rho(x^{**}, y^{**})$$

and, using 1.5, we obtain that

$$\rho(|f_n(x^{**})|, |f_n(y^{**})|) \leq \rho(x^{**}, y^{**}).$$

Since $\|y^{**}\| < \|x^{**}\|$, we can suppose that $\|y^{**}\| < |f_n(x^{**})|$ for all n. In consequence, we have that $|f_n(y^{**})| \leq \|y^{**}\| < |f_n(x^{**})|$ and we can apply 3.1 to obtain

$$\rho(|f_n(x^{**})|, \|y^{**}\|) \leq \rho(|f_n(x^{**})|, |f_n(y^{**})|).$$

Therefore, we obtain that $\rho(|f_n(x^{**})|, \|y^{**}\|) \leq \rho(x^{**}, y^{**})$ and, taking limits when $n \to \infty$, we get

$$\rho(\|x^{**}\|, \|y^{**}\|) \leq \rho(x^{**}, y^{**}).$$

□

It is clear that Proposition 3.1.2 can be extended to other algebras of functions. In particular, we can consider elements $x^{**}, y^{**} \in \overline{B}_{E^{**}}$ in the algebra $A_u(B_E)$.

3.1.2 The classical interpolating theorems in H^∞

L. Carleson [Car58], W.K. Hayman [Hay58] and D.J. Newman [New59] studied sufficient conditions for sequences $(x_n) \subset \mathbf{D}$ to be interpolating for H^∞ at the end of the 50's. In this paragraph we recall their results.

The works of W.K. Hayman and D.J. Newman are deeply related and it is easy to conclude the following result [Hof62],

Theorem 3.1.3 (Hayman-Newman)**.** *Let (z_n) be a sequence in $\mathbf{D}$. A sufficient condition for (z_n) to be interpolating in H^∞ is the existence of $0 < c < 1$ such that*

$$\frac{1 - |z_{n+1}|}{1 - |z_n|} < c. \tag{3.4}$$

A sequence which satisfies condition 3.4 is said to increase exponentially to the unit circle. It is easy to show [Hof62] that this condition is also necessary if $z_n \geq 0$ for all $n \in \mathbb{N}$ and (z_n) is increasing. In addition,

Corollary 3.1.4. *Let $(z_n) \subset \mathbf{D}$ be a sequence such that $\lim_{n\to\infty} |z_n| = 1$. Then, there exists a subsequence $(z_{n_k})_k$ which is interpolating for H^∞.*

The main result of D.J. Newman in the study of interpolating sequences for H^∞ is the following,

Theorem 3.1.5 (Newman)**.** *A sequence $(z_n) \subset \mathbf{D}$ is interpolating for H^∞ if and only if the following two conditions are satisfied,*

$$\sum_{k=1}^{\infty} |f(z_k)|(1-|z_k|) < \infty \quad \textit{for any } f \in H^1 \quad \textit{and} \tag{3.5}$$

$$\textit{there exists } \delta > 0 \textit{ such that } \prod_{k\neq j} \rho(z_k, z_j) > \delta \quad \textit{for any } j \in \mathbb{N}. \tag{3.6}$$

L. Carleson improved Newman's result by showing that condition 3.5 could be removed, that is, condition 3.6 is sufficient for a sequence $(x_n) \subset \mathbf{D}$ to be interpolating for H^∞. From now on, we will refer to this condition as *Carleson condition*.

Theorem 3.1.6 (Carleson Interpolation Theorem)**.** *A sequence (z_n) in $\mathbf{D}$ is interpolating for H^∞ if and only if the Carleson condition 3.6 is satisfied.*

3.2 Interpolating Sequences for $H^\infty(B_E)$

In the spirit of paragraph 3.1.2 we focus on interpolating sequences for $A = H^\infty(B_E)$. Recall that $B_{E^{**}} \subset M_A$ by 1.6.7. In this section we will study sufficient conditions for a sequence $(x_n) \subset B_{E^{**}}$ to be interpolating for $H^\infty(B_E)$. In some particular cases, we will provide explicitly the interpolating functions.

Since we deal with Banach spaces E and sequences $(x_n) \subset B_E$, the natural extension of the Hayman-Newman condition is to consider the sequence $(\|x_n\|)$ increasing exponentially to 1.

To simplify some further statements, we introduce the following definition,

Definition 3.2.1. *Let A be a uniform algebra and (x_n) a sequence in M_A. The sequence (x_n) is said to satisfy the* generalized Carleson condition *if there exists $\delta > 0$ such that*

$$\prod_{k \neq j} \rho(x_k, x_j) \geq \delta \quad \text{for all } j \in \mathbb{N}. \tag{3.7}$$

Several questions arise. On one hand, we wonder if the Hayman-Newman condition for the norms and the generalized Carleson condition are sufficient for a sequence (x_n) to be interpolating for $H^\infty(B_E)$. On the other hand, B. Berndtsson, S-Y. A. Chang and K-C. Lin [BCL87] showed that the generalized Carleson condition is not necessary when we deal with $H^\infty(\mathbf{D}^2)$.

In order to find sufficient conditions for a sequence to be interpolating for $H^\infty(B_E)$, B. Berndtsson [Ber85] studied the generalized Carleson condition for sequences in $H^\infty(B_n)$, that is, when we deal with the finite dimensional Hilbert space $(\mathbb{C}^n, \|\cdot\|_2)$, whose unit ball is denoted by B_n. He found that this condition was sufficient for a sequence to be interpolating for $H^\infty(B_n)$. He showed this result by means of a construction provided by P. Jones in [Jon83] to give a new approach to the Carleson Interpolation Theorem.

In [BCL87], it is also proved that the generalized Carleson condition is sufficient for a sequence (x_n) to be interpolating for $H^\infty(\mathbf{D}^n)$, that is, when we deal with the finite dimensional Banach space $(\mathbb{C}^n, \|\cdot\|_\infty)$.

P. Galindo, T. W. Gamelin and M. Lindström extended the result given by B. Berndtsson for finite dimensional Hilbert spaces to any complex Hilbert space H in [GGL08]. Nevertheless, their proof does not provide explicitly the interpolating functions.

Before the main results of this chapter, let us recall that from the results on interpolation for uniform algebras from Chapter 2 we obtain

Proposition 3.2.2. *Let $A = H^\infty(B_E)$ and $(x_n) \subset M_A$. Then, the following conditions are equivalent,*

a) There exists a sequence (f_n) of Beurling functions for (x_n).
b) The sequence (x_n) is interpolating for $H^\infty(B_E)$.
c) The sequence (x_n) is c_0−interpolating for $H^\infty(B_E)$.

Proof. The equivalences are clear since the algebra $A = H^\infty(B_E)$ satisfies assumptions of Corollary 2.3.4 by Montel's Theorem. □

The following lemma includes some inequalities which can be found in [Ber85] and [BCL87].

Lemma 3.2.3. *Let $h(t)$ be a positive non-increasing function on $(0,\infty)$, and let (c_j) a sequence of non negative real numbers such that the series $\sum_{n=1}^{\infty} c_n$ converges. Then,*

$$\sum_{j=1}^{\infty} c_j h\left(\sum_{k\geq j} c_k\right) \leq \int_0^{\infty} h(t)dt.$$

In particular, if $h(t) = \min\left(1, 1/t^2\right)$, then

$$\sum_{j=1}^{\infty} c_j h\left(\sum_{k\geq j} c_k\right) \leq 2. \tag{3.8}$$

Moreover, suppose that $h(t) = \exp(-t)$ and there exists a sequence $(S_j)$of sets of $\mathbb{N}$ satisfying the property that if $k \notin S_j$, then $j \in S_k$. Then,

$$\sum_{j=1}^{\infty} c_j h\left(\sum_{k\in S_j} c_k\right) \leq 2e. \tag{3.9}$$

3.2.1 The case of Hilbert spaces

We will improve the result given by P. Galindo, T. W. Gamelin and M. Lindström by providing the Beurling functions required in Corollary 2.3.2.

Recall that the set of *biholomorphic automorphisms* on $\mathbf{D}$ is denoted by $Aut(\mathbf{D})$. It is well-known that this set is generated by rotations and by *Möbius transformations* $m_a : \mathbf{D} \longrightarrow \mathbf{D}$ defined for any $a \in \mathbf{D}$ by

$$m_a(z) = \frac{z+a}{1+\bar{a}z}. \tag{3.10}$$

For any $a \in \mathbf{D}$, the function m_a satisfies

i) $m_a(-a) = 0$ and

ii) m_a is defined on $\overline{\mathbf{D}}$ and $|m_a(e^{i\theta})| = 1$.

The analogues of Möbius transformations on a complex Hilbert space H are given by $M_a : B_H \longrightarrow B_H$ for any $a \in H$. This function is defined by the analytic map

$$M_a(x) = (\sqrt{1-\|a\|^2}q_a + p_a)(m_a(x)) \qquad \text{for any } a \in B_H \tag{3.11}$$

where $m_a : H \longrightarrow H$ is given by the analytic mapping

$$m_a(x) = \frac{z+a}{1+ < x, a >}, \tag{3.12}$$

the function $p_a : H \longrightarrow H$ denotes the orthogonal projection on H whose rank is the one-dimensional subspace spanned by a and $q_a : H \longrightarrow H$ is the orthogonal complement $q_a = Id_H - p_a$. It is also known [GR84] that the pseudohyperbolic distance defined in 1.7.3 for $H^\infty(B_H)$, H a Hilbert space, satisfies

$$\rho(x,y) = \|M_{-y}(x)\| \tag{3.13}$$

and, therefore, it is easy to deduce that

$$\rho(x,y)^2 = 1 - \frac{(1-\|x\|^2)(1-\|y\|^2)}{|1-<x,y>|^2} \tag{3.14}$$

just making some calculations.

As we have mentioned above, B. Berndtsson proved [Ber85] that the Carleson condition is sufficient for a sequence to be interpolating when we deal with finite-dimensional Hilbert spaces and the proof of this result is based on the P. Jones construction [Jon83] of Beurling functions. P. Galindo, T. Gamelin and M. Lindström pointed out [GGL08] that the Carleson condition is also sufficient for a sequence when we deal with infinite dimensional Hilbert spaces. Indeed, it is sufficient to notice that finite subsets of the sequence are interpolating and the constants of interpolation are uniform by the B. Berndtsson's work in [Ber85] and, then, a normal families argument to pass to a limit as $n \to \infty$ is applied. Our aim here is to give an explicit formula for the Beurling functions and the interpolating functions. First, we provide a lemma which includes some calculations related to the automorphisms M_a.

Lemma 3.2.4. *Let $x, y \in B_H$ and $M_{-y} : H \longrightarrow H$ the corresponding automorphism defined as in 3.11. Then, we have that*

$$<M_{-y}(x), M_{-y}(z)>= 1 - \frac{(1-<x,z>)(1-<y,y>)}{(1-<x,y>)(1-<y,z>)}.$$

Proof. Since for any $x \in B_H$ we have

$$M_{-y}(x) = \left(\sqrt{1-\|y\|^2}q_{-y} + p_{-y}\right)(m_{-y}(x)),$$

we obtain that $<M_{-y}(x), M_{-y}(z)>=$

$$<\left(\sqrt{1-\|y\|^2}q_{-y} + p_{-y}\right)(m_{-y}(x)), \left(\sqrt{1-\|y\|^2}q_{-y} + p_{-y}\right)(m_{-y}(z))>=$$

$$(1-\|y\|^2) < q_{-y}(m_{-y}(x)), q_{-y}(m_{-y}(z)) > + < p_{-y}(m_{-y}(x)), p_{-y}(m_{-y}(z)) >=$$

$$\frac{(1-\|y\|^2) < q_{-y}(x-y), q_{-y}(z-y) > + < p_{-y}(x-y), p_{-y}(z-y) >}{(1- < x,y >)(1- < y,z >)}$$

by 3.12 just making some calculations. Since we have that $p_a + q_a = Id_H$ for any $a \in H$, we obtain that $< M_{-y}(x), M_{-y}(z) >=$

$$\frac{< x-y, z-y > -\|y\|^2 < q_{-y}(x-y), q_{-y}(z-y) >}{(1- < x,y >)(1- < y,z >)}.$$

The complement of the orthogonal projection is given by

$$q_{-y}(x) = x - \frac{< x,y >}{< y,y >} y,$$

hence $q_{-y}(x-y) = q_{-y}(x)$ and $q_{-y}(z-y) = q_{-y}(z)$.

Moreover,

$$< q_{-y}(x), q_{-y}(z) >=< x - \frac{< x,y >}{< y,y >} y, z - \frac{< z,y >}{< y,y >} y >=$$

$$< x,z > - \frac{1}{\|y\|^2} < x,y >< y,z > -$$

$$\frac{1}{\|y\|^2} < x,y >< y,z > + \frac{1}{\|y\|^2} < x,y >< y,z >=$$

$$< x,z > - \frac{1}{\|y\|^2} < x,y >< y,z >= \frac{< x,z >< y,y > - < x,y >< y,z >}{\|y\|^2}.$$

Therefore, $< M_{-y}(x), M_{-y}(z) >=$

$$\frac{< x-y, z-y > -\|y\|^2 \frac{<x,z><y,y>-<x,y><y,z>}{\|y\|^2}}{(1- < x,y >)(1- < y,z >)} =$$

$$\frac{< x-y, z-y > - < x,z >< y,y > + < x,y >< y,z >}{(1- < x,y >)(1- < y,z >)}.$$

Since $< x-y, z-y >=< x,z > - < x,y > - < y,z > + < y,y >$, we have that the numerator $< x-y, z-y > - < x,z >< y,y > + < x,y >< y,z >$ equals to

$$< x,z > - < x,y > - < y,z > + < y,y > - < x,z >< y,y > + < x,y >< y,z > .$$

Adding and subtracting 1 and arranging terms, we obtain that the numerator equals to

$$(1- < x,y >)(1- < y,z >) - (1- < x,z >)(1- < y,y >).$$

Therefore, dividing by the denominator, we have that

$$< M_{-y}(x), M_{-y}(z) >= \frac{(1- < x,y >)(1- < y,z >) - (1- < x,z >)(1- < y,y >)}{(1- < x,y >)(1- < y,z >)} =$$

$$1 - \frac{(1- < x,z >)(1- < y,y >)}{(1- < x,y >)(1- < y,z >)},$$

and the lemma is proved. □

We will also need some technical lemmas. For the first one, we will need Proposition 5.1.2 in [Rud80], which states as follows,

Lemma 3.2.5. *Let a,b,c points in the unit ball of a finite dimensional Hilbert space. Then,*

$$|1- < a,b > | \leq (\sqrt{|1- < a,c > |} + \sqrt{|1- < b,c > |})^2$$

Then, we obtain the following lemma which is an extension of Lemma 5 in [Ber85] to any complex Hilbert space,

Lemma 3.2.6. *Let H be a complex Hilbert space and $x_1,x_2,x_3 \in B_H$. Then,*

$$|1- < x_1,x_2 > | \leq 2(|1- < x_1,x_3 > | + |1- < x_2,x_3 > |)$$

and

$$1 - | < x_1,x_2 > | \leq 2(1 - | < x_1,x_3 > | + 1 - | < x_2,x_3 > |).$$

Proof. Let $x_1,x_2,x_3 \in \overline{B}_H$ and set the tridimensional space $H_1 = \text{span}\{x_1,x_2,x_3\}$. We have that H_1 is itself a Hilbert space and we can consider an orthonormal basis $\{e_1,e_2,e_3\}$ of H_1. Consider for $j = 1,2,3$ the vectors $y_j = (y_j^1, y_j^2, y_j^3)$ given by the components of x_j in that basis. It is clear that these vectors are in the unit Euclidean ball of $\mathbb{C}^3$ and $< x_j,x_k >=< y_j,y_k >$, so we apply Lemma 3.2.5 to deduce

$$|1- < x_1,x_2 > | \leq (\sqrt{|1- < x_1,x_3 > |} + \sqrt{|1- < x_2,x_3 > |})^2 =$$

$$|1- < x_1,x_3 > | + |1- < x_2,x_3 > | + 2\sqrt{|1- < x_1,x_3 > |}\sqrt{|1- < x_2,x_3 > |}.$$

Applying the arithmetic-geometric means inequality, we have $|1- < x_1,x_2 > | \leq$

$$|1- < x_1,x_3 > | + |1- < x_2,x_3 > | + 2\frac{|1- < x_1,x_3 > | + |1- < x_2,x_3 > |}{2} =$$

$$2(|1-<x_1,x_3>|+|1-<x_2,x_3>|).$$

To prove the other result, notice that

$$1-|<x_j,x_k>| = \min_{\theta\in[0,2\pi)} |1-e^{i\theta}<x_j,x_k>|.$$

We have that

$$1-|<x_1,x_3>| = |1-e^{i\alpha}<x_1,x_3>|$$

and

$$1-|<x_2,x_3>| = |1-e^{i\beta}<x_2,x_3>|$$

for some $\alpha,\beta\in[0,2\pi)$. Then, applying the inequality above, we have that

$$1-|<x_1,x_2>| = 1-|<e^{i\alpha}x_1,e^{i\beta}x_2>| \leq |1-<e^{i\alpha}x_1,e^{i\beta}x_2>| \leq$$

$$2(|1-e^{i\alpha}<x_1,x_3>|+|1-e^{i\beta}<x_2,x_3>|) = 2(1-|<x_1,x_3>|+1-|<x_2,x_3>|).$$

□

Then, we obtain the following lemma which is an extension of Lemma 6 in [Ber85]. The result was proved by B. Berndtsson for finite dimensional Hilbert spaces. We do not provide a proof since it does not depend on the finite dimension of H but only on the previous lemmas which have been proved for any Hilbert space H.

Lemma 3.2.7. *Let H be a Hilbert space and $x_k,x_j\in B_H$. If $\|x_k\|\geq\|x_j\|$, then*

$$\frac{1-|<x_k,x>|^2}{1-|<x_k,x_j>|^2} \geq \frac{1}{8}\frac{1-\|x_k\|^2}{1-|<x_j,x>|^2}. \tag{3.15}$$

The following lemma is just a calculation,

Lemma 3.2.8. *Let $h(t)=\min\{1,1/t^2\}$. Then, the function $x^2\exp(-xt/8)$ is bounded by $256h(t)/e^2$ for $0\leq x\leq 1$ and $t>0$.*

We will also need the following lemma,

Lemma 3.2.9. *Let $\{x_n\}\subset B_H$ and $\delta>0$ satisfying*

$$\prod_{k\neq j}\rho(x_k,x_j)\geq\delta. \tag{3.16}$$

Then, we have that

$$\sum_{k\neq j}^{\infty}(1-\|x_k\|^2) \leq (1+2\log\frac{1}{\delta})\frac{1+\|x_j\|}{1-\|x_j\|} \qquad \forall j\in\mathbb{N}. \tag{3.17}$$

Proof. Taking squares and logarithms in 3.16 we obtain

$$-\sum_{k\neq j}^{\infty} \log\rho(x_k,x_j)^2 \leq -2\log\delta.$$

By (3.2), we have that $1-\rho(x_k,x_j)^2 \leq -\log\rho(x_k,x_j)^2$ for any $k\neq j$, so bearing in mind (3.14), we obtain

$$\sum_{k\neq j}^{\infty} \frac{(1-\|x_k\|^2)(1-\|x_j\|^2|)}{|1-<x_k,x_j>|^2} \leq -2\log\delta.$$

In consequence,

$$\sum_{k\neq j}^{\infty}(1-\|x_k\|^2) = \sum_{k\neq j}^{\infty} \frac{(1-\|x_k\|^2)(1-\|x_j)\|^2)}{|1-<x_k,x_j>|^2} \frac{|1-<x_k,x_j>|^2}{1-\|x_j\|^2} \leq$$

$$-2(\log\delta)\frac{(1+\|x_j\|)^2}{1-\|x_j\|^2} = 2\left(\log\frac{1}{\delta}\right)\frac{1+\|x_j\|}{1-\|x_j\|}$$

and the lemma is proved. □

Now we are ready to prove the result for complex Hilbert spaces. In addition, we will provide an upper estimate for the constant of interpolation depending only on δ and the sequence (x_n).

Theorem 3.2.10. *Let H be a Hilbert space and (x_n) a sequence in B_H. Suppose that there exists $\delta>0$ such that (x_n) satisfies the generalized Carleson condition for δ. Then, there exists a sequence of Beurling functions (F_k) for (x_n). In particular, the sequence (x_n) is interpolating for $H^\infty(B_H)$ and the constant of interpolation is bounded by*

$$\frac{2048}{e\delta}(1+2\log\frac{1}{\delta})^2.$$

Proof. Define, for any $k,j\in\mathbb{N}$, $k\neq j$, the function $g_{k,j}: H\longrightarrow\mathbb{C}$ given by

$$g_{k,j}(x) = <M_{-x_k}(x), M_{-x_k}(x_j)>.$$

For each $j\in\mathbb{N}$ we define the function $B_j: B_H\longrightarrow\mathbb{C}$ by

$$B_j(x) = \prod_{k\neq j} g_{k,j}(x).$$

First we check that the infinite product converges uniformly on

$$rB_H = \{x \in B_H : \|x\| \leq r\}$$

for fixed $0 < r < 1$. Let $x \in rB_H$. We have, by Lemma 3.2.4, that

$$1 - g_{k,j}(x) = 1 - < M_{-x_k}(x), M_{-x_k}(x_j) > = \frac{(1 - < x, x_j >)(1 - < x_k, x_k >)}{(1 - < x, x_k >)(1 - < x_k, x_j >)}.$$

It is easy that $|1 - < x, x_j > | \leq 1 + r$, $|1 - < x, x_k > | \geq 1 - r$ and

$$|1 - < x_k, x_j > | \geq 1 - \|x_k\|\|x_j\| \geq 1 - \|x_j\|.$$

Then, we have that

$$|1 - g_{k,j}(x)| \leq \frac{1+r}{1-r}\frac{1 - \|x_k\|^2}{1 - \|x_j\|},$$

so for any $j \in \mathbb{N}$, the series $\sum_{k \neq j} |1 - g_{k,j}(x)|$ is uniformly convergent on rB_H by Lemma 3.2.9. In particular, the infinite product $\prod_{k \neq j} g_{k,j}(x)$ converges uniformly on compact sets, so $B_j \in H^\infty(B_H)$, as we wanted.

Note that for $x \in B_H$,

$$|B_j(x)| = \prod_{k \neq j} |g_{k,j}(x)| = \prod_{k \neq j} | < M_{-x_k}(x), M_{-x_k}(x_j) > | \leq \prod_{k \neq j} \|M_{-x_k}(x)\|\|M_{-x_k}(x_j)\| \leq 1$$

so $\|B_j\|_\infty \leq 1$. It is clear that $B_j(x_k) = 0$ for $k \neq j$ since $M_{-x_k}(x_k) = 0$ and, according to 3.13, we have that

$$|B_j(x_j)| = \prod_{k \neq j} |g_{k,j}(x_j)| = \prod_{k \neq j} | < M_{-x_k}(x_j), M_{-x_k}(x_j) > | =$$

$$\prod_{k \neq j} \|M_{x_k}(x_j)\|^2 = \prod_{k \neq j} \rho(x_k, x_j)^2 \geq \delta^2.$$

Consider, for any $j \in \mathbb{N}$, the function $q_j : B_H \longrightarrow \mathbb{C}$ defined by

$$q_j(x) = \left(\frac{1 - \|x_j\|^2}{1 - < x, x_j >}\right)^2,$$

which clearly satisfies $q_j \in H^\infty(B_H)$ for any $j \in \mathbb{N}$. Define $A_j : B_H \longrightarrow \mathbb{C}$ by

$$A_j(x) = \sum_{\{k : \|x_k\| \geq \|x_j\|\}} \frac{(1 - \|x_k\|^2)(1 - \|x_j\|^2)}{1 - | < x_k, x_j > |^2} \frac{1 + < x_k, x >}{1 - < x_k, x >}.$$

These functions are analytic since for $0<r<1$ and $x\in rB_H$ we have that

$$|A_j(x)| \le \sum_{\{k:\|x_k\|\ge\|x_j\|\}} \frac{(1-\|x_k\|^2)(1-\|x_j\|^2)}{1-\|x_j\|^2}\frac{1+r}{1-r} \le \frac{1+r}{1-r}\sum_{\{k:\|x_k\|\ge\|x_j\|\}}(1-\|x_k\|^2)$$

so the series converges uniformly on rB_H and hence $A_j\in H(B_H)$. Moreover, $\exp\left(-A_j\right)$ belongs to $H^\infty(B_H)$ since $|\exp\left(-A_j\right)| = \exp\left(-\Re e\, A_j\right)$ and $\Re e\, A_j>0$ by the formula 3.3. Consider, for any $j\in\mathbb{N}$, the analytic function $F_j : B_H \longrightarrow \mathbb{C}$ defined by

$$F_j(x) = \frac{B_j(x)}{B_j(x_j)} q_j(x)^2 \exp\left(-\frac{1}{1+2\log 1/\delta}(A_j(x)-A_j(x_j))\right).$$

It is clear that $F_j(x_j)=1$ and $F_j(x_k)=0$ for any $k\neq j$. We will find $M>0$ such that $\sum_{j=1}^\infty |F_j(x)|\le M$ for any $x\in B_H$.

We also have by 3.3 that

$$\Re e\, A_j(x) = \sum_{\{k:\|x_k\|\ge\|x_j\|\}} \frac{(1-\|x_k\|^2)(1-\|x_j\|^2)(1-|<x_k,x>|^2)}{(1-|<x_k,x_j>|^2)(|1-<x_k,x>|^2)}.$$

In particular, for $x=x_j$, we obtain

$$\Re e\, A_j(x_j) = \sum_{\{k:\|x_k\|\ge\|x_j\|\}} \frac{(1-\|x_k\|^2)(1-\|x_j\|^2)(1-|<x_k,x_j>|^2)}{(1-|<x_k,x_j>|^2)(|1-<x_k,x_j>|^2)}.$$

Using the formula 3.14 we obtain that

$$\Re e\, A_j(x_j) = \sum_{\{k:\|x_k\|\ge\|x_j\|\}} (1-\rho^2(x_k,x_j)) = 1 + \sum_{\{k:\|x_k\|>\|x_j\|\}} (1-\rho^2(x_k,x_j))$$

and, by 3.2, we have that

$$\Re e\, A_j(x_j) \le 1 - \sum_{\{k:\|x_k\|>\|x_j\|\}} \log\rho(x_k,x_j)^2 \le 1 - \sum_{k\neq j}\log\rho(x_k,x_j)^2 \le 1+2\log\frac{1}{\delta}.$$

Moreover, to estimate $\Re e\, A_j(x)$ from below we use Lemma 3.2.7 and we obtain that

$$\Re e\, A_j(x) \ge \frac{1}{8}\frac{1-\|x_j\|^2}{1-|<x_j,x>|^2} \sum_{\{k:\|x_k\|\ge\|x_j\|\}} \frac{(1-\|x_k\|^2)^2}{|1-<x_k,x>|^2}.$$

We define

$$b_k(x) = \frac{1-\|x_k\|^2}{1-|<x_k,x>|^2}.$$

Then, we have that

$$\Re e\, A_j(x) \geq \frac{1}{8} b_j(x) \sum_{\{k:\|x_k\|\geq\|x_j\|\}} |q_k(x)|.$$

Since $1-|<x_j,x>|^2 = (1-|<x_j,x>|)(1+|<x_j,x>|) \leq 2|1-<x_j,x>|$, we have that

$$|q_j(x)| \leq \left|\frac{1-\|x_j\|^2}{1-<x_j,x>}\right|^2 \leq 4\left(\frac{1-\|x_j\|^2}{1-|<x,x_k>|^2}\right)^2 = 4b_j(x)^2.$$

Therefore, we obtain that

$$\sum_{k=1}^{\infty} |F_j(x)| \leq \frac{4e}{\delta}|q_j(x)|b_j(x)^2 \exp\left(\frac{-1}{8(1+2\log 1/\delta)}\, b_j(x) \sum_{\{k:\|x_k\|\geq\|x_j\|\}} |q_k(x)|\right).$$

We denote

$$C(\delta) = \frac{1}{1+2\log 1/\delta}$$

and $h(t) = \min(1, 1/t^2)$ for $t > 0$. Since $0 \leq b_k(x) \leq 1$, by Lemma 3.2.8 we have that $\sum_{j=1}^{\infty} |F_j(x)|$ is bounded by

$$\frac{1024}{e\delta C(\delta)^2} \sum_{k=1}^{\infty} |q_j(x)| h\left(\sum_{\{k:\|x_k\|\geq\|x_j\|\}} |q_k(x)|\right).$$

Since $\int_0^\infty h(t)dt = 2$, we apply 3.8 in Lemma 3.2.3 to obtain

$$\sum_{j=1}^{\infty} |F_j(x)| \leq \frac{2048}{e\delta}(1+2\log\frac{1}{\delta})^2.$$

Then, (F_j) is a sequence of Beurling functions and, by Corollary 2.3.2, we have that (x_j) is interpolating for $H^\infty(B_H)$. □

The sequence of Beurling functions in Theorem 3.2.10 is bounded by $\frac{2048}{e\delta}(1+2\log\frac{1}{\delta})^2$, so this is an upper bound for the constant of interpolation of (x_n).

3.2.2 The case of Banach spaces

We have shown that the generalized Carleson condition 3.7 is sufficient for a sequence $(x_n) \subset B_H$ to be interpolating for $H^\infty(B_H)$ when we deal with complex Hilbert spaces. In general Banach spaces, there are no known explicit formulas for the pseudohyperbolic distance. This makes it difficult to study such condition in this case and, hence, to prove whether it is sufficient for a sequence to be interpolating for $H^\infty(B_E)$. Thus, we introduce a stronger condition. We are interested in sequences (x_n) whose sequence of norms $(\|x_n\|)$ satisfies the Carleson condition 3.6. By the Carleson interpolation Theorem 3.1.6, it is clear that this condition is equivalent to $(\|x_n\|)$ being interpolating for H^∞. In addition, it is stronger than the generalized Carleson condition 3.7 since $\rho(\|x_k\|,\|x_j\|) \le \rho(x_k,x_j)$ by 3.1.2. Therefore, we wonder if this condition is sufficient for a sequence to be interpolating for $H^\infty(B_E)$. We will prove that this is true. We begin giving a number of lemmas.

Lemma 3.2.11. *Let E be a complex Banach space and (x_n) a sequence in $B_{E^{**}}$. Consider the sequence of norms $(\|x_n\|)$ and suppose that 0 is not a cluster point of $(\|x_n\|)$. Then, we have the following results,*

1) *For $k, j \in \mathbb{N}$ and $\alpha_{k,j} > 0$, there exists a functional $T_{k,j} \in E^*$ such that $\|T_{k,j}\| \le 1$ and*

$$\rho(T_{k,j}(x_k), T_{k,j}(x_j)) \ge \rho(\|x_k\|,\|x_j\|) - \alpha_{k,j}. \tag{3.18}$$

Moreover, there exists $m > 0$, which depends only on the sequence (x_n), such that

$$m \le |T_{k,j}(x_k)| \text{ for indexes } k, j \in \mathbb{N} \text{ which satisfy } \|x_k\| > \|x_j\|. \tag{3.19}$$

2) *If, further, $\lim \|x_n\| = 1$, then for any $\alpha_k > 0$, there exists a functional $L_k \in E^*$ such that $\|L_k\| \le 1$ and*

$$\rho(L_k(x_k), L_k(x_j)) \ge \rho(\|x_k\|,\|x_j\|) - \alpha_k \text{ for } j \in \mathbb{N} \text{ such that } \|x_k\| \ge \|x_j\| \tag{3.20}$$

and

$$L_k(x_k) \ge \|x_j\| \quad \text{for indexes } j \in \mathbb{N} \text{ such that } \|x_k\| > \|x_j\|. \tag{3.21}$$

Proof. 1) Since 0 is not a cluster point of (x_n), it is clear that there exists $m > 0$ so that

$$m < \inf_{n\in\mathbb{N}} \{\|x_n\| : \|x_n\| \neq 0\}.$$

For every $k \in \mathbb{N}$, there exists a sequence $\{T_n^k\}_n$ in $\overline{B}_{E^*}$ such that

$$\lim_{n\to\infty} |T_n^k(x_k)| = \|x_k\|. \tag{3.22}$$

If $x_k \neq 0$, we may assume as well that $|T_n^k(x_k)| \geq m$ for any $n \in \mathbb{N}$. Set $j \in \mathbb{N}$. If $\|x_k\| = \|x_j\|$, then $\rho(\|x_k\|, \|x_j\|) = 0$, so 3.18 is satisfied by any $\alpha_{k,j} > 0$. So, suppose that $\|x_k\| > \|x_j\|$. When considering the pseudohyperbolic distance between $\|x_k\|$ and $\|x_j\|$, there are two possibilities:

i) Either, there is $m_1 \in \mathbb{N}$ so that $\rho(|T_{m_1}^k(x_k)|, |T_{m_1}^k(x_j)|) \geq \rho(\|x_k\|, \|x_j\|)$. In this case, by 1.5, we choose $T_{k,j} = T_{m_1}^k$ and then (3.18) also holds for arbitrary $\alpha_{k,j} \geq 0$.

ii) Or such m_1 does not exist, that is,

$$\rho(|T_n^k(x_k)|, |T_n^k(x_j)|) < \rho(\|x_k\|, \|x_j\|) \quad \text{for any } n.$$

Since $\|x_j\| < \|x_k\|$ there is $m_2 \geq m_1$ such that $\|x_j\| \leq |T_n^k(x_k)|$ for $n \geq m_2$ by 3.22. Thus, after using (3.1), we have

$$\rho(|T_n^k(x_k)|, |T_n^k(x_j)|) \geq \rho(|T_n^k(x_k)|, \|x_j\|).$$

Consequently,

$$\rho(|T_n^k(x_k)|, \|x_j\|) \leq \rho(|T_n^k(x_k)|, |T_n^k(x_j)|) < \rho(\|x_k\|, \|x_j\|) \quad \text{for any } n \geq m_2.$$

Hence, by the choice in (3.22),

$$\lim_{n\to\infty} \rho(|T_n^k(x_k)|, |T_n^k(x_j)|) = \rho(\|x_k\|, \|x_j\|).$$

Therefore, for any $\alpha_{k,j} > 0$ there exists $m_3 > m_2$ such that

$$\rho(|T_n^k(x_k)|, |T_n^k(x_j)|) \geq \rho(\|x_k\|, \|x_j\|) - \alpha_{k,j} \quad \text{for } n \geq m_3,$$

and, by 1.5, we obtain

$$\rho(T_n^k(x_k), T_n^k(x_j)) \geq \rho(\|x_k\|, \|x_j\|) - \alpha_{k,j} \quad \text{for } n \geq m_3,$$

so we can choose $T_{k,j} = T_{m_3}^k$ and (3.18) holds in this case as well .

2) Assume now that $\lim_{n\to\infty} \|x_n\| = 1$ and fix $k \in \mathbb{N}$. Observe that there are only a finite number of terms x_j in the sequence satisfying $\|x_j\| < \|x_k\|$. For such $j \in \mathbb{N}$ we consider the set

$$B_j = \left\{ n : \rho(|T_n^k(x_k)|, |T_n^k(x_j)|) < \rho(\|x_k\|, \|x_j\|) \right\}.$$

In case B_j is a finite set, we may find $n_j \in \mathbb{N}$ so that

$$\rho(|T_n^k(x_k)|, |T_n^k(x_j)|) \geq \rho(\|x_k\|, \|x_j\|)) \geq \rho(\|x_k\|, \|x_j\|) - \alpha_k \quad \text{for any } n \geq n_j. \tag{3.23}$$

Whenever B_j is not finite, we argue as in ii) above to show that

$$\lim_{n\in B_j} \rho(|T_n^k(x_k)|, |T_n^k(x_j)|) = \rho(\|x_k\|, \|x_j\|).$$

Therefore, an n_j can be found so that

$$\rho(|T_n^k(x_k)|, |T_n^k(x_j)|) \geq \rho(\|x_k\|, \|x_j\|) - \alpha_k \quad \text{for any } n \in B_j \text{ such that } n \geq n_j.$$

Note that this inequality also holds for $n \notin B_j$. Consequently,

$$\rho(|T_n^k(x_k)|, |T_n^k(x_j)|) \geq \rho(\|x_k\|, \|x_j\|) - \alpha_k \quad \text{for any } n \geq n_j.$$

Since we are considering a finite number of j, we have that $\max\{n_j\}$ is finite. In addition, $\lim_{n\to\infty} |T_n^k(x_k)| = \|x_k\|$ and $\max\{\|x_j\| : \|x_j\| < \|x_k\|\} < \|x_k\|$, so we may take $n_0 \geq \max\{n_j\}$ big enough so that

$$|T_{n_0}^k(x_k)| > \|x_j\| \quad \text{for any } j \in \mathbb{N} \text{ such that } \|x_j\| < \|x_k\|.$$

By setting $L_k = T_{n_0}^k$ and using (1.5), we are done. □

We continue with a result similar to Lemma 3.2.9.

Lemma 3.2.12. *Let $(x_n) \subset B_{E^{**}}$ and $\delta > 0$ such that $(\|x_n\|)$ satisfies the Carleson condition 3.6. Then, for any $0 < s < 1$, there exists a sequence of positive numbers $(\alpha_{k,j})$ such that the set of functionals $(T_{k,j})$ found in Lemma 3.2.11 satisfy*

$$\prod_{k\neq j} \rho(T_{k,j}(x_k), T_{k,j}(x_j)) \geq (1-s)\delta \tag{3.24}$$

and

$$\sum_{k=1}^{\infty} (1 - |T_{k,j}(x_k)|^2) \leq \left(1 + 2\log\frac{1}{(1-s)\delta}\right) \frac{1+\|x_j\|}{1-\|x_j\|} \qquad \forall j \in \mathbb{N}. \tag{3.25}$$

Proof. Since $0 < s < 1$, a sequence $(\beta_k) \subset (0,1)$ can be found such that $\prod(1-\beta_k) \geq 1-s$ and then put $\alpha_{k,j} = \beta_k \rho(\|x_k\|, \|x_j\|)$. Now, apply Lemma 3.2.11 to find the functionals $(T_{k,j})$ such that

$$\rho(T_{k,j}(x_k), T_{k,j}(x_j)) \geq \rho(\|x_k\|, \|x_j\|) - \alpha_{k,j} = (1-\beta_k)\rho(\|x_k\|, \|x_j\|).$$

Hence, considering the corresponding infinite product, we obtain 3.24.

Let us prove the other inequality. Taking squares and logarithms in 3.24, we obtain that

$$-\sum_{k\neq j}\log\left|\frac{T_{k,j}(x_j)-T_{k,j}(x_k)}{1-\overline{T_{k,j}(x_k)}T_{k,j}(x_j)}\right|^2 \leq -2\log(1-s)\delta.$$

Moreover, by 3.2, we obtain

$$1-\left|\frac{T_{k,j}(x_j)-T_{k,j}(x_k)}{1-\overline{T_{k,j}(x_k)}T_{k,j}(x_j)}\right|^2 \leq -\log\left|\frac{T_{k,j}(x_j)-T_{k,j}(x_k)}{1-\overline{T_{k,j}(x_k)}T_{k,j}(x_j)}\right|^2 \quad \text{for any } k\neq j$$

and bearing in mind 1.6, it results

$$\sum_{k\neq j}\frac{(1-|T_{k,j}(x_k)|^2)(1-|T_{k,j}(x_j)|^2|)}{|1-\overline{T_{k,j}(x_k)}T_{k,j}(x_j)|^2} \leq -2\log(1-s)\delta.$$

As consequence,

$$\sum_{k=1}^{\infty}(1-|T_{k,j}(x_k)|^2) = \sum_{k=1}^{\infty}\frac{(1-|T_{k,j}(x_k)|^2)(1-|T_{k,j}(x_j)|^2)}{|1-\overline{T_{k,j}(x_k)}T_{k,j}(x_j)|^2}\frac{|1-\overline{T_{k,j}(x_k)}T_{k,j}(x_j)|^2}{1-|T_{k,j}(x_j)|^2} \leq$$

$$(1-2\log(1-s)\delta)\frac{(1+\|x_j\|)^2}{1-\|x_j\|^2} = \left(1+2\log\frac{1}{(1-s)\delta}\right)\frac{1+\|x_j\|}{1-\|x_j\|}.$$

□

Lemma 3.2.13. *Let* $\{x_n\}\subset B_{E^{**}}$ *and* $\delta>0$ *such that the sequence* $(\|x_n\|)$ *satisfies the Carleson condition 3.6. Then, for any* $0<s<1$*, there exists a sequence* (α_k) *of positive numbers such that the set of functionals* (L_k) *found in Lemma 3.2.11 satisfy*

$$\prod_{\{k:\|x_k\|>\|x_j\|\}}\rho(L_k(x_k),L_k(x_j)) \geq (1-s)\delta \tag{3.26}$$

and

$$\sum_{\{k:\|x_k\|>\|x_j\|\}}(1-|L_k(x_k)|^2) \leq \left(1+2\log\frac{1}{(1-s)\delta}\right)\frac{1+\|x_j\|}{1-\|x_j\|} \quad \textit{for any } j\in\mathbb{N}. \tag{3.27}$$

Proof. The proof follows the same pattern as Lemma 3.2.12. Simply choose

$$\alpha_k = \min_{\{j:\|x_j\|<\|x_k\|\}} \{\beta_k \rho(\|x_k\|, \|x_j\|)\}$$

and pick L_k from Lemma 3.2.11. □

Now we proceed to show our main result in this section.

Theorem 3.2.14. *Let (x_n) be a sequence in $B_{E^{**}}$ and suppose that the sequence $(\|x_n\|)$ satisfies the Carleson condition 3.6. Then, for any $0 < s < 1$, there exists a sequence of Beurling functions $(F_j) \subset H^\infty(B_E)$ for (x_n) depending on s. In particular, the sequence (x_n) is interpolating for $H^\infty(B_E)$ and the constant of interpolation can be chosen to be bounded by*

$$\frac{4e^2}{(1-s)\delta}\left(1 + 2\log\frac{1}{(1-s)\delta}\right).$$

Proof. Fix $0 < s < 1$. Let $\{T_{k,j}\}$ be the set of functionals furnished by Lemma 3.2.12. We define the function $g_{k,j} : B_{E^{**}} \longrightarrow \mathbb{C}$ by

$$g_{k,j}(x) = a_{k,j}\frac{T_{k,j}(x_k) - T_{k,j}(x)}{1 - \overline{T_{k,j}(x_k)}T_{k,j}(x)} \qquad \forall k, j \in \mathbb{N} \;\; k \neq j.$$

where $a_{k,j}$ is given by

$$a_{k,j} = \begin{cases} \frac{\overline{T_{k,j}(x_k)}}{|T_{k,j}(x_k)|} & \text{if } T_{k,j}(x_k) \neq 0 \\ -1 & \text{if } T_{k,j}(x_k) = 0. \end{cases}$$

For each $j \in \mathbb{N}$ we define

$$B_j(x) = \prod_{k\neq j} g_{k,j}(x).$$

Firstly, in order to check that $B_j : B_{E^{**}} \longrightarrow \mathbb{C}$ is analytic, we prove that this infinite product converges uniformly on $rB_{E^{**}}$ for fixed $0 < r < 1$. Recall that, for indexes k, j such that $\|x_k\| > \|x_j\|$, there is a constant $m > 0$ which satisfies $|T_{k,j}(x_k)| \geq m > 0$. Then, we have

$$1 - g_{k,j}(x) = \frac{1}{|T_{k,j}(x_k)|}\left[1 - \frac{|T_{k,j}(x_k)|^2 - \overline{T_{k,j}(x_k)}T_{k,j}(x)}{1 - \overline{T_{k,j}(x_k)}T_{k,j}(x)}\right] + 1 - \frac{1}{|T_{k,j}(x_k)|} =$$

$$\frac{1-|T_{k,j}(x_k)|}{|T_{k,j}(x_k)|}\left[\frac{1+|T_{k,j}(x_k)|}{1-\overline{T_{k,j}(x_k)}T_{k,j}(x)}-1\right] \text{ for } \|x_k\|>\|x_j\|.$$

Moreover, $|T_{k,j}(x)| \leq \|x\| \leq r < 1$, so we get

$$|1-g_{k,j}(x)| \leq \frac{1-|T_{k,j}(x_k)|}{|T_{k,j}(x_k)|}\left[\frac{2}{1-r}+1\right] \text{ for } \|x_k\|>\|x_j\|.$$

Since $\lim_k \|x_k\| = 1$, there exists $k_j \in \mathbb{N}$ such that $\|x_k\| > \|x_j\|$, for $k \geq k_j$ and, recalling (3.19) in Lemma 3.2.11, we obtain

$$|1-g_{k,j}(x)| \leq \frac{1}{m}\left[\frac{2}{1-r}+1\right](1-|T_{k,j}(x_k)|) \qquad \forall k \geq k_j.$$

Now we use inequality (3.25) to show that the series $\sum_{k=1}^\infty |1-g_{k,j}(x)|$ is uniformly convergent on $\{x : \|x\| \leq r\}$, as we wanted. In particular, $\prod_{k\neq j} g_{k,j}(x)$ converges uniformly on compact sets, so $B_j \in H^\infty(B_{E^{**}})$.

Note that $\|B_j\|_\infty \leq 1$, $B_j(x_k) = 0$ for $k \neq j$ and, according to (3.24),

$$|B_j(x_j)| = \prod_{k\neq j} \rho(T_{k,j}(x_k), T_{k,j}(x_j)) \geq (1-s)\delta.$$

Next we take the sequence (L_k) found in Lemma 3.2.13 and define the function $A_j : B_{E^{**}} \longrightarrow \mathbb{C}$ by

$$A_j(x) = \sum_{\{k:\|x_k\|\geq\|x_j\|\}} \left[\frac{1+\overline{L_k(x_k)}L_k(x)}{1-\overline{L_k(x_k)}L_k(x)}\right](1-|L_k(x_k)|^2).$$

We prove that A_j is analytic on $B_{E^{**}}$. Indeed, since $|L_k(x)| \leq \|x\|$ and $|L_k(x_k)| \leq 1$, we have that

$$|A_j(x)| \leq \sum_{\{k:\|x_k\|\geq\|x_j\|\}} \left[\frac{1+\|x\|}{1-\|x\|}\right](1-|L_k(x_k)|^2) \leq \left(1+2\log\frac{1}{(1-s)\delta}\right)\frac{1+\|x_j\|}{1-\|x_j\|}\frac{1+\|x\|}{1-\|x\|}$$

where we used (3.27). Hence, the convergence of the series is uniform on $rB_{E^{**}}$ and therefore A_j is analytic on $B_{E^{**}}$.

Consider the mappings $q_j \in H^\infty(B_{E^{**}})$ defined by

$$q_j(x) = \left[\frac{1-|L_j(x_j)|^2}{1-\overline{L_j(x_j)}L_j(x)}\right]^2$$

and denote by $C(s,\delta)$ the constant given by

$$\frac{1}{2\left(1+2\log\frac{1}{(1-s)\delta}\right)}.$$

Let $F_j \in H^\infty(B_{E^{**}})$ be given by

$$F_j(x) = \frac{B_j(x)}{B_j(x_j)} q_j(x) \exp\left(-C(s,\delta)\left(A_j(x) - A_j(x_j)\right)\right).$$

Recall that the Aron-Berner extension is continuous whenever both $H^\infty(B_E)$ and $H^\infty(B_{E^{**}})$ are endowed with the topology of the uniform convergence on balls of radii less than 1 (see [ACG91] (10.1) p. 86). Consequently, each of the functions $g_{k,j}, B_j, q_j$ and F_j is the Aron-Berner extension of its restriction to B_E.

We claim that

$$F_j(x_k) = \delta_{j,k} \text{ for } j,k \in \mathbb{N} \text{ and that} \tag{3.28}$$

$$\text{there exists } M > 0 \text{ such that } \sum |F_j(x)| \leq M. \tag{3.29}$$

Condition (3.28) is trivially verified. To prove (3.29), we first recall that $|B_j(x_j)| \geq (1 - s)\,\delta$. Then,

$$\sum_{j=1}^{\infty} |F_j(x)| \leq \frac{1}{(1-s)\delta} \sum_{j=1}^{\infty} |q_j(x)| \exp\left(-C(s,\delta)\Re e\,\left(A_j(x) - A_j(x_j)\right)\right).$$

Applying 3.3 in Lemma 3.1.1 we obtain

$$\Re e\, A_j(x) = \sum_{\{k:\|x_k\|\geq\|x_j\|\}} \frac{(1-|L_k(x_k)|^2|L_k(x)|^2)(1-|L_k(x_k)|^2)}{|1-\overline{L_k(x_k)}L_k(x)|^2}. \tag{3.30}$$

If $\|x_k\| > \|x_j\|$, then $|L_k(x_j)| \leq \|x_j\| \leq L_k(x_k)$ by (3.21). Now, applying the inequality $1-\alpha\beta \leq 2(1-\beta)$ for $\alpha \geq \beta$, it turns out that $1 - L_k(x_k)^2|L_k(x_j)|^2 \leq 2(1-|L_k(x_j)|^2)$. Thus

$$\Re e\, A_j(x_j) \leq 2 \sum_{\{k:\|x_k\|\geq\|x_j\|\}} \frac{(1-|L_k(x_j)|^2)(1-|L_k(x_k)|^2)}{|1-\overline{L_k(x_k)}L_k(x_j)|^2} =$$

$$2 \sum_{\{k:\|x_k\|\geq\|x_j\|\}} \left[1 - \left|\frac{L_k(x_j) - \|x_k\|}{1-\overline{L_k(x_j)}\|x_k\|}\right|^2\right].$$

Since $\|x_k\| \neq \|x_j\|$ for any indexes $k \neq j$ because of the Carleson condition for (x_n), we have that $\|x_k\| = \|x_j\|$ is only satisfied for one index, that is, $k = j$. Hence,

$$\Re e\, A_j(x_j) \leq 2 \sum_{\{k:\|x_k\|>\|x_j\|\}} \left[1 - \left|\frac{L_k(x_j) - \|x_k\|}{1 - \overline{L_k(x_j)}\|x_k\|}\right|^2\right] + 2\left[1 - \left|\frac{L_j(x_j) - \|x_j\|}{1 - \overline{L_j(x_j)}\|x_j\|}\right|^2\right] \leq$$

$$\sum_{\{k:\|x_k\|>\|x_j\|\}} 2\left[1 - \rho(L_k(x_j), L_k(x_k))^2\right] + 2$$

after bearing in mind (1.6). Next, using inequality (3.2) in Lemma 3.1.1 we obtain

$$\Re e\, A_j(x_j) \leq -2\log \prod_{\{k:\|x_k\|>\|x_j\|\}} \rho(L_k(x_j), L_k(x_k))^2 + 2 \leq 2(1 + 2\log\frac{1}{(1-s)\delta})$$

and then

$$\sum_{j=1}^{\infty} |F_j(x)| \leq \frac{e}{(1-s)\delta} \sum_{j=1}^{\infty} \left|q_j(x)\right| \exp\left(-C(s,\delta)\Re e\, A_j(x)\right).$$

Therefore, bearing in mind (3.30), we have that $\sum_{j=1}^{\infty} |F_j(x)|$ is bounded by

$$\frac{e}{(1-s)\delta} \sum_{j=1}^{\infty} \left|\frac{1 - |L_j(x_j)|^2}{1 - \overline{L_j(x_j)}L_j(x)}\right|^2 \exp\left(-C(s,\delta) \sum_{\{k:\|x_k\|\geq\|x_j\|\}} \left|\frac{1 - |L_k(x_k)|^2}{1 - \overline{L_k(x_k)}L_k(x)}\right|^2\right).$$

To apply inequality 3.9 in Lemma 3.2.3 we put

$$c_j = C(s,\delta) \left|\frac{1 - |L_j(x_j)|^2}{1 - \overline{L_j(x_j)}L_j(x)}\right|^2$$

and then we obtain that

$$\sum_{j=1}^{\infty} |F_j(x)| \leq \frac{2e}{(1-s)\delta}\left(1 + 2\log\frac{1}{(1-s)\delta}\right) 2e = \frac{4e^2}{(1-s)\delta}\left(1 + 2\log\frac{1}{(1-s)\delta}\right).$$

□

The converse of Theorem 3.2.14 is false for any Banach space. To show this, consider a vector $x \in S_E$ and $(\lambda_n) \subset]0,1[$ an interpolating sequence for H^∞ such that $\lambda_1 = 1/2$. Let $\phi \neq 0$ be a linear mapping whose norm is 1 and $\phi(x) = 1$. Since the sequence $\{-\lambda_1, \lambda_1, \ldots, \lambda_n, \ldots\}$ is interpolating for H^∞, we obtain that the sequence $\{-\lambda_1 x, \lambda_1 x, \ldots, \lambda_n x, \ldots\}$, is interpolating for $H^\infty(B_E)$. However $\|-\lambda_1 x\| = \|\lambda_1 x\| = 1/2$, so the Carleson condition 3.6 for $(\|x_n\|)$ clearly fails.

We have the following corollaries,

Corollary 3.2.15. *Let (x_n) be a sequence in $B_{E^{**}}$. If $(\|x_n\|)$ is an interpolating sequence for H^∞, then (x_n) is an interpolating sequence for $H^\infty(B_E)$.*

Proof. If $(\|x_n\|)$ is an interpolating sequence for H^∞, the Carleson condition for $(\|x_n\|)$ is satisfied. Then, Theorem 3.2.14 shows that $\{x_n\}_{n=1}^\infty$ is an interpolating sequence for $H^\infty(B_E)$. □

Corollary 3.2.16. *Let (x_n) be a sequence in $B_{E^{**}}$ and $0 < c < 1$ such that*

$$\frac{1 - \|x_{k+1}\|}{1 - \|x_k\|} < c \tag{3.31}$$

Then (x_n) is interpolating for $H^\infty(B_E)$.

Proof. Since $(\|x_n\|)$ satisfies the Hayman-Newman condition, then $(\|x_n\|)$ is interpolating for H^∞. Thus, apply Corollary 3.2.15 and we are done. □

In addition, from Corollary 3.2.16 we derive the following interpolation result due to R. M. Aron, B. Cole and T. Gamelin [ACG91] which generalizes Corollary 3.1.4.

Theorem 3.2.17. *Let (x_n) be a sequence in $B_{E^{**}}$ satisfying $\lim_{n\to\infty} \|x_n\| = 1$. Then there is an interpolating subsequence for $H^\infty(B_E)$.*

Proof. Since $\lim_{n\to\infty} \|x_n\| = 1$, there exists a subsequence of $(\|x_n\|)$ which increases exponentially to 1, that is, satisfying the Hayman-Newman condition. By Corollary 3.2.16, we obtain that this subsequence is interpolating for $H^\infty(B_E)$. □

3.2.3 Further results

As we have mentioned above, in [BCL87] the authors proved that the generalized Carleson condition is sufficient for a sequence (x_k) to be interpolating for $H^\infty(\mathbf{D}^n)$. In addition, they noticed that if the constant of interpolation was independent of dimension n, then for arbitrary uniform algebras A and finite sequences $(x_k)_{k=1}^N \subset M_A$ satisfying the generalized Carleson condition, we would obtain that $(x_k)_{k=1}^N$ is an interpolating sequence for A with the constant of interpolation depending on δ only, and not on the number of points N in the sequence. Following this idea, we have

Proposition 3.2.18. *Suppose that any sequence $(x_n) \subset B_{c_0}$ satisfying the generalized Carleson condition 3.7 is interpolating for $H^\infty(B_{c_0})$ with interpolation constant depending only on δ. Then, for any dual uniform algebra $A = X^*$, all sequences $(x_n) \subset X \cap M_A$ satisfying the generalized Carleson condition 3.7 are linear interpolating sequences for A with constant of interpolation depending only on δ.*

Proof. Assume there is $\delta > 0$ such that $\prod_{j \neq k} \rho_A(x_j, x_k) > \delta$ for all k. For $k, j \in \mathbb{N}$ there exists $f_{k,j} \in A$ such that $f_{k,j}(x_k) = 0$, $\|f_{k,j}\| \leq 1$ and

$$\rho_A(x_j, x_k) \geq |f_{j,k}(x_j)| \geq (1 - \frac{1}{2^{j+k}})\rho_A(x_j, x_k).$$

Fix $n \in \mathbb{N}$ and define $\phi : M_A \longrightarrow B_{c_0}$ according to

$$\phi(u) := (u(f_{1,1}), u(f_{1,2}), u(f_{2,1}), \ldots, u(f_{n,n}), 0, 0, \ldots).$$

Then, by the expression for the pseudohyperbolic distance 1.8 for c_0, we have that

$$\rho_{H^\infty(B_{c_0})}(\phi(u), \phi(v)) = \max_{j,k}\{\rho(u(f_{j,k}), v(f_{j,k})\}.$$

Thus

$$\rho_{H^\infty(B_{c_0})}(\phi(x_r), \phi(x_s)) = \max_{j,k}\{\rho(f_{j,k}(x_r), f_{j,k}(x_s))\}) \geq$$

$$\rho(f_{s,r}(x_r), f_{s,r}(x_s)) \geq (1 - \frac{1}{2^{r+s}})\rho_A(x_r, x_s).$$

Hence

$$\prod_{r \neq s}^{n} \rho_{H^\infty(B_{c_0})}(\phi(x_r), \phi(x_s)) \geq \prod_{r \neq s}^{n}(1 - \frac{1}{2^{r+s}})\rho_A(x_r, x_s) \geq$$

$$\prod_{r \neq s}^{\infty}(1 - \frac{1}{2^{r+s}})\prod_{r \neq s}^{n}\rho_A(x_r, x_s) \geq \prod_{r \neq s}^{\infty}(1 - \frac{1}{2^{r+s}})\delta \geq \prod_{r=1}^{\infty}(1 - \frac{1}{2^r})\delta$$

which is greater than $C\delta$ since the product converges. Then, the hypothesis guarantees that the finite sequences $(\phi(x_j))_{j=1}^n$ are interpolating sequences for $H^\infty(B_{c_0})$ with interpolation constant M depending only on δ.

Therefore, for given $(\alpha_j)_j^n$ in the unit ball of ℓ_∞, there is $F \in H^\infty(B_{c_0})$ such that $\|F\| \leq M$ and $(F \circ \phi)(x_j) = F(\phi(x_j)) = \alpha_j$. Since F actually depends only of a finite number of variables, it turns out that $F \circ \phi \in A$. Hence $(x_j)_{j=1}^n$ interpolates $(\alpha_j)_j^n$ by means of $(F \circ \phi)$ and $\|F \circ \phi\| \leq M$. By Theorem 2.3.3, the sequence (x_n) is interpolating for A and, in addition, the constant of interpolation is bounded by M^2. □

Notice that for finite dimensional Banach spaces E, interpolating sequences for $H^\infty(B_E)$ converge in norm to 1. Indeed, let E be a finite dimensional Banach space and $(x_n) \subset B_E$ an interpolating sequence for $H^\infty(B_E)$. If $\|x_n\|$ does not converge to 1, then, passing to a subsequence if necessary, we have that $(x_n) \subset rB_E$ for some $0 < r < 1$ and x_n should converge to some $x_0 \in B_E$ since $r\overline{B_E} \subset B_E$ is a compact set. If we deal with infinite dimensional Banach spaces, we can extend this result as follows,

Proposition 3.2.19. *Consider the following statements:*

a) All polynomials on E are weakly continuous on bounded sets.
b) If $(x_n) \subset B_E$ is a interpolating sequence for $H^\infty(B_E)$, then $\|x_n\|$ converges to 1.
c) The space E does not contain copies of ℓ_1.

Then a) implies b) and b) implies c). In addition, if all polynomials on E are weakly sequentially continuous, then c) implies a) and, therefore, the three statements are equivalent.

Proof. $b) \Rightarrow c)$ If E contains a copy of ℓ_1, then it is well-known that ℓ_2 is a quotient of E. Let $q : E \longrightarrow \ell_2$ be the quotient map and denote by e_n the n^{th} unit vector of the canonical basis of ℓ_2. Let $(y_n) \subset E$ be a bounded sequence such that $q(y_n) = e_n$ for any $n \in \mathbb{N}$. Let $C > 0$ be such that $\|y_n\| \leq \frac{C}{2}$ and set $x_n = \frac{y_n}{C}$.

We check that (x_n) is a c_0 interpolating sequence. Indeed, take $(\alpha_n) \in c_0$. Let $P \in P(^2E)$ the polynomial defined by

$$P(x) = \sum_{n=1}^{\infty} C^2 \alpha_n x_n^2 \qquad \text{for } x = (x_n) \in \ell_2.$$

The polynomial $P \circ q : E \to \mathbb{C}$ also belongs to $P(^2E)$. In addition,

$$(P \circ q)(x_j) = P(\frac{q(y_j)}{C}) = \frac{P(e_j)}{C^2} = \alpha_j.$$

Then, (x_j) is c_0-interpolating but $\|x_n\| \leq \frac{1}{2}$. This contradicts $b)$.

$a) \Rightarrow b)$ Suppose that the conclusion fails. We can suppose, passing to subsequences if necessary, that $||x_j|| \leq r < 1$ for all j. By the assumptions on E, every $f \in H^\infty(B_E)$ is weakly uniformly continuous on rB_E. Hence every $f \in H^\infty(B_E)$ extends to an analytic function $\hat{f}$ on $B_{E^{**}}$ which is w^*-continuous on $rB_{E^{**}}$.

By Rosenthal's ℓ_1 Theorem, either (x_j) has a subsequence equivalent to the unit basis of ℓ_1 or it has a weak Cauchy subsequence. The first alternative cannot hold since otherwise we could find, in the same way we have done in $(b) \Rightarrow (c)$, a polynomial which would be non weakly continuous on the unit ball. So we can suppose that (x_j) is a weak Cauchy sequence. If we consider (x_j) as a sequence in E^{**}, we get that (x_j) is a w^*-convergent sequence in E^{**} since $B_{E^{**}}$ is w^*-compact. Let $x_j \to x$ in $(rB_{E^{**}}, w(E^{**}, E^*))$. Then $f(x_j) \to \hat{f}(x)$ for all $f \in H^\infty(B_E)$.

On the other hand, since (x_j) is c_0-interpolating, it is interpolating by Theorem 2.3.3, which contradicts the former assertion.

It is easy that $c) \Rightarrow a)$ under the assumption that all polynomials are weakly sequentially continuous since if E does not contain copies of ℓ_1, all weakly sequentially continuous polynomials are weakly continuous on bounded sets by Proposition 1.5.1 c). □

Recall (see [ACG91]) that for $E = \ell_p$, $1 < p < +\infty$, any sequence $(\lambda_j e_j)$ with $0 < \inf|\lambda_j|$ and $|\lambda_j| < 1$ is c_0-interpolating for $H^\infty(B_E)$, so (c) may not imply either (b) or (a) in the above proposition if the assumption on polynomials is not satisfied. Let us also remark that for (c) to imply (a) it suffices that E enjoys the Dunford-Pettis property. In [CGG99], several examples of spaces satisfying that all polynomials are weakly sequentially continuous and lacking the *DP* property are exhibited; let us mention among them the dual S^* of the Schreier space (see [CGG99]) and the predual $d_*(w)$ of some Lorentz sequence space $d(w;1)$.

3.3 Separability in $A_\infty(B_E)$ and $A_u(B_E)$

3.3.1 Separability in $A_\infty(B_E)$

J. Globevnik studied in [Glo78] the existence of interpolating sequences for $A_\infty(B_E)$ when E belongs to a big class of infinite dimensional Banach spaces. In particular, he proved such existence for the class of all infinite dimensional reflexive Banach spaces. In this paper, he asked if this result can be extended to all infinite dimensional Banach spaces. We answer this question affirmatively by proving the existence of interpolating sequences for $A_\infty(B_E)$ when we deal with infinite dimensional non-reflexive Banach spaces.

The general result given by J. Globevnik was given for infinite dimensional Banach spaces such that there exists a sequence $(x_n) \subset S_E$ of strongly exposed points with no cluster points. He proved that strongly exposed points of $\overline{B}_E$ are strongly peak points of $A_\infty(B_E)$ and this allowed him to give the following result,

Theorem 3.3.1. *[Glo78] Let E be a complex Banach space and (x_n) a sequence of strongly peak points for $A_\infty(B_E)$ with no cluster points. For any $\alpha = (\alpha_n)_n \in \ell^\infty$ there exists a function $\phi \in A_\infty(B_E)$ such that $\phi(x_n) = \alpha_n$ for any $n \in \mathbb{N}$.*

This theorem solves the interpolation problem for the class of Banach spaces mentioned above. It states as follows,

Theorem 3.3.2. *Let E be an infinite dimensional complex Banach space whose unit sphere S_E contains a sequence (x_n) of strongly exposed points of $\overline{B}_E$ with no cluster points. Then (x_n) is an interpolating sequence for $A_\infty(B_E)$.*

It is clear that the class of Banach spaces which satisfies the conditions of the theorem contains the reflexive spaces. The following theorem proves this result for any infinite dimensional Banach space.

Theorem 3.3.3. *Let E be an infinite dimensional complex Banach space. Then, there exist interpolating sequences for $A_\infty(B_E)$.*

Proof. If E is reflexive, the problem is solved by Theorem 3.3.2. Otherwise, since E is not reflexive, by the James theorem there exists a functional $L \in E^*$ such that $\|L\| = 1$ which does not attain its norm on $\overline{B}_E$. Moreover, there exists $(x_n) \subset S_E$ such that $|L(x_n)| \to 1$. By Corollary 3.1.4, there exists a subsequence $(L(x_{n_k}))_k$ which is interpolating for H^∞. Set $(\alpha_n)_n \in \ell_\infty$. There exists a function $h \in H^\infty$ such that $h(L(x_{n_k})) = \alpha_k$. Consider the function $g : \overline{B}_E \longrightarrow \mathbb{C}$ defined by

$$g(x) = h \circ L(x) \quad \text{for any } x \in \overline{B}_E.$$

We claim that g belongs to $A_\infty(B_E)$. Indeed, for $x \in \overline{B}_E$, there exists $\delta_x > 0$ such that $|L(x)| < 1 - \delta_x$. Then,

$$|L(y)| < 1 \text{ for any } y \in B(x, \delta_x)$$

so g is analytic on $B(x, \delta_x)$ for any $x \in \overline{B}_E$. Therefore, g is analytic and bounded on an open neighbourhood of $\overline{B}_E$ and, in particular, $g \in A_\infty(B_E)$. Thus (x_{n_k}) is interpolating for $A_\infty(B_E)$. □

From this result we obtain the following corollary which characterizes the separability of $A_\infty(B_E)$ in terms of the finite dimension of E.

Corollary 3.3.4. *Let E be a complex Banach space. The following assertions are equivalent:*

i) *$A_\infty(B_E)$ is non-separable.*
ii) *E is infinite dimensional.*
iii) *$A_\infty(B_E)$ contains interpolating sequences.*

Proof. If E is finite-dimensional, then $A_\infty(B_E) = A_u(B_E)$ is separable. If E is infinite dimensional, by Theorem 3.3.3 we have that there exist interpolating sequences for $A_\infty(B_E)$. If an algebra contains an interpolating sequence, then it is not separable. □

3.3.2 Separability in $A_u(B_E)$

There are several conditions on E which assure that the algebra $A_u(B_E)$ is separable. In this section we will recall these conditions and will give a rather trivial characterization of it that will lead us to study the metrizability of the polynomial topologies.

Notice that a necessary condition to assure that $A_u(B_E)$ is separable is the separability of E^* since this is a closed subspace of $A_u(B_E)$.

It is known (see the proof of $b \to c$ in Proposition 3.2.19) that if $\ell_1 \subset E$, then, there exists an interpolating sequence for $A_u(B_E)$ and, therefore, the algebra $A_u(B_E)$ is not separable. The converse is false. Indeed, given $1 < p < \infty$ and the algebra $A = A_u(B_{\ell_p})$, we have that the canonical basis (e_n) is an interpolating sequence for A but $\ell_1 \not\subset \ell_p$ since ℓ_p is reflexive. This example also shows that the non containment of ℓ_1 copies is not sufficient to assure that $A_u(B_E)$ is separable.

It is also clear that $A_u(B_E)$ is separable if E^* is separable and $P_f(E)$ is dense in $A_u(B_E)$. This condition is satisfied by c_0 and the Tsirelson space T^* by Proposition 1.5.1 and 1.6.4.

In order to give other conditions for $A = A_u(B_E)$ to be separable, we will deal with the metrizability of the spectrum M_A for the polynomial topology. It is well-known that a Banach algebra is separable if and only if M_A is metrizable. Indeed, it is sufficient to notice that M_A is a subset of $\overline{B}_{A^*}$ and the Gelfand topology in M_A is the restriction of the $w(A^*,A)$-topology to M_A. From this, we obtain the following result,

Proposition 3.3.5. *Let E be a complex Banach space and $A = A_u(B_E)$. Then, A is separable if and only if M_A is $w(A^*,P(E))$-metrizable.*

Proof. As we have mentioned above, A is separable if and only if M_A is $w(A^*,A)$-metrizable. Since $P(E)$ is a dense set in $A_u(B_E)$, the Hausdorff topology $w(A^*,P(E))$ coincides on M_A with the finest compact $w(A^*,A)$. □

This proposition leads us to study the metrizability for the polynomial topology of bounded sets of M_A, in particular those of B_E. If we replace M_A by a smaller set L in Proposition 3.3.5, the metrizability may not lead to the separability for the seminorm $\|\cdot\|_L$. Indeed, the following example shows that there exist Banach spaces E and closed bounded separable subsets $L \subset B_E$ such that L is $\tau(E,P(^nE))$-metrizable but $(P^n(E),\|\cdot\|_L)$ is not separable. Indeed, consider $E = \ell_2$ and $L = \{e_n/2 : n \in \mathbb{N}\}$. It is clear that L is a closed bounded separable $\tau(\ell_2,P(^2\ell_2))$-metrizable subset of B_{ℓ_2} since it has no cluster points. However,

$(P(^2\ell_2), \|\cdot\|_L)$ is not separable since it contains ℓ_∞. Indeed, set $\alpha = (\alpha_n) \in \ell_\infty$ and consider the polynomial $P_\alpha \in P(^2\ell_2)$ given by

$$P_\alpha(x) = 4\sum_{n=1}^{\infty} \alpha_n x_n^2.$$

The map $T : \ell_\infty \longrightarrow (P(^2\ell_2), \|\cdot\|_L)$ given by $T(\alpha) = P_\alpha$ is a linear isometry since

$$\|T(\alpha)\|_L = \sup_{n\in\mathbb{N}} \|\alpha\|_\infty.$$

At least, the metrizability for the polynomial topology of closed bounded absolutely convex separable subsets L does lead to the separability of the dual space $(E^*, \|\cdot\|_L)$.

Proposition 3.3.6. *Let E be a real or complex Banach space and L a closed bounded absolutely convex separable set in E which is $\tau(E, P(E))$-metrizable. Then, $(E^*, \|\cdot\|_L)$ is separable.*

Proof. Since L is a bounded set, we can assume without loss of generality that $L \subset B_E$. Since L is $\tau(E, P(E))-$metrizable, there exists a countable basis of $\tau(E, P(E))-$neighbourhoods (V_n) of 0 in L. Therefore, there exists a sequence (F_n) of finite subsets of $P(E)$ such that the sequence (V_n) is given by

$$V_n = \{x \in L : |P(x) - P(0)| \leq 1 \text{ for any } P \in F_n\}.$$

Set $G_n = \{dP_x : x \in L,\ P \in F_n\}$. Since the mapping $x \in E \mapsto dP_x \in E^*$ is continuous, G_n is a norm separable set in E^* for all $n \in N$. Let G be the norm closure in E^* of $span(\cup_{n=1}^{\infty} G_n)$. It is clear that G is a separable subspace of E^* and G is $\|\cdot\|_L$-separable as well.

We want to prove that $E^* = G + L^\circ$. For this, let $g \in E^*$ and consider U the $\tau(E, P(E))$-neighbourhood of 0 defined by

$$U = \{x \in L : |g(x)| \leq 1\}.$$

Then, we find $m \in \mathbb{N}$ such that $V_m \subset U$. Moreover, we have that $G_m^\circ \cap L \subset \{g\}^\circ$. Indeed, for $z \in G_m^\circ \cap L$, it follows from an application of the mean value theorem to each of the polynomials $P \in F_m$ in the segment $[0, z] \subset L$, that there exists $0 < t < 1$ so that $|P(z) - P(0)| \leq |dP_{tz}(z)| \leq 1$ since $tz \in L$. Therefore, $|P(z) - P(0)| \leq 1$ for all $P \in F_m$, that is, $z \in V_m$. Hence, $|g(z)| \leq 1$ and, therefore, $z \in \{g\}^\circ$.

Further, since $\overline{\Gamma(G_m)}^{w*}$, the $w(E^*, E)-$closure of the absolutely convex hull of G_m, is also a $w(E^*, E)-$compact set and L° is a $w(E^*, E)-$closed set, their sum is an absolutely convex $w(E^*,E)-$closed set to which g belongs to, since, otherwise, if $g \notin \overline{\Gamma(G_m)}^{w*} + L^{\circ}$, we may appeal to the Hahn-Banach theorem to get some $x \in E$ such that $|g(x)| > 1$ and $|\phi(x)| < 1$ for all $\phi \in \overline{\Gamma(G_m)}^{w*} + L^{\circ}$, that is $x \in L^{\circ\circ} = L$, and $x \in G_n^{\circ}$, contradicting the relation $G_m^{\circ} \cap L \subset \{g\}^{\circ}$. Hence, $g \in G + L^{\circ}$.

Finally, for any $p \in \mathbb{N}$, we have that $pg \in G + L^{\circ}$, so there are $\alpha \in G$ and $\beta \in L^{\circ}$ such that $pg = \alpha + \beta$, so

$$\left\| g - \frac{\alpha}{p} \right\|_L = \left\| \frac{\beta}{p} \right\|_L \leq \frac{1}{p}.$$

Therefore, G is $\|\cdot\|_L$-dense in E^*, thus $(E^*, \|\cdot\|_L)$ is a separable space. □

The separability of L cannot be removed in Proposition 3.3.6 as we show in the following example.

Consider the real Hilbert space $H = \ell_2(\mathbb{R})$. We have that a net (x_i) which $\tau(H, P(H))-$converges to x is also norm convergent since the expression $\|x_i - x\|^2 = \|x_i\|^2 + \|x\|^2 - 2 < x_i, x >$ tends to 0 because $< \cdot, x >$ and the norm are actually polynomials. Therefore, the norm and the polynomial topology coincide and we obtain that the unit ball B_H is $\tau(H, P(H))-$metrizable. Nevertheless, we have that $(\ell_2(\mathbb{R})^*, \|\cdot\|_{B_H}) = \ell^2(\mathbb{R})$, which is not separable.

Chapter 4

Applications to Composition Operators on $H^\infty(B_E)$

In this chapter, we will deal with composition operators C_φ on $H^\infty(B_E)$. First, we will deal with the spectra of composition operators on $H^\infty(B_E)$; results on interpolation studied in Chapter 3 will be used to describe the spectra of such operators. Then, we will study the class of Radon-Nikodým composition operators. In order to do this, we will give some results on Asplund sets, which are deeply related to this class of operators.

4.1 A Lemma

The next lemma is closely related to Proposition 2 in [GLR99]. We prove it as an application of Corollary 3.2.16.

Lemma 4.1.1. *Let $\varphi : B_E \longrightarrow B_F$ an analytic map and suppose that there is no $0 < r < 1$ such that $\varphi(B_E) \subset rB_F$. Then, there exist linear operators $T : H^\infty(B_E) \longrightarrow \ell^\infty$ and $S : \ell^\infty \longrightarrow H^\infty(B_F)$ such that*

$$T \circ C_\varphi \circ S = Id_{\ell^\infty}.$$

Proof. Given C_φ, we find, by hypothesis, a sequence $(x_n) \subset B_E$ such that

$$\lim_{n\to\infty} \|\varphi(x_n)\| = 1.$$

It is easy to find a subsequence of $(\varphi(x_n))$ which converges fast enough in order to satisfy the generalized Hayman-Newman condition. We still denote this subsequence by $(\varphi(x_n))$. By Corollary 3.2.17, the sequence $(\varphi(x_n))$ is interpolating for $A = H^\infty(B_F)$. In particular, there exists a sequence $(f_k) \subset H^\infty(B_F)$ such that

$$f_k(\varphi(x_n)) = \delta_{kn} \quad \text{for all } n, k \in \mathbb{N}$$

and

$$\sum_{n=1}^{\infty} |f_n(x)| \leq M \quad \text{for all } x \in M_A.$$

Define the operator $S : \ell^\infty \longrightarrow H^\infty(B_F)$ by $S((\alpha_k)) = \sum_{k=1}^{\infty} \alpha_k f_k$ and the operator $T : H^\infty(B_E) \longrightarrow \ell^\infty$ by $T(f) = (f(x_n))_{k=1}^{\infty}$, which is well-defined, linear and continuous. It is clear that

$$T \circ C_\varphi \circ S((\alpha_k)_k) = T \circ C_\varphi(\sum_{k=1}^{\infty} \alpha_k \circ f_k) =$$

$$T(\sum_{k=1}^{\infty} \alpha_k \circ f_k \circ \varphi) = \left(\sum_{k=1}^{\infty} (\alpha_k \circ f_k \circ \varphi)(x_n)\right) = (\alpha_n)_n$$

and, therefore, $T \circ C_\varphi \circ S = Id_{\ell^\infty}$. □

4.2 Spectra of Composition Operators on $H^\infty(B_E)$

In the study of the spectra of composition operators C_φ on $H^\infty(B_E)$, we are led to consider the sequence of the iterates of φ, that is, the sequence (φ_n) where $\varphi_n : B_E \longrightarrow B_E$ is the map given by

$$\varphi_n = \varphi \circ \overset{n}{\cdots} \circ \varphi.$$

If C_φ is power compact, the operator C_φ^n is compact for some $n \in \mathbb{N}$. It is clear that $C_\varphi^n = C_{\varphi_n}$ and, hence, by Theorem 1.8.3, we have that there exists $0 < r < 1$ such that $\varphi_n(B_E) \subset rB_E$. If E is finite dimensional, it is clear that the converse is also true since $\varphi(B_E)$ is always relatively compact. Nevertheless, this fails in the infinite dimensional case. Just consider $\varphi(x) = x/2$ for which $\varphi_n(x) = x/2^n$, and thus C_φ^n is not compact. Indeed, we need in addition to consider $\varphi(B_E)$ to be a relatively compact set in B_E when E is infinite dimensional.

We introduce the following notation,

Definition 4.2.1. *Let $\varphi : B_E \longrightarrow B_E$ an analytic map. A finite or infinite sequence $(x_k)_{k\geq 0} \subset B_E$ is said to be an* iteration sequence *for φ if $\varphi(x_k) = x_{k+1}$.*

H. Kamowitz studied the spectrum of composition operators on $A(\mathbf{D})$ and H^p in [Kam73] and [Kam75]. L. Zheng completed the description of the spectrum of composition operators $C_\varphi : H^\infty \longrightarrow H^\infty$ under the assumption that the symbol φ has an interior fixed point.

Theorem 4.2.2. *Let $\varphi : \mathbf{D} \longrightarrow \mathbf{D}$ be a non constant, analytic self-map and suppose that φ is not an automorphism. If there is $z_0 \in \mathbf{D}$ such that $\varphi(z_0) = z_0$, then either*

$$\sigma(C_\varphi) = \overline{\mathbf{D}}, \qquad \text{if } C_\varphi \text{ is not power compact.}$$

or

$$\sigma(C_\varphi) = \left\{\varphi'(z_0)^k : k \in \mathbb{N}\right\} \cup \{0,1\} \quad \text{if } C_\varphi \text{ is power compact.}$$

P. Galindo, T. W. Gamelin and M. Lindström extended Theorem 4.2.2 to Banach spaces for power compact composition operators $C_\varphi : H^\infty(B_E) \longrightarrow H^\infty(B_E)$ in [GGL08]. Indeed, they described the spectrum of C_φ as follows,

Theorem 4.2.3. *If C_φ is power compact, then φ has a fixed point $x_0 \in B_E$ such that $\|\varphi'(x_0)\| < 1$ and the spectrum of C_φ consists of $\lambda = 0$ and $\lambda = 1$, together with all possible products $\lambda = \lambda_1 \cdots \lambda_k$, where $k \geq 1$ and the λ_j's are eigenvalues of $\varphi'(x_0)$.*

P. Galindo, T. W. Gamelin and M. Lindström [GGL08] proved that the result given by L. Zheng for the non power compact case also remains true for the Euclidean unit ball of $\mathbb{C}^n$ (see Theorem 4.2.7 below). So we are led to study spectra of non power compact composition operators on $H^\infty(B_E)$. Just bearing in mind the finite dimensional case, we are led, by Theorem 1.8.3 a), to the case that $\varphi_n(B_E)$ is not strictly inside B_E for all $n \in \mathbb{N}$ since, otherwise, C_φ is power compact because $\varphi(B_E)$ is relatively compact in E. For simplicity, we will refer to such situation by saying that φ satisfies the approaching condition,

Definition 4.2.4. *Let $\varphi : B_E \longrightarrow B_E$ be an analytic map. We say that $\varphi : B_E \longrightarrow B_E$ satisfies the* approaching condition *if $\varphi_n(B_E)$ is not strictly inside B_E for any $n \in \mathbb{N}$.*

Notice that Theorem 1.8.3 a) also tells us that whenever $\varphi(B_E)$ is relatively compact, C_φ is non power compact if and only if φ satisfies the approaching condition.

We recall some useful calculations:

Suppose that the symbol $\varphi : B_E \longrightarrow B_E$ satisfies $\varphi(0) = 0$ and $\|\varphi'(0)\| < 1$. Consider the analytic function $h : \mathbf{D} \longrightarrow \mathbf{D}$ given by

$$h(\lambda) = L(\varphi(\lambda w))/\lambda,$$

where $w \in E$ and $L \in E^*$ satisfy $\|w\| = ||L|| = 1$. It is easy that each such h satisfies $|h| \leq 1$ and $|h(0)| \leq \|\varphi'(0)||$. A normal families argument shows that for each $s < 1$, there exists $a < 1$ such that any such h satisfies $|h(\lambda)| \leq a$ for $|\lambda| \leq s$. Taking the supremum over L and setting $x = \lambda w$, we obtain

$$\|\varphi(x)\| \leq a\|x\|, \quad \text{for } x \in E, \quad \|x\| \leq s. \tag{4.1}$$

Hence,

$$\frac{1 - \|\varphi(x)\|}{1 - \|x\|} \geq \frac{1 - a\|x\|}{1 - \|x\|}, \quad \text{for } x \in B_E, \quad 0 < \|x\| < s, \tag{4.2}$$

so given $0 < r < s < 1$, there exists $\varepsilon > 0$ such that

$$\frac{1 - \|\varphi(x)\|}{1 - \|x\|} \geq 1 + \varepsilon, \quad x \in B_E, \quad r < \|x\| < s.$$

When we deal with the open unit disk $\mathbf{D}$ this estimate remains valid even as $||x|| \to 1$ (see Lemma 7.33 in [CM95]),

Lemma 4.2.5. *If φ is an analytic map, not an automorphism, of the unit disk into itself and $\varphi(0) = 0$, then for $0 < r < 1$, there is $\varepsilon > 0$ such that*

$$\frac{1 - |\varphi(z)|}{1 - |z|} > 1 + \varepsilon \quad \textit{for all } z \textit{ with } |z| \geq r.$$

This result is typically obtained by using Julia's lemma (see Lemma 2.41 in [CM95]) and angular derivatives. This kind of estimate will be called *Julia-type estimate*.

We say that a subset $W \subset B_E$ *approaches S_E compactly* if any sequence $(z_n) \subset W$ such that $\|z_n\| \to 1$ has a convergent subsequence.

The following Julia-type estimate for Hilbert spaces was shown in [GGL08],

Theorem 4.2.6. *Let H be a Hilbert space and $\varphi : B_H \longrightarrow B_H$ an analytic map satisfying $\varphi(0) = 0$ and $\|\varphi'(0)\| < 1$. Suppose that W approaches S_H compactly. Then, for any $0 < \delta < 1$, there exists $\varepsilon > 0$ such that*

$$\frac{1 - \|\varphi(x)\|}{1 - \|x\|} \geq 1 + \varepsilon, \qquad \textit{for all } x \in W \textit{ such that } \|x\| \geq \delta.$$

This Julia-type estimate allowed them to conclude the result mentioned above which extends Theorem 4.2.2 in the non power compact case to Hilbert spaces,

Theorem 4.2.7. *Let H be a Hilbert space. Let $\varphi : B_H \longrightarrow B_H$ an analytic map satisfying $\varphi(0) = 0$, $\|\varphi'(0)\| < 1$ and $\varphi(B_H)$ is a relatively compact subset of H. Suppose that φ satisfies the approaching condition. Then, the spectrum of C_φ coincides with the closed unit disk, that is, $\sigma(C_\varphi) = \overline{\mathbb{D}}$.*

4.2.1 Results on the spectrum of C_φ

In this section, we aim to extend Theorem 4.2.7 under the assumption that φ satisfies a Julia-type estimate as in Theorem 4.2.6.

In [Zhe02] and [GGL08], an interpolation result is necessary in order to get the description of the spectrum of C_φ (see also Lemma 7.34 in [CM95]). The interpolation result given in [GGL08] uses the Julia-type estimate 4.2.6 and states as follows,

Proposition 4.2.8. *Let H be a Hilbert space, and let $\varphi : B_H \longrightarrow B_H$ be an analytic map such that $\varphi(0) = 0$ and $\|\varphi'(0)\| < 1$ and suppose that $\varphi(B_H)$ is a relatively compact subset of H. Let $\delta > 0$. Then there exists a constant $M \geq 1$ which depends only on δ such that any finite iteration sequence $\{z_0, z_1, \dots, z_N\}$ satisfying $z_0 \in \varphi(B_H)$ and $||z_N|| \geq \delta$ is an interpolating sequence for $H^\infty(B_H)$ with constant of interpolation not greater than M.*

When we deal with Banach spaces, under the assumption that φ satisfies a Julia-type estimate, we will also get an interpolation result which extends Proposition 4.2.8 in order to give a Theorem which describes the spectrum of C_φ for the non power compact case. This interpolation result will be proved by appealing to Corollary 3.2.16,

Lemma 4.2.9. *Let E be a Banach space and let $\varphi : B_E \longrightarrow B_E$ be an analytic self-map such that $\varphi(0) = 0$ and $\|\varphi'(0)\| < 1$. Suppose that there exist $\delta > 0$ and $\varepsilon > 0$ such that*

$$\frac{1 - \|\varphi(x)\|}{1 - \|x\|} \geq 1 + \varepsilon, \quad \text{for all } x \in \varphi(B_E) \text{ such that } \|x\| \geq \delta. \tag{4.3}$$

Then, there exists a constant $M \geq 1$ which depends only on ε, such that any finite iteration sequence $\{x_0, x_1, \dots, x_N\}$ satisfying $x_0 \in \varphi(B_E)$ and $\|x_N\| \geq \delta$ is an interpolating sequence for $H^\infty(B_E)$ with constant of interpolation not greater than M.

Proof. Let $(x_k)_{k=0}^N$ be a finite iteration sequence satisfying $x_0 \in \varphi(B_E)$ and $\|x_N\| \geq \delta$. Recall that $x_{k+1} = \varphi(x_k)$ for any $0 \leq k \leq N-1$ and consider the sequence

$$\{x_N, x_{N-1}, \dots, x_1, x_0\}.$$

By inequality 4.3, we have that

$$\frac{1 - \|x_k\|}{1 - \|x_{k+1}\|} = \frac{1 - \|x_k\|}{1 - \|\varphi(x_k)\|} \leq \frac{1}{1+\varepsilon} \text{ for any } 0 \leq k \leq N-1.$$

Then, the assumption in Corollary 3.2.16 is satisfied by the finite sequence $\{x_N, x_{N-1}, \dots, x_0\}$ (note the reversal of the order). Thus, this sequence is interpolating and its constant of interpolation depends only on ε. □

In addition, we will need the following lemmas.

The first one is an improvement of Lemma 7.17 in [CM95].

Lemma 4.2.10. *Let E and F be Banach spaces. Let $C : E \oplus F \longrightarrow E \oplus F$ be a linear operator which leaves F invariant and for which $C|_E : E \longrightarrow E \oplus F$ is a compact operator. If the operator C has the matrix representation*

$$C = \begin{pmatrix} X & 0 \\ Y & Z \end{pmatrix} \tag{4.4}$$

with respect to this descomposition, then $\sigma(C) = \sigma(X) \cup \sigma(Z)$.

Proof. Let $\lambda \notin \sigma(C)$ and suppose that

$$(C - \lambda I)^{-1} = \begin{pmatrix} T & R \\ U & V \end{pmatrix}. \tag{4.5}$$

Then,

$$\begin{pmatrix} X - \lambda I_{11} & 0 \\ Y & Z - \lambda I_{22} \end{pmatrix} \begin{pmatrix} T & R \\ U & V \end{pmatrix} = \begin{pmatrix} I_{11} & 0 \\ 0 & I_{22} \end{pmatrix} \tag{4.6}$$

which implies that $(X - \lambda I_{11})T = I_{11}$. If $\lambda \neq 0$, since $X : E \longrightarrow E$ is compact, the Fredholm alternative 1.1.4 holds, so $X - \lambda I_{11}$ is surjective if and only if $X - \lambda I_{11}$ is injective; hence $X - \lambda I_{11}$ is invertible. If $\lambda = 0$, then I_{11} is a compact operator, so E is finite dimensional, and again $X - \lambda I_{11}$ is invertible. Thus in any case we obtain that $R = 0$. This gives that $(Z - \lambda I_{22})V = I_{22}$. Multiplying the opposite order gives that $V(Z - \lambda I_{22}) = I_{22}$, so the operator $Z - \lambda I_{22}$ is invertible and obtain

$$\sigma(X) \cup \sigma(Z) \subset \sigma(C).$$

The converse inclusion is proved as in Lemma 7.17 in [CM95]. We prove it by the sake of completeness. If X and Z are invertible, it is easy to see that

$$\begin{pmatrix} X & 0 \\ Y & Z \end{pmatrix}^{-1} = \begin{pmatrix} X^{-1} & 0 \\ -Z^{-1}YX^{-1} & Z^{-1} \end{pmatrix},$$

so C is invertible. □

Denote by $P_n f$ the n-th term of the Taylor series of the analytic function $f \in H^\infty(B_E)$ at 0. Set

$$H^\infty_m(B_E) = \{f \in H^\infty(B_E) : P_n f = 0 \text{ for } n = 0, 1, \ldots, m-1\}.$$

In other words, a function in $H^\infty(B_E)$ belongs to $H_m^\infty(B_E)$ if the first $m-1$ terms of its Taylor series at 0 vanish. Denoting by $P(^{<m}E)$ the subspace of polynomials of degree less than m, it is clear that $H^\infty(B_E)$ is isomorphic to $H_m^\infty(B_E) \oplus P(^{<m}E)$.

In order to apply Lemma 4.2.10, we show that C_φ leaves invariant the space $H_m^\infty(B_E)$ under the assumption that $\varphi(0) = 0$.

Lemma 4.2.11. *Let $\varphi : B_E \longrightarrow B_E$ be an analytic map such that $\varphi(0) = 0$. Then C_φ leaves invariant the space $H_m^\infty(B_E)$.*

Proof. Let $f \in H_m^\infty(B_E)$ and fix $x \in B_E$. It is easy, from the Taylor series expansion of φ at 0, that the function $g : \mathbf{D} \longrightarrow \mathbb{C}$ defined by $g(\lambda) = \varphi(\lambda x)$ satisfies $g(\lambda) = \lambda h_x(\lambda)$ for a particular analytic function h_x which depends on x and λ. Set $f \in H_m^\infty(B_E)$. We have that

$$(f \circ \varphi)(\lambda x) = \sum_{n\geq m} P_n f(\varphi(\lambda x)) = \sum_{n\geq m} P_n f(\lambda h_x(\lambda)) = \sum_{n\geq m} \lambda^n P_n f(h_x(\lambda))$$

and there is no non-null term of degree less than m in this series expansion. Therefore, if $\sum_n Q_n$ is the Taylor series of $f \circ \varphi$, there must be no non-null term of degree less than m in

$$\sum_n Q_n(\lambda x) = \sum_n \lambda^n Q_n(x)$$

since both polynomials on λ, $\sum_n \lambda^n Q_n(x)$ and $\sum_{n\geq m} \lambda^n P_n f(h_x(\lambda))$ must be the same. Thus $Q_n(x) = 0$ for $n = 0, 1, \ldots, m-1$ and, therefore, $C_\varphi(f) \in H_m^\infty(B_E)$. □

Then, we have the main result which describes the spectrum of C_φ for the non power compact case. The proof is an adaptation of Theorem 4.2.7.

Theorem 4.2.12. *Let $\varphi : B_E \longrightarrow B_E$ be an analytic map satisfying $\varphi(0) = 0$, $\|\varphi'(0)\| < 1$ such that $\varphi(B_E)$ is a relatively compact subset of E. Suppose that φ satisfies the approaching condition and the following Julia-type estimate: For any $0 < \delta < 1$, there exists $\varepsilon > 0$ such that*

$$\frac{1 - \|\varphi(x)\|}{1 - \|x\|} \geq 1 + \varepsilon, \quad \textit{for all } x \in \varphi(B_E) \textit{ such that } \|x\| \geq \delta. \tag{4.7}$$

Then, the spectrum of C_φ coincides with the closed unit disk $\overline{\mathbf{D}}$.

Proof. Since the spectrum $\sigma(C_\varphi)$ is a closed subset of $\overline{\mathbf{D}}$, it is sufficient to prove that $\mathbf{D} \setminus \{0\} \subset \sigma(C_\varphi)$. Therefore, we deal with $\lambda \in \mathbf{D}$ such that $0 < |\lambda| < 1$.

Notice that for every $m \in \mathbb{N}$, each norm bounded subset of $P(^mE)$ is relatively compact for the compact-open topology by Montel theorem. Since $\overline{\varphi(B_E)}$ is a compact set in E, we conclude that the linear operator

$$C_\varphi|_{P(^{<m}E)} : P(^{<m}E) \longrightarrow H^\infty(B_E)$$

is compact by Theorem 1.8.3 a).

As we mentioned above, for any $m \in \mathbb{N}$ we have that

$$H^\infty(B_E) = P(^{<m}E) \oplus H^\infty_m(B_E).$$

Let C_m denote the restriction of C_φ to $H^\infty_m(B_E)$, which is an invariant subspace of C_φ by Lemma 4.2.10. Then, if $\lambda \in \sigma(C_m)$ for some $m \in \mathbb{N}$, we will get $\lambda \in \sigma(C_\varphi)$ by Lemma 4.2.11.

Notice that $\lambda \in \sigma(C_m)$ if $C_m - \lambda I$ is not invertible and, for this, it is sufficient to show that $(C_m - \lambda I)^*$ is not bounded from below, so we will prove this assertion.

Given an iteration sequence $(x_k)_{k=0}^\infty$, in view of inequality 4.1, there exists $0 < a < 1$ such that

$$\|\varphi(x)\| \leq a\|x\| \quad \text{for any } \|x\| \leq \|x_0\|. \tag{4.8}$$

Therefore, we get

$$\|x_k\| = \|\varphi_k(x_0)\| \leq a^k \|x_0\|$$

so the norms of the elements of any iteration sequence $\{x_0, x_1, \dots,\}$ decrease to 0.

Fix $0 < \delta < 1$. For any iteration sequence $(x_k)_{k=0}^\infty \subset \varphi(B_E)$, we denote by $N = N(x_0)$ the largest integer such that $\|x_N\| > \delta$. Since the approaching condition guarantees that $\varphi_k(B_E)$ is not contained in the ball δB_E for all $k \geq 1$, we have that for any arbitrarily large $N \in \mathbb{N}$, we can find $x_0 \in B_E$ such that $N = N(x_0)$.

Since $0 < \sqrt{\delta} < 1$, by inequality 4.1, there exists $0 < c < 1$ satisfying

$$\|\varphi(x)\| \leq c\|x\|, \quad \text{for any } \|x\| \leq \sqrt{\delta}. \tag{4.9}$$

We can suppose without loss of generality that c is close to 1, so we choose c such that $c \geq \sqrt{\delta}$. In addition, we can assume that $c = 1/(1+\varepsilon)$ and get

$$\frac{1 - \|x\|}{1 - \|\varphi(x)\|} \leq c \text{ for any } \|x\| \geq \delta \tag{4.10}$$

since we can take ε closer to 0 and inequality 4.7 remains true.

We claim that for any such interation sequence $(x_k)_{k\geq 0}^{\infty}$ we have

$$\|x_{N+1}\| \leq c\|x_N\|. \tag{4.11}$$

Indeed, we have two possibilities. If $\|x_N\| \leq \sqrt{\delta}$, it is exactly 4.9; if otherwise $\|x_N\| > \sqrt{\delta}$, inequality 4.11 is also satisfied since, otherwise, we have that

$$\|x_{N+1}\| > c\|x_N\| \geq \sqrt{\delta}\sqrt{\delta} = \delta,$$

a contradiction since N is the last index k such that $\|x_k\| > \delta$.

For any $n > N+1$, we also have that $\|x_{n+1}\| \leq c\|x_n\|$ since $\|x_n\| \leq \|x_N\|$, so we obtain by induction that

$$\|x_{N+k}\| \leq c^k\|x_N\| \quad \text{for any } k \geq 0.$$

Let $\{x_k\}_{k=0}^{\infty}$ be an iteration sequence in $\varphi(B_E)$. Notice that for $f \in H_m^\infty(B_E)$, we have that $|f(x)| \leq ||f||_\infty ||x||^m$ for all $x \in B_E$ by the maximum modulus principle. Hence, if we choose m so large that $c^m < |\lambda|$, we obtain

$$\sum_{k=N+1}^{\infty} \frac{|f(x_k)|}{|\lambda|^{k+1}} \leq \sum_{k=N+1}^{\infty} \frac{\|f\|_\infty \|x_k\|^m}{|\lambda|^{k+1}} \leq \|f\|_\infty \frac{\|x_N\|^m}{|\lambda|^{N+1}} \sum_{k=1}^{\infty} \frac{c^{km}}{|\lambda|^k}.$$

Therefore, we define the linear functional L on $H_m^\infty(B_E)$ given by

$$L(f) = \sum_{k=0}^{\infty} \frac{f(x_k)}{\lambda^{k+1}}, \qquad f \in H_m^\infty(B_E)$$

and we obtain an estimate for the tail of the series,

$$\left| \sum_{k=N+1}^{\infty} \frac{f(x_k)}{\lambda^{k+1}} \right| \leq \|f\|_\infty \frac{\|x_N\|^m}{|\lambda|^{N+1}} \frac{c^m}{|\lambda| - c^m}, \qquad f \in H_m^\infty(B_E). \tag{4.12}$$

Now choose an $m-$homogeneous polynomial P satisfying $\|P\| = 1$ and $|P(x_N)| = \|x_N\|^m$. By Lemma 4.2.9, there is an interpolation constant $M = M(c)$ and $g \in H^\infty(B_E)$ such that $\|g\| \leq M$, $g(x_k) = 0$ for $0 \leq k < N$, and $g(x_N) = 1$. Then $P \cdot g \in H_m^\infty(B_E)$ satisfies $\|P \cdot g\| \leq M$, and using the estimate 4.12 for $f = P \cdot g$, we obtain

$$|L(P \cdot g)| = \left| \sum_{k=0}^{\infty} \frac{(P \cdot g)(x_k)}{\lambda^{k+1}} \right| = \left| \frac{(P \cdot g)(x_N)}{\lambda^{N+1}} + \sum_{k=N+1}^{\infty} \frac{(P \cdot g)(x_k)}{\lambda^{k+1}} \right| \geq$$

$$\left|\frac{(P\cdot g)(x_N)}{\lambda^{N+1}}\right| - \left|\sum_{k=N+1}^{\infty}\frac{(P\cdot g)(x_k)}{\lambda^{k+1}}\right| \geq \frac{||x_N||^m}{|\lambda|^{N+1}} - \frac{||x_N||^m}{|\lambda|^{N+1}}\frac{Mc^m}{|\lambda|-c^m}.$$

Since

$$\frac{Mc^m}{|\lambda|-c^m} \to 0 \text{ when } m \to \infty,$$

we can choose m so that, in addition to $c^m < |\lambda|$, we have

$$\frac{Mc^m}{|\lambda|-c^m} < \frac{1}{2}.$$

Then, since $|L(P\cdot g)| \leq M\|L\|$ and $\|x_N\| \geq \delta > 1/4$, we get

$$M\|L\| \geq |L(P\cdot g)| \geq \frac{\|x_N\|^m}{2|\lambda|^{N+1}} \geq \frac{1}{2\cdot 4^m|\lambda|^{N+1}}. \tag{4.13}$$

Next observe that for $f \in H_m^{\infty}(B_E)$,

$$((\lambda I - C_m)^*(L))(f) = \lambda L(f) - L(C_m(f)) = \lambda L(f) - L(f\circ\varphi) =$$

$$\lambda\sum_{k=0}^{\infty}\frac{f(x_k)}{\lambda^{k+1}} - \sum_{k=0}^{\infty}\frac{f(x_{k+1})}{\lambda^{k+1}} = f(x_0).$$

Hence

$$\|(\lambda I - C_m)^*(L)\| \leq 1. \tag{4.14}$$

As we have mentioned above, we can form iteration sequences for which N is arbitrarily large, hence by 4.13 for which $\|L\|$ is arbitrarily large. In view of 4.14, we show then that $(\lambda I - C_m)^*$ is not bounded below. Consequently $(\lambda I - C_m)^*$ is not invertible, and neither then is $\lambda I - C_m$, so that $\lambda \in \sigma(C_m)$. □

4.2.2 A Julia-Type Estimate for $C_0(X)$ spaces

In Theorem 4.2.12 we need to assume that a Julia-type estimate for E is satisfied in order to describe the spectrum of C_{φ}. As we recalled in Theorem 4.2.6, this estimate exists when we deal with Hilbert spaces. In this section we will give a Julia-type estimate for $C_0(X)$ spaces, in particular for the n-fold space $\mathbb{C}^n$ and c_0.

Recall that the pseudohyperbolic distance for $A = H^{\infty}(B_E)$ with $E = C_0(X)$, X a locally compact space, is given by formula 1.8:

$$\rho(x,y)=\sup_{t\in X}\rho(x(t),y(t)) \quad \text{for all } x,y\in B_E. \tag{4.15}$$

We will also need the following lemma,

Lemma 4.2.13. *The following holds:*

a) Let $\{\alpha_i : i\in I\}$ be a set of real numbers such that $0<\alpha_i<1$. Then,

$$\sup_{i\in I}\left\{\frac{1}{1-\alpha_i^2}\right\}=\frac{1}{1-\sup_{i\in I}\alpha_i^2}. \tag{4.16}$$

b) The function $h(x)=(1-x)/(1+x)$ is decreasing in $[0,1)$.

Then, we state the main result,

Theorem 4.2.14. *Let $E=C_0(X)$ and consider an analytic map $\varphi : B_E \longrightarrow B_E$ such that $\varphi(0)=0$, $\|\varphi'(0)\|<1$. Suppose that W approaches S_E compactly. Then, the following Julia-type estimate holds: For any $\delta>0$, there exists $\varepsilon>0$ such that*

$$\frac{1-\|\varphi(x)\|}{1-\|x\|}\geq 1+\varepsilon \quad \textit{for any } x\in W \textit{ such that } \|x\|\geq\delta. \tag{4.17}$$

Proof. Suppose that the estimate (4.17) fails. Then, we can choose a sequence $(x_n)\subset W$, $\|x_n\|\geq\delta$ such that

$$\frac{1-\|\varphi(x_n)\|}{1-\|x_n\|}\to 1.$$

In particular, $\|x_n\|\to 1$ and $\|\varphi(x_n)\|\to 1$ when $n\to\infty$. By the assumption on W, we may suppose, passing to a subsequence if necessary, that $x_n\to x_0$, for $x_0\in S_E$.

Fix $z\in B_E$. We denote $\varphi_t(x)=\varphi(x)(t)$ for $x\in B_E$ and $t\in X$. Then, by 4.15 and 1.6, we have that

$$\frac{1-\|\varphi(x_n)\|^2}{1-\rho^2(\varphi(z),\varphi(x_n))}=\sup_{t\in X}\left\{\frac{|1-\overline{\varphi_t(z)}\varphi_t(x_n)|^2}{1-|\varphi_t(z)|^2}\frac{1-\|\varphi(x_n)\|^2}{1-|\varphi_t(x_n)|^2}\right\}. \tag{4.18}$$

Since

$$\frac{|1-\overline{\varphi_t(z)}\varphi_t(x_n)|^2}{1-|\varphi_t(z)|^2}\geq\frac{(1-|\varphi_t(z)|)^2}{1-|\varphi_t(z)|^2}=\frac{1-|\varphi_t(z)|}{1+|\varphi_t(z)|}\geq\frac{1-||\varphi(z)||}{1+||\varphi(z)||}$$

we have that

$$\sup_{t\in X}\left\{\frac{|1-\overline{\varphi_t(z)}\varphi_t(x_n)|^2}{1-|\varphi_t(z)|^2}\frac{1-\|\varphi(x_n)\|^2}{1-|\varphi_t(x_n)|^2}\right\}\geq\frac{1-||\varphi(z)||}{1+||\varphi(z)||}\sup_{t\in X}\frac{1-\|\varphi(x_n)\|^2}{1-|\varphi_t(x_n)|^2}.$$

Therefore, by (4.18) and using Lemma 4.2.13 a), we obtain

$$\frac{1-\|\varphi(x_n)\|^2}{1-\rho^2(\varphi(z),\varphi(x_n))}\geq\frac{1-||\varphi(z)||}{1+||\varphi(z)||}\sup_{t\in X}\frac{1-\|\varphi(x_n)\|^2}{1-|\varphi_t(x_n)|^2}=\frac{1-||\varphi(z)||}{1+||\varphi(z)||}. \tag{4.19}$$

On the other hand, using the contraction property 1.4 we have

$$\frac{1-\|\varphi(x_n)\|^2}{1-\rho^2(\varphi(z),\varphi(x_n))}\leq\frac{1-\|\varphi(x_n)\|^2}{1-\rho^2(z,x_n)}\leq$$

$$\frac{1-\|\varphi(x_n)\|^2}{1-\|x_n\|^2}\sup_{t\in X}\left\{\frac{|1-\overline{z(t)}x_n(t)|^2}{1-|z(t)|^2}\frac{1-\|x_n\|^2}{1-|x_n(t)|^2}\right\}. \tag{4.20}$$

Fix $0<r<1$ and set $z=rx_0$. Consider for any $n\in\mathbb{N}$ the continuous function $g_n : X\longrightarrow\mathbb{R}$ defined by

$$g_n(t)=\frac{|1-\overline{rx_0(t)}x_n(t)|^2}{1-r^2|x_0(t)|^2}.$$

It is easy to prove that g_n converges uniformly to the function $g : X\longrightarrow\mathbb{R}$ given by

$$g(t)=\frac{(1-r|x_0(t)|^2)^2}{1-r^2|x_0(t)|^2}$$

so passing to a subsequence we can consider that

$$g_n(t)\leq g(t)+\varepsilon \text{ for all } t\in X. \tag{4.21}$$

Consider the continuous function $h : [0,1]\longrightarrow\mathbb{R}$ given by

$$h(u)=\frac{(1-ru^2)^2}{1-r^2u^2}.$$

We have that for $\varepsilon>0$ there exists $0<u_0<1$ such that

$$h(u)\leq\frac{1-r}{1+r}+\varepsilon \text{ for any } u_0\leq u<1. \tag{4.22}$$

Therefore, by (4.21) and (4.22) we have

$$g_n(t) \leq \frac{1-r}{1+r} + 2\varepsilon \text{ for any } |x_0(t)| \geq u_0.$$

Consider the supremum from inequality (4.20) and pay attention to items with $|x_0(t)| < u_0$. We have

$$\frac{|1-\overline{x_0(t)}x_n(t)|^2}{1-r^2|x_0(t)|^2}\frac{1-\|x_n\|^2}{1-|x_n(t)|^2} \leq \frac{|1-r\overline{x_0(t)}x_n(t)|^2}{1-r^2|x_0(t)|^2}\frac{1-\|x_n\|^2}{1-u_0^2} \leq \frac{4(1-\|x_n\|^2)}{(1-u_0^2)(1-r^2)}.$$

On the other hand, if we pay attention to items with $|x_0(t)| \geq u_0$ we obtain that

$$\frac{1-\|\varphi(x_n)\|^2}{1-\rho^2(\varphi(rx_0),\varphi(x_n))} \leq \tag{4.23}$$

$$\frac{1-\|\varphi(x_n)\|^2}{1-\|x_n\|^2}\sup_{|x_0(t)|\geq u_0}\left\{\frac{|1-r\overline{x_0(t)}x_n(t)|^2}{1-|rx_0(t)|^2}\frac{1-\|x_n\|^2}{1-|x_n(t)|^2}, \frac{4(1-\|x_n\|^2)}{(1-u_0^2)(1-r^2)}\right\} \leq$$

$$\frac{1-\|\varphi(x_n)\|^2}{1-\|x_n\|^2}\max_{|x_0(t)|\geq u_0}\left\{\frac{|1-r\overline{x_0(t)}x_n(t)|^2}{1-|rx_0(t)|^2}, \frac{4(1-\|x_n\|^2)}{(1-u_0^2)(1-r^2)}\right\}.$$

If we let $n \longrightarrow \infty$ in the last term, we obtain

$$\limsup_{n\to\infty}\frac{1-\|\varphi(x_n)\|^2}{1-\rho^2(\varphi(rx_0),\varphi(x_n))} \leq \frac{(1-r)}{1+r} + 2\varepsilon. \tag{4.24}$$

Consequently, by (4.19) and (4.24), we obtain

$$\frac{(1-r)}{1+r} + 2\varepsilon \geq \frac{1-||\varphi(rx_0)||}{1+||\varphi(rx_0)||}$$

and hence, if ε tends to 0, we obtain that $||\varphi(rx_0)|| \geq r = ||rx_0||$, a contradiction. □

By Theorems 4.2.12 and 4.2.14, we obtain an extension of Theorem 4.2.2 for non power compact composition operators C_φ.

Theorem 4.2.15. *Let E be a $C_0(X)$ space and consider an analytic map $\varphi : B_E \longrightarrow B_E$ such that $\varphi(0) = 0$, $\|\varphi'(0)\| < 1$ and $\varphi(B_E)$ is relatively compact in B_E. Suppose that φ satisfies the approaching condition. Then, $\sigma(C_\varphi) = \overline{\mathbf{D}}$.*

Proof. Apply Theorem 4.2.14 to $W = \varphi(B_E)$, which approaches S_E compactly since $\varphi(B_E)$ is relatively compact and does not lie inside B_E. Then, we find the Julia-type estimate necessary to apply Theorem 4.2.12 and conclude the result. □

In particular, the result is valid for c_0 and we obtain the following corollary for the n-fold space $\mathbb{C}^n$,

Corollary 4.2.16. *Let* $\varphi : \mathbf{D}^n \longrightarrow \mathbf{D}^n$ *an analytic map such that* $\varphi(0) = 0$ *and* $\|\varphi'(0)\| < 1$. *If* $C_\varphi : H^\infty(\mathbf{D}^n) \longrightarrow H^\infty(\mathbf{D}^n)$ *is non-power compact, then* $\sigma(C_\varphi) = \overline{\mathbf{D}}$.

4.3 Radon-Nikodým Composition Operators

In this section we aim to study Radon-Nikodým composition operators. In order to do it we use the Asplund sets. The notion of Asplund set in a Banach space E arises in studying differentiability properties of the norm of E. At same time, there exists a strong duality between the Asplund property and the Radon-Nikodým property on Banach spaces. Further references can be found in [Asp68], [Ste81], [Fit80] and [Bgi83].

We begin with some background on the Radon-Nikodým property.

4.3.1 Background on the Radon-Nikodým Property

In order to introduce the class of Radon-Nikodým operators, we will give some background on the Radon-Nikodým property for Banach spaces. For further results, see [Saa80], [Ste81] and [DU77].

A *measurable space* is a pair (Ω, Σ), such that Ω is a set and Σ a σ-algebra of subsets of Ω. The triple (Ω, Σ, μ) is a *measure space* if (Ω, Σ) is a measurable space and μ is a finite nonnegative measure on Σ. All the measures treated are countably additive unless the contrary is explicitly stated. The classical *Radon-Nikodým Theorem* states as follows:

Theorem 4.3.1 (Radon-Nikodým). *Let* (Ω, Σ, μ) *a finite measure space and* $m : \Sigma \to \mathbb{R}$ *a finite measure which is absolutely continuous with respect to* μ $(m << \mu)$. *Then, there exists a* μ*-integrable function* $f : \Omega \longrightarrow \mathbb{R}$, *denoted by* $dm/d\mu$, *such that*

$$\int_A f d\mu = m(A) \text{ for any } A \in \Sigma.$$

In order to replace $\mathbb{R}$ by a general Banach space E in both instances above, we recall the following:

Let Σ be a $\sigma-$algebra and E a Banach space. A map $m : \Sigma \longrightarrow E$ such that $m(\emptyset) = 0 \in X$ is called a *vector measure* (or an E-valued measure) if it is countably additive; that is, whenever $\{A_j\}_{j=1}^{\infty}$ is a sequence of pairwise disjoint sets in E, then

$$m(\cup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} m(A_j).$$

Such a measure is said to be *absolutely continuous* with respect to μ ($m << \mu$) if for $A \in \Sigma$, the condition $\mu(A) = 0$ implies $m(A) = 0$.

Now we give the concept of Bochner integrability, which generalizes measure theory to vector measures.

Denote by χ_A the characteristic function of a subset $A \subset E$. A function $s : \Omega \to E$ is called a *simple function* if s can be written in the form $\sum_{i=1}^{n} x_i \chi_{A_i}$ for distinct $x_i \in E$, $i = 1, \ldots, n$ and a finite partition $\{A_i\}_{i=1}^{n}$ of Ω chosen from Σ.

Definition 4.3.2. *A function $f : \Omega \longrightarrow E$ is said to be μ-Bochner integrable if there exists a sequence (s_n) of simple functions such that*

i) $\lim_n s_n(\omega) = f(\omega) \qquad \mu - a.e.$

ii) $\lim_n \int_{\Omega} \|f(\omega) - s_n(\omega)\| d\mu(\omega) = 0.$

The set of Bochner integrable functions is denoted by $L_X^1(\Omega, \Sigma, \mu)$ or simply $L_X^1(\mu)$. When f is Bochner integrable, the limit $\lim_n \int_A s_n d\mu$ exists independently of the particular choice of (s_n) and the integral is defined by

$$\int_A f d\mu = \lim_n \int_A s_n \, d\mu.$$

Definition 4.3.3. *A closed, bounded, convex subset C of a Banach space E is called a* Radon-Nikodým set *if, for any probability space (Ω, Σ, P) and any E-valued vector measure $m : \Sigma \longrightarrow E$ such that the average of m is in C (i.e. $m(A)/P(A)$ is in C for every $A \in \Sigma$), there exists a P-Bochner integrable function $f : \Omega \longrightarrow E$ such that*

$$m(A) = \int_A f dP \qquad \text{for all } A \in \Sigma.$$

A Banach space E is said to have the *Radon-Nikodým property* if each E-valued measure on Σ which is of finite total variation and satisfies $m << \mu$, admits an $f \in L_X^1(\Omega, \Sigma, \mu)$ such that

$$\int_A f d\mu = m(A) \text{ for each } A \in \Sigma.$$

The concept of Radon-Nikodým operator is the following [GS84],

Definition 4.3.4. *The linear operator $T : E \longrightarrow F$ is said to be a* Radon-Nikodým operator *if for every finite measure space $(\Omega, \Sigma, \lambda)$ and for every vector measure $G : \Sigma \longrightarrow E$ such that $\|G(B)\| \le \lambda(B)$ for all $B \in \Sigma$, there exists a Bochner integrable function $f : \Omega \longrightarrow F$ such that*

$$T \circ G(B) = \int_B f d\lambda \qquad \text{for all } B \in \Sigma.$$

It is well-known that E has the Radon-Nikodým property if and only if I_E is a Radon-Nikodým operator [Bgi83].

It is also well-known [Pie80] that the set of Radon-Nikodým operators is a closed ideal of operators. In connection with Radon-Nikodým operators, the strong Radon-Nikodým operators are defined as follows,

Definition 4.3.5. *We say that $T : E \longrightarrow F$ is a* strong Radon-Nikodým operator *if $\overline{T(B_E)}$ is a Radon-Nikodým set.*

If $T : E \longrightarrow F$ is a strong Radon-Nikodým operator, then T is a Radon-Nikodým operator by [Edg80]. The converse is not true since any quotient Q from ℓ_1 onto c_0 is a Radon-Nikodým operator but is not a strong Radon-Nikodým operator [GS84]. Nevertheless, it follows from [Ste81] the following result which states that these concepts are equivalent for adjoint maps,

Theorem 4.3.6. *Let $T : E \longrightarrow F$ be a bounded linear operator. Then $T^* : F^* \longrightarrow E^*$ is a strong Radon-Nikodým operator if and only if it is a Radon-Nikodým operator.*

4.3.2 Asplund sets

This section is devoted to the study of the Asplund property. We will give some background on Asplund sets and operators and will study some related properties. Then, we will give conditions for polynomials between Banach spaces to preserve the Asplund property of some sets and will derive some consequences.

We recall the definition of S. Fitzpatrick [Fit80] of Asplund set. He showed that this definition is equivalent to the earlier one related to the differentiability of the norm of E. Recall that, given a subset $A \subset E$, the seminorm $\|\cdot\|_A : E^* \longrightarrow [0, \infty)$ of E^* is given by

$$\|f\|_A = \sup_{x \in A} |f(x)|.$$

Definition 4.3.7. *Let E be a Banach space and $A \subset E$. A bounded set $D \subset E$ is said to be an* Asplund set *if, for any countable set $A \subset D$, the seminormed space $(E^*, \|\cdot\|_A)$ is separable. The space E is said to be an* Asplund space *if its unit ball B_E is an Asplund set.*

Proposition 4.3.8. *Let $D \subset E$. Then, D is an Asplund set if and only if the closure of its absolutely convex hull $\overline{\Gamma}(D)$ is an Asplund set.*

There are many characterizations of Asplund spaces [Bgi83]. The following one will be used later,

Proposition 4.3.9. *The Banach space E is Asplund if and only if, for any separable subspace $F \subset E$, we have that F^* es separable.*

By 4.3.9, it is clear that any reflexive Banach space is an Asplund space. Furthermore, if E^* is separable, then E is also an Asplund space. In addition, we have the following proposition:

Proposition 4.3.10. *Let $T : E \longrightarrow F$ a linear operator and $D \subset E$ a bounded set. If D is an Asplund set, then $T(D)$ is an Asplund set as well.*

We wonder if it is possible to extend Proposition 4.3.10 to nonlinear mappings. In particular, we are interested in studying analytic maps from E to F. The following example shows that this extension cannot be done in general.

Example 4.3.11. *Set $E = \ell_2$ and $F = \ell_1$. Consider the polynomial $P \in P(^2\ell_2, \ell_1)$ given by*

$$P((x_n)_{n=1}^{\infty}) = (x_n^2)_{n=1}^{\infty}.$$

Then P does not transform Aslpund sets into Asplund sets.

To show this, consider the set $D = \{e_n : n \in \mathbb{N}\}$, which is is an Asplund set in ℓ_2. However, $P(D) = D$ is not an Asplund set in ℓ_1; indeed, $\ell_1^* = \ell_\infty$ and the space $(\ell_\infty, \|\cdot\|_D)$ is not separable since for $f = (f_n) \in \ell_\infty$ we have

$$\|f\|_D = \sup_{x \in D} |f(x)| = \sup_{n \in \mathbb{N}} |f_n| = \|f\|_\infty$$

so $\|\cdot\|_D$ is the usual norm for ℓ_∞ and, therefore, $(\ell_\infty, \|\cdot\|_D)$ is not separable. □

Therefore, we aim to find sufficient conditions for analytic mappings to transform Asplund sets into Asplund sets. This will lead us to characterize Radon-Nikodým composition operators C_φ in terms of $\varphi(B_E)$. The following lemma (see [Bgi83]) will be needed,

Lemma 4.3.12. *Let E be a Banach space. Then,*

a) The sum and the union of a finite number of Asplund sets in E is an Asplund set.

b) Let (D_n) be a sequence of Asplund sets in E and let (t_n) be a sequence of positive numbers such that $\lim_n t_n = 0$. Then, the set

$$D = \bigcap_{n=1}^{\infty} (D_n + t_n B_E)$$

is an Asplund set.

Let U be a subset of E. A subset $D \subset U$ is said to be U-bounded if D is bounded and $d(D, E \setminus U) > 0$. A function $f : U \longrightarrow F$ is said to be of bounded type if it maps U-bounded sets into bounded sets.

Proposition 4.3.13. *Let E and F be Banach spaces and $D \subset E$ an Asplund set.*

a) *Suppose that $P(^kE) = \overline{P_f(^kE)}$ for some $k \in \mathbb{N}$. If $P : E \longrightarrow F$ is a k-homogeneous polynomial, then $P(D)$ is an Asplund set.*

b) *Suppose that $P(^kE) = \overline{P_f(^kE)}$ for any $k \in \mathbb{N}$. If $f : U \subset E \longrightarrow F$ is an analytic function of bounded type and D is U- bounded, then $f(D)$ is an Asplund set.*

Proof. We can suppose, without loss of generality, that $D \subset B_E$, since the class of Asplund sets is stable under translations and homotheties.

a) Set $A \subset P(D)$ a countable set, that is, $A = P(C)$ for some countable set $C \subset D$. Since D is Asplund, we have that $(E^*, \|\cdot\|_C)$ is separable, so there exists a countable set S which is dense in $(E^*, \|\cdot\|_C)$. Consider the adjoint mapping $P^* : F^* \longrightarrow P(^kE)$ given by

$$P^*(\varphi) = \varphi \circ P.$$

Clearly, the mapping $P^* : (F^*, \|\cdot\|_A) \longrightarrow (P(^kE), \|\cdot\|_C)$ is a linear isometry. Therefore, $(F^*, \|\cdot\|_A)$ is separable if $P^*(F^*) \subset (P(^kE), \|\cdot\|_C)$ is separable and this will be a consequence of the separability of the seminormed space $(P(^kE), \|\cdot\|_C)$. This space is separable since the algebra generated by S is dense in $(P_f(^kE), \|\cdot\|_C)$, which is, again, dense in $(P(^kE), \|\cdot\|_C)$ since $\overline{P_f(^kE)} = P(^kE)$ by assumption and the norm topology is finer than the $\|\cdot\|_C$ - topology.

b) By 4.3.12 a) and a) in this Proposition, any polynomial defined on E maps D into an Asplund set. Set $n \in \mathbb{N}$. Since f is of bounded type, there exists a polynomial P_n such that $\|f(x) - P_n(x)\| \leq 1/n$ for any $x \in D$. Therefore,

$$f(D) \subset P_n(D) + \frac{1}{n} B_F,$$

and, hence, $f(D)$ is an Asplund set by 4.3.12 b). □

The following example shows that the $G^\infty(B_E)$, the predual of $H^\infty(B_E)$ defined in paragraph 1.6.5 is not an Asplund space regardless of the Banach space E.

Example 4.3.14. *The space $G^\infty(B_E)$ is not Asplund.*

Indeed, we know that $c_0 \subset H^\infty(B_E)$ by Theorem 1.3.4 and, then, $\ell_1 \subset G^\infty(B_E)$ by Theorem 4 in [BP58]. Therefore, $G^\infty(B_E)$ has not the Asplund property by Proposition 4.3.9 since $\ell_1^* = \ell_\infty$ is not separable.

This remark leads us to show that Proposition 4.3.13 b) does not hold for non U-bounded sets. To show this, consider the map $\delta : \mathbf{D} \to G^\infty$ given by $\delta(x) = \delta_x$, the evaluation at x. Then $\delta(\mathbf{D}) = \{\delta_x : x \in \mathbf{D}\}$ is not an Asplund set since the closure of its absolutely convex hull is B_{G^∞} by Proposition 1.3.

The following result gives a sufficient condition for all functions in the algebra $A_u(B_E,F)$ to transform Asplund sets into Asplund sets. This result can be extended to other algebras of analytic functions on B_E.

Proposition 4.3.15. *Let E and F be complex Banach spaces and suppose that $A_u(B_E)$ is separable. Then, any $f \in A_u(B_E,F)$ transforms Asplund sets into Asplund sets.*

Proof. We denote by A the algebra $A_u(B_E,F)$. Pick $f \in A$ and consider the restriction $f^* := C_f|_{F^*} : F^* \longrightarrow A_u(B_E)$ given by $f^*(y^*)(x) = y^*(f(x))$ for any $x \in E$. Let $D \subset B_E$ be an Asplund set. To show that $f(D) \subset F$ is also an Asplund set, consider the space $(F^*, \|\cdot\|_{f(D)})$ and $B \subset f(D)$ a countable set. There exists a countable set $C \subset D$ such that $f(C) = B$. In consequence, $(A_u(B_E), \|\cdot\|_C)$ is separable since $\|g-h\|_C \leq \|g-h\|_\infty$ for any $g,h \in A_u(B_E)$. Since $f^* : (F^*, \|\cdot\|_{f(D)}) \to (A_u(B_E), \|\cdot\|_D)$ is a linear isometry, we have that $(F^*, \|\cdot\|_{f(C)})$ is separable and, therefore, $f(D)$ is an Asplund set. □

Remark 4.3.16. *We have the following results,*

(i) *There exist Banach spaces E such that E^* is separable, so E is Asplund, but there exist functions $f \in A_u(B_E,F)$ which do not transform Asplund sets into Asplund sets. To show this, it is sufficient to consider Example 4.3.11.*

(ii) *There exist Banach spaces E whose dual spaces E^* are not separable but any $f \in A_u(B_E,F)$ transforms Asplund sets into Asplund sets for any Banach space F. Consider, for instance, $E = c_0(\Gamma)$ for Γ an uncountable set, whose dual space $\ell_1(\Gamma)$ is not separable. However, any function $f \in A_u(B_E,F)$ transforms Asplund sets into Asplund sets by Proposition 4.3.13 b).*

4.3.3 Radon-Nikodým Composition Operators

Now we recall the duality between the Radon-Nikodým property and the Asplund property. Then, we will give some conditions for composition operators to be Radon-Nikodým operators.

Definition 4.3.17. *The linear operator $T : E \longrightarrow F$ is said to be an* Asplund operator *if $T(B_E)$ is an Asplund set.*

The following result can be found in [Bgi83],

Theorem 4.3.18. *Let $T : E \longrightarrow F$ be a linear operator. Then,*

T is Asplund if and only if T^* is (strong) Radon-Nikodým.

Notice that condition on strongness on T^* can be removed since, as we mentioned in the background, an adjoint operator T^* is strong Radon-Nikodým if and only if it is Radon-Nikodým.

In the following corollary we apply this theorem to $C_\varphi : H^\infty(B_F) \longrightarrow H^\infty(B_E)$. It is easy that the range of $C_\varphi^*|_{G^\infty(B_E)}$ is contained in $G^\infty(B_F)$ since for any $u \in G^\infty(B_E)$, the composition $u \circ C_\varphi$ is still τ_c-continuous on $B_{G^\infty(B_F)}$. We denote by C^φ the restriction $C_\varphi{}^*|_{G^\infty(B_E)}$. Since $(C^\varphi)^* = (C_\varphi{}^*|_{G^\infty(B_E)})^* = C_\varphi$, we obtain

Corollary 4.3.19. *A composition operator $C_\varphi : H^\infty(B_F) \longrightarrow H^\infty(B_E)$ is Radon-Nikodým if and only if the operator $C^\varphi : G^\infty(B_E) \longrightarrow G^\infty(B_F)$ is Asplund.*

The following result makes it easier the study of the Asplundness of the operator C^φ.

Proposition 4.3.20. *The operator C^φ is Asplund if and only if the set $\{\delta_{\varphi(x)} : x \in B_E\}$ is Asplund in $G^\infty(B_F)$.*

Proof. Set $B = B_{G^\infty(B_E)}$. The set $C_\varphi{}^*(B) = \{\mu \circ C_\varphi : \mu \in B\}$ is Asplund if and only if $\Gamma(\{\delta_x \circ C_\varphi : x \in B_E\})$ is Asplund if and only if $\{\delta_x \circ C_\varphi : x \in B_E\}$ is Asplund if and only if $\{\delta_{\varphi(x)} : x \in B_E\}$ is Asplund. □

Notice that Example 4.3.14 shows that $\{\delta_{\varphi(x)} : x \in B_E\}$ being an Asplund set is not equivalent to $\varphi(B_E)$ being an Asplund set.

In the following result, we apply Proposition 4.1.1.

Proposition 4.3.21. *Let $C_\varphi : H^\infty(B_F) \longrightarrow H^\infty(B_E)$ be a Radon-Nikodým operator. Then, there exists $0 < r < 1$ such that $\varphi(B_E) \subset rB_F$.*

Proof. If such an $0 < r < 1$ does not exist, we apply Proposition 4.1.1 and find linear operators $T : \ell_\infty \longrightarrow H^\infty(B_E)$ and $S : H^\infty(B_E) \longrightarrow \ell_\infty$ such that $S \circ C_\varphi \circ T = Id_{\ell_\infty}$. This is not possible since the class of Radon-Nikodým operators is an operator ideal and Id_{ℓ_∞} is not Radon-Nikodým. □

Now we present a characterization of Radon-Nikodým composition operators in terms of the Asplund property and the condition given in the previous proposition.

Theorem 4.3.22. *The composition operator $C_\varphi : H^\infty(B_F) \longrightarrow H^\infty(B_E)$ is Radon-Nikodým if and only if there exists $0 < r < 1$ such that $\varphi(B_E) \subset rB_F$ and $(P(F), \|\cdot\|_A)$ is separable for any countable set $A \subset \varphi(B_E)$.*

Proof. To prove the sufficient condition it is enough to show that $\delta_{\varphi(B_E)} = \{\delta_x : x \in \varphi(B_E)\}$ is an Asplund set in $G^\infty(B_F)$ by Proposition 4.3.20 and Corollary 4.3.19. Let $0 < r < 1$ such that $\varphi(B_E) \subset rB_F$. Let $A \subset \varphi(B_E)$ a countable set. This set can be described as $A = \varphi(C)$ for some countable set $C \subset B_E$. Recall that the Taylor series of $f \in H^\infty(B_F)$ converge uniformly to f on rB_F and, therefore, f is uniformly approximable by polynomials on $\varphi(C)$. In consequence, $P(F)$ is dense in $(H^\infty(B_F), \|\cdot\|_{\varphi(C)})$. Since $(P(F), \|\cdot\|_{\varphi(C)})$ is separable and $(G^\infty(B_F))^* = H^\infty(B_F)$, it follows that $((G^\infty(B_F))^*, \|\cdot\|_{\varphi(C)})$ is separable. Then, $\delta_{\varphi(B_E)}$ is an Asplund set in $G^\infty(B_F)$ and C_φ is Radon-Nikodým.

Now we prove the necessary conditions. Let C_φ Radon-Nikodým. Then, by Proposition 4.3.21, there exists $0 < r < 1$ such that $\varphi(B_E) \subset rB_F$. To show the other condition, let $A \subset \varphi(B_E)$ a countable set, that is, $A = \varphi(C)$ for some countable set $C \subset B_E$. Since $C^\varphi(B_{G^\infty(B_E)})$ is an Asplund set, the space $(H^\infty(B_F), \|\cdot\|_{\varphi(C)})$ is separable. Therefore, since $P(F) \subset H^\infty(B_F)$, it follows that $(P(F), \|\cdot\|_{\varphi(C)})$ is a subspace of the seminormed space $(H^\infty(B_F), \|\cdot\|_{\varphi(C)})$. Then, $(P(F), \|\cdot\|_A)$ is separable. □

Corollary 4.3.23. *Let $\varphi : B_E \longrightarrow B_F$ be an analytic map.*

a) Suppose that $P(^kF) = \overline{P_f(^kF)}$ for any $k \in \mathbb{N}$. If $\varphi(B_E)$ is an Asplund set and there exists $0 < r < 1$ such that $\varphi(B_E) \subset rB_F$, then the composition operator $C_\varphi : H^\infty(B_F) \longrightarrow H^\infty(B_E)$ is Radon-Nikodým.

b) Suppose that the algebra $A_u(B_F)$ is separable. If $\varphi(B_E)$ is an Asplund set and there exists $0 < r < 1$ such that $\varphi(B_E) \subset rB_F$, then the composition operator $C_\varphi : H^\infty(B_F) \longrightarrow H^\infty(B_E)$ is Radon-Nikodým.

Proof. a) In Proposition 4.3.13 a), we show that for any countable set $C \subset \varphi(B_E)$, the space $(P(^kF), \|\cdot\|_C)$ is separable. Then,

$$(P(F), \|\cdot\|_C) = \bigcup_{k \in \mathbb{N}} (P(^kF), \|\cdot\|_C)$$

is also separable and, therefore, we can use the previous theorem to get the result.

b) Since $\varphi(B_E) \subset rB_F$ is an Asplund set and the function $\delta|_{rB_F}$ belongs to $A_u(rB_F, G^\infty(B_F))$, we have, if $A_u(B_F)$ is separable, that the set $\delta_{\varphi(B_E)}$ is also an Asplund set by Corollary 4.3.15 and, therefore, the composition operator C_φ is Radon-Nikodým. $\square$

Chapter 5

Hankel-type Operators

Hankel operators were first studied acting on the Hardy space H^2 and they were extended to act on closed subspaces of $C(K)$ spaces, in particular, on uniform algebras. Several properties of these subspaces can be described through these operators, which will be called Hankel-type operators. Their weak compactness and their compactness led to introduce the concept of *tightness* of a uniform algebra. The complete continuity of the Hankel operators led to define the so-called Bourgain algebras, which are closely related to the Dunford-Pettis property.

5.1 Background

Let us introduce some basic results and notation related to Hankel operators. For further results, see [Pow82], [Zhu90] and [Pel98].

5.1.1 The Hankel-type operators

Let H^2 denote the usual Hardy space of functions on the circle $\partial\mathbf{D}$ and consider the Cauchy projection $\mathscr{C}$ from L^2 onto H^2 defined in paragraph 1.4.1. The classical *Hankel operators* correspond to functions $g \in L^2$ which act on H^2 by defining $S_g : H^2 \longrightarrow (H^2)^\perp$ by

$$S_f(g) = gf - \mathscr{C}(gf).$$

The aim of the Hankel-type operators is the extension of Hankel operators to closed subspaces of $C(K)$, in particular, to uniform algebras. Since such operators are multiplication operators into the orthogonal, the most natural candidate is to consider the following definition found in [CG82],

Definition 5.1.1. *Let A be a uniform algebra on K and $g \in C(K)$. The Hankel-type operator $S_g : A \longrightarrow C(K)/A$ is defined according to*

$$S_g(f) = gf + A.$$

It is obvious that Hankel-type operators are linear. They are also continuous since

$$\|S_g(f)\| = \|gf + A\| = \inf\{\|gf + x\|_\infty : x \in A\} \leq \|gf\|_\infty \leq \|g\|_\infty \|f\|_\infty.$$

When we deal with closed operator ideals $\mathscr{U}$, we define the sets

$$A_{\mathscr{U}} = \left\{g \in C(K) : S_g \in \mathscr{U}\right\} \text{ and } A_{\mathscr{U}^{**}} = \left\{g \in C(K) : S_g^{**} \in \mathscr{U}\right\}.$$

We have the following result,

Proposition 5.1.2. *Let $\mathscr{U}$ be a closed operator ideal. Then, $A_{\mathscr{U}}$ and $A_{\mathscr{U}^{**}}$ are closed subspaces of $C(K)$.*

Proof. It is easy to show that $A_{\mathscr{U}}$ and $A_{\mathscr{U}^{**}}$ are subspaces of $C(K)$. Then, we prove that $A_{\mathscr{U}}$ is closed. Recall that S_g is continuous and $\|S_g\| \leq \|g\|_\infty$. Let $\{g_n\}$ a sequence of functions of $A_{\mathscr{U}}$ converging to $g_0 \in C(K)$. By the inequality $\|S_g\| \leq \|g\|_\infty$ we deduce that S_{g_n} converges to S_{g_0} and, since $\mathscr{U}$ is a closed operator ideal, we obtain that $S_{g_0} \in \mathscr{U}$. As consequence, $g_0 \in A_{\mathscr{U}}$ and therefore the set is a closed subspace. □

Examples of closed operator ideals are the class of compact and weakly compact operators and the class of completely continuous operators. When we deal with these classes of operators, the sets $A_{\mathscr{U}}$ and $A_{\mathscr{U}^{**}}$ are, indeed, closed subalgebras of $C(K)$ as we will recall later.

5.1.2 The Dunford-Pettis property and Bourgain Algebras

The birthplace of the Dunford-Pettis property is a work of A. Grothendieck [Gro53] where he proved that weakly compact operators $T : C(K) \longrightarrow F$ are completely continuous for any Banach space F. This was an extension of the work done by N. Dunford and B. J. Pettis in [DP40] for weakly compact operators on $L^1(\mu)$.

Definition 5.1.3. *A Banach space E is said to have the* Dunford-Pettis property *(DP property) if any weakly compact operator $T : E \longrightarrow F$ is completely continuous for any Banach space F.*

It is well-known [Die80] that E has the DP property if and only if $\lim_n x_n^*(x_n) = 0$ for sequences $x_n \xrightarrow{w} 0$ in E and $x_n^* \xrightarrow{w} 0$ in E^*. Then, it is clear that E has the DPP if E^* also enjoys it. The reciprocal is false in general, as was shown by C. Stegall [Ste72]. The DP property is hereditary for complemented subspaces but not for closed subspaces in general [PS65]. It is also well-known that there are no reflexive infinite dimensional Banach spaces satisfying the DP property. For further results about the DP property, see [Die80].

If we deal with algebras of analytic functions, we find several results related to the DP property; J. Chaumat [Cha74] proved that the disk algebra $A(\mathbf{D})$ has the DP property and J. Bourgain proved it for H^∞ [Bou84b], the ball algebras $A(B_n)$ and the polydisk algebras $A(\mathbf{D}^n)$ [Bou84a]. We recall the main results of the Bourgain's work in [Bou84a], which will be connected to our results on Bourgain algebras given in section 5.2. The main theorem in [Bou84a] allows to prove that $A(B_n)$ and $A(\mathbf{D}^n)$ enjoy the DP property:

Theorem 5.1.4. *Let E be the ball algebra $A(B_n)$ or the polydisk algebra $A(\mathbf{D}^n)$. Then, any bounded sequence (x_n^*) in E^* either tends uniformly to zero on weakly compact subsets of E^{**} or $c_0 \subset E$ and (x_n^*) does not tend uniformly to zero on the c_0-basis (e_k).*

Besides,

Proposition 5.1.5. *If E satisfies the thesis of Theorem 5.1.4, then E^*, and hence E, have the DP property.*

To prove Theorem 5.1.4, J. Bourgain develops a procedure to obtain c_0-sequences. Recall that the product defined in $C(K)^{**}$ is given by the Arens product (see paragraph 1.3.2). We denote by (e_k) the canonical basis of c_0 or an isomorphic space to c_0. Bourgain's result states as follows,

Theorem 5.1.6. *Let E be a closed subspace of $C(K)$, (x_n^*) a bounded sequence in E^* and $\delta > 0$. Suppose that the following property holds,*

(P) For each $g \in C(K)$ and each $\varepsilon > 0$, there exists a sequence $(x_n^{**}) \subset B_{E^{**}}$ which is weakly convergent to 0 such that

i) $\limsup_n |x_n^{**}(x_n^*)| > \delta$

ii) $d(g \cdot x_n^{**}, E^{**}) < \varepsilon$ *for any $n \in \mathbb{N}$.*

Then, $c_0 \subset E$ and $\sup_n |x_n^*(e_k)| > \delta/2$.

Theorem 5.1.6 led him to prove that the ball algebras $A(B_n)$ satisfy Theorem 5.1.4, and hence the Dunford-Pettis property. Recall that the spectrum of the ball algebra $A(B_n)$ is given by $\overline{B}_n$. J. Bourgain's way to Theorem 5.1.6 was opened by

Proposition 5.1.7. *Let A be the ball algebra $A(B_n)$. Then, for any $x_n^{**} \xrightarrow{w} 0$ in A^{**} and $g \in C(\overline{B}_n)$, we have that* $\lim_{n\to\infty} d(g \cdot x_n^{**}, A) = 0$.

Proposition 5.1.7 proves that property (P) in Theorem 5.1.6 is satisfied when we deal with $A(B_n)$ and, hence, Theorem 5.1.4 as mentioned above. We obtain that $A(B_n)$ and $A(B_n)^*$ enjoy the DP property.

J. A. Cima and R. M. Timoney [CT87] studied the Bourgain's work and found a convenient reformulation of his results. They introduced the so-called *Bourgain algebras*, which are closely connected to Theorem 5.1.6 and Proposition 5.1.7. Such sets contain the symbols of Hankel-type operators on the algebra which are completely continuous. This will give a connection between tightness, the Dunford-Pettis property and the Bourgain algebras.

Definition 5.1.8. *Let X be a closed subspace of $C(K)$. Then, the sets defined by*

$$X_b = \left\{ g \in C(K) : S_g : X \longrightarrow C(K)/A \text{ is completely continuous } \right\} \text{ and}$$

$$X_B = \left\{ g \in C(K) : S_g^{**} : X^{**} \longrightarrow C(K)^{**}/A^{**} \text{ is completely continuous } \right\}.$$

are closed subalgebras of $C(K)$ and are called the Bourgain algebras of X.

We will refer to X_b as the big Bourgain algebra and X_B as the small Bourgain algebra. In addition, we have the following result [CT87],

Proposition 5.1.9. *Let A be a closed subalgebra of $C(K)$. Then, its Bourgain algebras A_b and A_B satisfy $A \subset A_B \subset A_b$.*

The J. A. Cima and R. M. Timoney reformulation can be summarized as follows,

Theorem 5.1.10. *Let X be a closed subspace of $C(K)$. Then,*

i) If $X_b = C(K)$ then X has the DPP.
ii) If $X_B = C(K)$ then both X^* and X have the DPP.

Theorem 5.1.11. *Let A be the ball algebra $A(B_n)$ on $K = \overline{B_n}$. Then, its Bourgain algebras A_b and A_B are both equal to $C(K)$. Hence, A^* and A have the DP property.*

A proof by induction using Theorem 5.1.4 allowed J. Bourgain to conclude that $A(\mathbf{D}^n)$ and its dual also enjoy the DP property. Nevertheless, the line of the proof is different to the one used for $A(B_n)$ because it was not clear whether the Bourgain algebras of $A(\mathbf{D}^n)$ were the whole $C(K)$ or not.

5.1.3 Tightness

The concept of tightness was introduced by B. J. Cole and T.W. Gamelin in [CG82]. It is related to Hankel-type operators on uniform algebras and reads as follows,

Definition 5.1.12. *Let K be a compact space. A closed subspace X of $C(K)$ is said to be* tight *if the Hankel-type operator S_g is weakly compact for any $g \in C(K)$.*

S. F. Saccone introduced a concept related to tightness in [Sac97]:

Definition 5.1.13. *A closed subspace X of $C(K)$ is said to be* strongly tight *if for any $g \in C(K)$, the Hankel-type operator S_g is compact.*

It is clear that a strongly tight subspace is also tight. Saccone also observed that infinite-dimensional reflexive spaces are tight in any $C(K)$ space they are embedded in, but can never be realized as strongly tight subspaces. However, there is no known example of a tight uniform algebra which fails to be strongly tight. If X is strongly tight, then X^*, hence X enjoy the DP property by Theorem 5.1.10. S. F. Saccone also proved that X enjoys the Pelczyński property (V) and its dual is weakly sequentially complete if X is tight.

The space $A^{**} + C(K)$ is always a closed subspace of $C(K)^{**}$. In [CG82] it is proved that A is tight if and only if $A^{**} + C(K)$ is a closed subalgebra of $C(K)^{**}$. In addition, this property is, roughly speaking, the abstract analogue of the solvability of a certain abstract $\bar{\partial}$-problem with a small gain in smoothness.

B. J. Cole and T. W. Gamelin proved that $A(\mathbf{D}^n)$ is not tight for $n \geq 2$. However, they proved that the algebra $A(D)$ is tight when D is a strictly pseudoconvex domain in $\mathbb{C}^n$ for which a certain $\bar{\partial}$-problem is appropriately solvable. Further results on tightness of the algebra generated by the weak*-continuous linear functionals on the closed unit ball $\overline{B}_E$ of a complex dual Banach space E were given by J. Jaramillo and A. Prieto in [JP93].

Most of the cases where we will study tightness involve uniform algebras. Tightness for a closed subalgebra A of $C(K)$ involves the compact space K. Therefore, unless it is otherwise specified, when we deal with the tightness of an algebra A, the algebra will be considered as a space of continuous functions on its spectrum M_A.

5.2 The Bourgain algebras of $A(\mathbf{D}^n)$

As mentioned above, neither from Cima and Timoney's work nor from Bourgain's work can we describe the Bourgain algebras of $A(\mathbf{D}^n)$. Indeed, these will be examples of algebras of analytic functions enjoying the DP property whose Bourgain algebras are themselves. For further results, see [Mir08].

Consider the case of the bidisk algebra, that is, $A = A(\mathbf{D}^2)$. Consider A as a closed subalgebra of $C(\overline{\mathbf{D}}^2)$. Given $h \in A_b$ and $\theta, \psi \in \mathbb{R}$, the function $h_{\theta,\psi} : \mathbb{C}^2 \longrightarrow \mathbb{C}$ defined by

$$h_{\theta,\psi}(z,w) = h(e^{i\theta}z, e^{i\psi}w)$$

belongs to A_b. This follows by noticing that the mapping $(z,w) \mapsto (e^{i\theta}z, e^{i\psi}w)$ is an automorphism that leaves A invariant. Moreover, we have the following result,

Proposition 5.2.1. *If α_0 and β_0 are integers, and the function $g(z,w)$ is defined by*

$$g(z,w) = \int_{[0,2\pi]^2} h_{\theta,\psi}(z,w)e^{-i\theta\alpha_0}e^{-i\psi\beta_0}d\theta d\psi,$$

we have that $g \in A_b$.

Proof. Fix $(z,w) \in \overline{\mathbf{D}}^2$. Then, since $g(z,w)$ is defined by a Riemann integral, there exists a sequence of partitions (P_n) such that

$$P_n = \left\{(\theta_{k_0}, \psi_{k_0}), \ldots, (\theta_{k_n}, \psi_{k_n})\right\}, \quad \theta_{k_0} = \psi_{k_0} = 0,\ \theta_{k_n} = \psi_{k_n} = 2\pi,$$

and $t_{k_i} \in (\theta_{k_i}, \theta_{n_{k+1}})$, $s_{k_i} \in (\psi_{k_i}, \psi_{k_{i+1}})$ for $i = 0, 1, \ldots, n-1$ such that

$$\lim_{n\to\infty} \sum_{k=0}^{k_n} h_{t_{k_i}, s_{n_i}}(z,w)e^{-it_{k_i}\alpha_0}e^{-is_{k_i}\beta_0}(\theta_{k_{i+1}} - \theta_{k_i})(\psi_{k_{i+1}} - \psi_{k_i}) =$$

$$\int_{[0,2\pi]^2} h_{\theta,\psi}(z,w)e^{-i\theta\alpha_0}e^{-i\psi\beta_0}d\theta d\psi.$$

Therefore, the integral is the limit of linear combinations of functions of type $h_{\theta,\psi}(z,w)$, which are in A_b as we mentioned above. Hence, g is the pointwise limit of a sequence in A_b and we conclude that g is the weak limit of this sequence in A_b by the Lebesgue Dominated Convergence Theorem. Since A_b is weakly closed by the Mazur's Theorem 1.1.1, we obtain that $g \in A_b$. □

These results are easy to extend to $A(\mathbf{D}^n)$ for any $n \geq 2$. Our main result about the Bourgain algebras of $A(\mathbf{D}^n)$ is the following,

Proposition 5.2.2. *Let A be the algebra $A(\mathbf{D}^n)$ considered as a subspace of $C(\bar{\mathbf{D}}^n)$ for $n \geq 2$. Then*

$$A_B = A_b = A.$$

Proof. We show that $A_b = A$ for $n = 2$. The proof for $A(\mathbf{D}^n)$ follows the same pattern. It is well-known that any infinite-dimensional Banach algebra contains a copy of c_0; thus $A(\mathbf{D})$ does not have the Schur property. So, there exists a sequence $\{f_n\} \subset A(\mathbf{D})$ such that $f_n \xrightarrow{w} 0$ and $\|f_n\| = 1$.

We know that $A \subset A_b$ and that A_b is a closed subalgebra of $C(\partial\mathbf{D}^2)$. Suppose that $A \subsetneq A_b$. Then we show that there exists a trigonometric monomial in $A_b \setminus A$; since $A \subsetneq A_b$, there exists a function $h \in A_b \setminus A$ and then $h(e^{i\theta}z, e^{i\psi}w)$ belongs to A_b for all real θ and ψ. Indeed, since $h \notin A$, we have that h restricted to the torus $\partial\mathbf{D}^2$ must have a nonzero Fourier coefficient with at least one negative index, say the coefficient a_{α_0,β_0} of $z^{\alpha_0}w^{\beta_0}$ and suppose, without loss of generality, that $\beta_0 < 0$. We consider the function

$$\int_{[0,2\pi]^2} h(e^{i\theta}z, e^{i\varphi}w)e^{-i\alpha_0\theta}e^{-i\beta_0\varphi}d\theta d\varphi$$

to get an element of A_b again which restricts on the torus to the function $g(z,w) = a_{\alpha_0,\beta_0}z^{\alpha_0}w^{\beta_0}$. Therefore, it is clear that the monomial g belongs to $A_b \setminus A$.

Put $\widetilde{f_n} = f_n/|a_{\alpha_0,\beta_0}|$. Since $g \in A_b$ and $\widetilde{f_n} \xrightarrow{w} 0$, there exist a sequence $(y_n) \subset A$ such that $\|g\widetilde{f_n} + y_n\| \to 0$. Set $g_n = g\widetilde{f_n} + y_n$ and consider the linear and continuous operator $P : C(\partial\mathbf{D}^2) \longrightarrow C(\partial\mathbf{D}^2)$ given by

$$P(f)(z,w) = \left(\int_{\partial\mathbf{D}} f(z,\eta)\eta^{-\beta_0}d\eta\right) w^{\beta_0}.$$

Then, we have

$$P(g_n)(z,w) = \left(\int_{\partial\mathbf{D}} \left(g(z,\eta)\widetilde{f_n}(z) + y_n(z,\eta)\right)\eta^{-\beta_0}d\eta\right) w^{\beta_0} =$$

$$\left(\int_{\partial\mathbf{D}} a_{\alpha_0,\beta_0}z^{\alpha_0}\eta^{\beta_0}\frac{f_n(z)}{|a_{\alpha_0,\beta_0}|}\eta^{-\beta_0}d\eta\right) w^{\beta_0} + \left(\int_{\partial\mathbf{D}} y_n(z,\eta)\eta^{-\beta_0}d\eta\right) w^{\beta_0}.$$

Therefore, we have

$$P(g_n)(z,w) = \frac{a_{\alpha_0,\beta_0}}{|a_{\alpha_0,\beta_0}|}\left(\int_{\partial\mathbf{D}} z^{\alpha_0}\eta^{\beta_0}f_n(z)\eta^{-\beta_0}d\eta\right) w^{\beta_0}.$$

Since we take the normalized Lebesgue measure on the circle, we have that

$$\|P(g_n)\| = \|f_n\|\left(\int_{\partial\mathbf{D}} d\eta\right)\|z^{\alpha_0}w^{\beta_0}\| = \|f_n\| = 1$$

which is a contradiction as we wanted. The result for A_B follows from the relationship $A \subset A_B \subset A_b$ given in Proposition 5.1.9. □

5.3 Tightness in some algebras of analytic functions

In this section, we will study tightness in some algebras of analytic functions. B. Cole and T. W. Gamelin proved that $A(\mathbf{D})$ is strongly tight. We begin giving an easy proof of this fact. Recall that the set of compact Hankel-type operators on a uniform algebra A, which we will denote by $A_{\mathscr{K}}$, is a closed subalgebra of $C(K)$ by [CG82] and [Sac95].

Theorem 5.3.1. *Let A be the disk algebra $A(\mathbf{D})$. Then, A is strongly tight.*

Proof. Since $A_{\mathscr{K}}$ is an algebra by the comments above and it is clear that $A \subset A_{\mathscr{K}}$, we only need to prove that the function $g : \overline{\mathbf{D}} \longrightarrow \mathbb{C}$ defined by $g(z) = \bar{z}$ belongs to $A_{\mathscr{K}}$ by the Stone-Weierstrass theorem. Let (x_n) be a bounded sequence of functions in A and define $y_n : \overline{\mathbf{D}} \longrightarrow \mathbb{C}$ by

$$y_n = \frac{x_n(z) - x_n(0)}{z}.$$

These functions belong to A, so

$$\bar{z}x_n(z) - \bar{z}x_n(0) = y_n(z) \in A$$

when we consider $z \in \partial\mathbf{D}$. In consequence,

$$\bar{z}x_n + A = \bar{z}x_n(0) + A.$$

The sequence $(x_n(0))$ is also bounded in $\mathbb{C}$, so there exists a convergent subsequence which converges to $z_0 \in \mathbb{C}$. We suppose, without loss of generality, that this sequence is $x_n(0)$ itself, that is, $x_n(0) \to z_0$. Then, the sequence given by

$$S_g(x_n) = \bar{z}x_n + A = \bar{z}x_n(0) + A$$

tends to $\bar{z}z_0 + A$ since $\bar{z}x_n(0) \to \bar{z}z_0$ and the quotient map $C(K) \to C(K)/A$ is continuous. Therefore, S_g is compact, $A_{\mathscr{K}} = C(K)$ and we conclude that $A(\mathbf{D})$ is strongly tight. □

As we mentioned in section 5.1.3, B. Cole and T. Gamelin proved that the polydisk algebras $A(\mathbf{D}^n)$ are not tight for $n \geq 2$ (see also [Sac95]). We offer two new approaches to this result. First, by Proposition 5.2.2, we know that the Bourgain algebra A_b of $A = A(\mathbf{D}^n)$ is A itself and not the whole $C(K)$. Since $A(\mathbf{D}^n)$ has the DP property by the Bourgain's work, if A were tight, then all the Hankel-type operators S_g would be weakly compact and, hence, completely continuous. Therefore, its Bourgain algebra A_b would be the whole $C(K)$, a contradiction. In general, we obtain [Mir08],

Corollary 5.3.2. *Let X be a closed subspace of $C(K)$. If X has the DP property, then*

X is tight if and only if the Bourgain algebras X_b and X_B are both equal to $C(K)$.

On the other hand, we give a new technique which allows us to extend the results about the tightness of $A(\mathbf{D}^n)$. Let E be any Banach space and consider F the Banach space $F = \mathbb{C} \times E$ endowed with the supremum norm $\|(z,x)\|_F = \sup\{|z|, \|x\|_E\}$. We prove that $A = A_u(B_E)$ is never tight on its spectrum [Mir08] and, therefore, in particular, the polydisk algebra is not tight on its spectrum. In addition, we can apply this result in algebras as $A_u(B_{c_0})$ and we also conclude that they are not tight on their spectra.

Theorem 5.3.3. *Let E be a Banach space and $F = \mathbb{C} \times E$ endowed with the supremum norm $\|(z,x)\|_F = \sup\{|z|, \|x\|_E\}$. Then $A = A_u(B_F)$ is not tight on its spectrum.*

Proof. We have that $\overline{B}_F = \overline{\mathbf{D}} \times \overline{B}_E \subseteq M_A$ by Proposition 1.6.7. To prove that A is not tight on its spectrum, we consider the function $g : \mathbf{D} \times B_E \longrightarrow \mathbb{C}$ defined by $g(z,x) = \bar{z}$. This function can be extended continuously to M_A and we still denote this function by g. Given $g \in C(M_A)$, we will show that the Hankel-type operator $S_g : A_u(B_F) \longrightarrow C(M_A)/A_u(B_F)$ is not weakly compact. For this, consider a functional $L \in E^*$ and $x_0 \in \overline{B}_E$ such that $\|x_0\| = 1$ and $\|L\| = L(x_0) = 1$. Let $\{f_n\} \subset A$ be the bounded sequence defined by $f_n(z,x) = L(x)^n$. If S_g was weakly compact, then there would exist a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and $h \in C(M_A)$ such that

$$S_g f_{n_k} \xrightarrow{w} h + A.$$

Recall that the dual space of $C(M_A)/A$ is isomorphic to

$$A^{\perp} = \{\mu \in C(M_A)^* : \mu(f) = 0 \text{ for any } f \in A\},$$

so we will refer indistinctly to $\mu(h+A)$ or $\mu(h)$ for any $h \in C(M_A)$ when we deal with $\mu \in (C(M_A)/A)^*$. Define the linear functional $L_b : C(M_A) \longrightarrow \mathbb{C}$ by

$$L_b(f) = f(\frac{1}{2}, e^{ib}x_0) - \frac{1}{2\pi i}\int_{\partial\mathbf{D}} \frac{f(z, e^{ib}x_0)}{z - 1/2} dz$$

for all $b \in [0, 2\pi)$. These functionals are well-defined since $f \in C(M_A)$ and $|z - 1/2| \geq 1/2$ for all $z \in \partial\mathbf{D}$, so, in consequence, the integral is well-defined. It is clear that they are linear and continuous since

$$|L_b(f)| \leq \|f\|_\infty + \frac{1}{2\pi}\|f\|_\infty \frac{1}{1/2}\int_{\partial\mathbf{D}} dz \leq 3\|f\|_\infty.$$

In addition, for any $f \in A_u(\partial\mathbf{D} \times B_E)$, we have that

$$f(\frac{1}{2}, e^{ib}x_0) = \frac{1}{2\pi i}\int_{\partial\mathbf{D}} \frac{f(z, e^{ib}x_0)}{z-1/2}dz \text{ for any } z \in \partial\mathbf{D}.$$

Indeed, the functions defined by $g_r(z) = f(z, re^{ib}x_0)$ converge uniformly to the function $g(z) = f(z, e^{ib}x_0)$ on $z \in \partial\mathbf{D}$ when $r \to 1^-$ for any $0 < r < 1$ and, therefore, the convergence is uniform when we divide by $z - 1/2$ since $|z - 1/2| \geq 1/4$. Then

$$\frac{1}{2\pi i}\lim_{r\to 1^-}\int_{\partial\mathbf{D}} \frac{f(z, re^{ib}x_0)}{z-1/2}dz = \frac{1}{2\pi i}\int_{\partial\mathbf{D}}\lim_{r\to 1^-} \frac{f(z, re^{ib}x_0)}{z-1/2}dz$$

and by the Cauchy Formula we obtain

$$f(\frac{1}{2}, e^{ib}x_0) = \lim_{r\to 1^-} f(\frac{1}{2}, re^{ib}x_0) = \lim_{r\to 1^-}\frac{1}{2\pi i}\int_{\partial\mathbf{D}} \frac{f(z, re^{ib}x_0)}{z-1/2}dz =$$

$$\frac{1}{2\pi i}\int_{\partial\mathbf{D}}\lim_{r\to 1^-} \frac{f(z, re^{ib}x_0)}{z-1/2}dz = \frac{1}{2\pi i}\int_{\partial\mathbf{D}} \frac{f(z, e^{ib}x_0)}{z-1/2}dz.$$

In consequence, we have that $L_b(f) = 0$ for any $f \in A$ and thus $L_b \in A^\perp$ for any $b \in [0, 2\pi)$. Since gf_{n_k} converges weakly to $L_b(h)$, we have

$$L_b(gf_{n_k}) \to L_b(h) \text{ for any } b \in [0, 2\pi).$$

The functionals L_b on gf_{n_k} are given by

$$L_b(gf_{n_k}) = \frac{1}{2}e^{in_kb}L(x_0)^{n_k} - \frac{1}{2\pi i}\int_{\partial\mathbf{D}} \frac{\bar{z}e^{in_kb}L(x_0)^{n_k}}{z-1/2}dz.$$

Since $L(x_0) = 1$ and $\bar{z} = 1/z$ for any $z \in \partial\mathbf{D}$, we have that

$$\int_{\partial\mathbf{D}} \frac{\bar{z}e^{in_kb}}{z-1/2}dz = e^{in_kb}\int_{\partial\mathbf{D}} \frac{1}{z(z-1/2)}dz = e^{in_kb}\int_{\partial\mathbf{D}}\left(-\frac{2}{z} + \frac{2}{z-1/2}\right)dz =$$

$$e^{in_kb}\left[-2\int_{\partial\mathbf{D}}\frac{1}{z}dz + 2\int_{\partial\mathbf{D}}\frac{1}{z-1/2}dz\right].$$

By the Cauchy Formula we have that both integrals are equal to $2\pi i$, so

$$\int_{\partial\mathbf{D}} \frac{\bar{z}e^{in_kb}}{z-1/2}dz = 0$$

and we obtain that $L_b(gf_{n_k}) = \frac{1}{2}e^{in_k b}$. In consequence, $e^{in_k b}/2 \to L_b(h)$ for any $b \in [0, 2\pi)$, so $|L_b(h)| = 1/2$ and therefore

$$e^{i(n_{k+1}-n_k)b} = \frac{\frac{1}{2}e^{in_{k+1}b}}{\frac{1}{2}e^{in_k b}} \to \frac{L_b(h)}{L_b(h)} = 1.$$

Set $m_k = n_{k+1} - n_k$ and consider, passing to a subsequence if necessary, that $m_k < m_{k+1}$. We obtain that

$$e^{im_k b} \to 1$$

and, by the Lebesgue Dominated Convergence Theorem, we have

$$\lim_{k\to\infty} \int_0^{2\pi} e^{im_k b} db = \int_0^{2\pi} 1 db. \tag{5.1}$$

The first integral equals to

$$\int_0^{2\pi} e^{im_k b} db = \left[\frac{1}{im_k} e^{im_k b}\right]_0^{2\pi} = e^{2\pi m_k} - e^0 = 1 - 1 = 0 \text{ for any } k \in \mathbb{N}$$

so equality 5.1 is a contradiction. We conclude that S_g is not weakly compact and, therefore, the algebra A is not tight on its spectrum. □

Now, we turn on attention to the tightness for $H^\infty(B_E)$. Recall that L^∞ denotes the space of bounded measurable complex-valued functions on the unit circle $\partial\mathbf{D}$, where the measure considered is the normalized Lebesgue measure on $\partial\mathbf{D}$. It is well-known that this space becomes a Banach algebra endowed with the norm given by the essential supremum (see [Hof62]). It is also easy that H^∞ is a closed subalgebra of L^∞. Furthermore, the spectrum M_{L^∞} is the Shilov boundary of H^∞. We also have that L^∞ is isometrically isomorphic to $C(\partial H^\infty)$ via the Gelfand transformation. For further results about H^∞ as a Banach algebra, see [Hof62].

The following result given by J. A. Cima, S. Janson and K. Yale [CJY89] about the big Bourgain algebra of H^∞ will be needed to prove Theorem 5.3.7,

Theorem 5.3.4. *Let H^∞ be a closed subalgebra of L^∞. Then the Bourgain algebra $(H^\infty)_b$ equals to $H^\infty + C(\partial\mathbf{D})$.*

Let B be a uniform algebra and L a closed subset of its spectrum M_B. The algebra $B_{|L}$ is given by the restrictions to L of elements in B. We will also need the following proposition for Theorem 5.3.7,

Proposition 5.3.5. *Let A, B be uniform Banach algebras. Suppose that A is tight on its spectrum and let L be a closed subset of the spectrum M_B such that $B_{|L}$ is closed in $C(L)$. If there is an onto homomorphism $T : A \longrightarrow B_{|L}$, then $B_{|L}$ is tight on L.*

Proof. Since T is onto, the restricted adjoint $T^* : L \longrightarrow M_A$ is a one to one continuous mapping. We consider the continuous linear multiplicative mapping $T^{**} : C(M_A) \to C(L)$ given by $T^{**}(h) = h \circ T^t$ for all $h \in C(M_A)$. It is clear that for any $a \in A$, this operator on the Gelfand transform $\hat{a}$ equals to $T(a)$, that is, $T^{**}(\hat{a}) = T(a)$. Observe also that the linear mapping $\Psi : C(M_A)/A \longrightarrow C(L)/B_{|_L}$ defined by

$$\Psi(h+A) = T^{**}(h) + B_{|_L}$$

is well-defined and continuous as well, since

$$|||T^{**}(h) + B_{|_L}||| = \inf_{b \in B_{|_L}} ||T^{**}(h) + b|| = \inf_{a \in A} ||T^{**}(h) + T^{**}(a)|| \leq$$

$$||T^{**}|| \inf_{a \in A} ||h + a|| = ||T^{**}|| \cdot |||h + A|||.$$

Since T is onto, by the Bartle-Graves selection principle 1.1.3, there is a continuous mapping $S : B_{|_L} \to A$ such that $T \circ S = id_{B_{|_L}}$.

We prove that S_g is weakly compact for any $g \in C(L)$. Given $g \in C(L)$, the function $g \circ (T^*)^{-1}$ is continuous on the compact subset $T^*(L)$ of M_A since T^* is one to one. Hence, $g \circ (T^*)^{-1}$ may be extended to M_A in a continuous way. Denote this extension by $\tilde{g}$. Thus, $\tilde{g} \circ T^* = g$. Now we show that

$$\hat{b}g = T^{tt}(\hat{S(b)}\tilde{g}) \text{ for any } b \in B_{|_L}.$$

Indeed, if $u \in L$, we have

$$(\hat{b}g)(u) = u(b) \cdot g(u) = u(T(S(b))) \cdot g(u) = (u \circ T)(S(b)) \cdot g(u) =$$

$$\hat{S(b)}(u \circ T) \cdot (\tilde{g} \circ T^t)(u) = \hat{S(b)}(T^t(u)) \cdot \tilde{g}(T^t(u)) = (\hat{S(b)} \cdot \tilde{g})(T^t(u)) = T^{tt}(\hat{S(b)}\tilde{g})(u).$$

Thus we have shown that $\Psi \circ S_{\tilde{g}} \circ S = S_g$ and therefore we obtain that S_g is weakly compact. □

To prove the main theorem on tightness for $H^\infty(B_E)$, we summarize in the following lemma some results related to the spectrum of H^∞ and $H^\infty + C(\partial \mathbf{D})$. We will denote by H_c^∞ the space $H^\infty + C(\partial \mathbf{D})$. These results can be found in [Gar81].

Lemma 5.3.6. *We have the following results,*

a) The spectrum of H_c^∞ is given by $M_{H^\infty} \setminus \mathbf{D}$.

b) Consider ∂H^∞, the Shilov boundary of H^∞. Then, there are elements on M_{H^∞} which are not in $\mathbf{D} \cup \partial H^\infty$.

In addition, recall also that J. Bourgain proved that H^∞ has the Dunford-Pettis in [Bou84b]. Then, we obtain the result for $H^\infty(B_E)$,

Theorem 5.3.7. *Let E be a Banach space. Then, $H^\infty(B_E)$ is not tight on its spectrum regardless the Banach space E.*

Proof. First we prove that H^∞ is not tight on its spectrum. By Lemma 5.3.6 a), the spectrum of H_c^∞ is given by $M_{H^\infty} \setminus \mathbf{D}$. We claim that H_c^∞ does not coincide with L^∞. Indeed, if these spaces coincide, their spectra would also coincide and, as we have mentioned, the spectrum of L_∞ is the Shilov boundary ∂H^∞ of H^∞. Thus $\partial H^\infty = M_{H^\infty} \setminus \mathbf{D}$; that is, $M_{H^\infty} = \partial H^\infty \cup \mathbf{D}$. However, we know by Lemma 5.3.6 b) that there are elements in $M_{H^\infty} \setminus (\partial H^\infty \cup \mathbf{D})$, a contradiction.

Since H^∞ has the DP property, every weakly compact Hankel-type operator $S_g : H^\infty \longrightarrow C(\partial H^\infty)/H^\infty$ is completely continuous. Therefore, we have that

$$\left\{ g \in C(\partial H^\infty) : S_g \text{ is weakly compact } \right\} \subset (H^\infty)_b .$$

However, the Bourgain algebra $(H^\infty)_b$ of H^∞ in L^∞ is H_c^∞ by Theorem 5.3.4. In consequence, H^∞ is not tight on ∂H^∞, neither in its spectrum by Proposition 5.3.4.

Now, let E be a Banach space and consider the algebra $H^\infty(B_E)$. Pick $e \in E$ such that $\|e\| = 1$ and set $\varphi \in E^*$, $\|\varphi\| = 1$, such that $\varphi(e) = 1$. Consider the mapping $i : \mathbf{D} \to B_E$ given by $i(z) = ze$ and the composition operator $C_i : H^\infty(B_E) \longrightarrow H^\infty$ given by

$$C_i(x) = x \circ i.$$

This homomorphism is onto since for all $f \in H^\infty$ we have that $f \circ \varphi$ belongs to $H^\infty(B_E)$ and $C_i(f \circ \varphi) = f \circ \varphi \circ i = f$. If $H^\infty(B_E)$ was tight on its spectrum then, by Proposition 5.3.5 above, H^∞ would be a tight algebra on M_{H^∞}, a contradiction. $\square$

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