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Chan Hong-Mo Tsou Sheung Tsun J Bordes and Lukas Nellen

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NEW EXPRESSIONS FOR STRING LOOP AMPLITUDES LEADING TO AN ULTRA-SIMPLE CONCEPTION OF STRING DYNAMICS

CHAN Hong-Mo

**Rutherford Appleton Laboratory,
Chilton, Didcot, Oxon, OX11 0QX, England.**

TSOU Sheung Tsun

**Mathematical Institute
24-29 St Giles, Oxford, OX1 3LB, England**

J BORDES

**Dipartament di Física Téorica, Universitat di Valencia
Valencia, Spain**

Lukas NELLEN¹

**Department of Theoretical Physics, Oxford University
1 Keble Road, Oxford OX1 3NP, England**

Abstract

New expressions are derived for string loop amplitudes as overlap integrals of string wave functionals. They are shown to take the form of exchange terms coming from the Bose-Einstein symmetrisation between string segments. One is thus led to the ultra-simple conception that string theory is basically free, and that ‘string interactions’ are due merely to the fact that strings are composite objects with Bose-Einstein segments as constituents.

¹present address. Inst. für Theoretische Kernphysik, Universität Bonn, Nussallee 14-16,
D-5300 Bonn 1, Germany

1 Introduction

In ordinary quantum mechanics of particles, transition amplitudes can be obtained as Feynman integrals over world-paths; they can also be evaluated as overlap integrals between initial and final state wave functions. Although the two methods are equivalent, in practical calculations, the second is often much more convenient.

For string theory, transition amplitudes are usually given as Polyakov integrals over world-sheets¹⁾. However, by analogy with particle theory, one expects that they can also be expressed as overlap integrals between string wave functionals. Indeed, in a recent paper²⁾, it was shown that tree amplitudes of open strings are expressible as generalised traces similar to the group theory trace factors commonly used to incorporate internal symmetry in string theory³⁾. In particular, for the four-string amplitude, one arrives at a formula of the following form:

$$\mathcal{A}_{tree} = Tr[A.\Omega B.\Omega C.\Omega D] \quad (1.1)$$

where²

$$\Omega = \int_{-\infty}^{\infty} d\lambda e^{\lambda(M+2)}, \quad M = L_1 - L_{-1}, \quad (1.2)$$

L_1 and L_{-1} being the standard Virasoro operators, and A, B, C, D are ∞ -dimensional half-string matrices after Witten⁴⁾. For example, in terms of the standard wave functional $\mathbf{A}[\mathbf{X}]$ of the full-string $\mathbf{X}(\sigma), \sigma = 0 \rightarrow \pi$, the half-string matrix A is defined as:

$$\begin{aligned} A[X_1, X_2] &= \mathbf{A}[\mathbf{X}], \\ X_1(\sigma) &= \mathbf{X}(\sigma); \sigma = 0 \rightarrow \pi/2, \\ X_2(\sigma) &= \mathbf{X}(\pi - \sigma); \sigma = \pi/2 \rightarrow \pi, \end{aligned} \quad (1.3)$$

and the matrix product $.$ is defined as:

$$A.B = \int \delta X_2 \delta Y_1 \delta[X_2 - Y_1] A[X_1, X_2] B[Y_1, Y_2], \quad (1.4)$$

while the trace of a matrix is defined as:

$$Tr A = \int \delta X_1 \delta X_2 \delta[X_1 - X_2] A[X_1, X_2]. \quad (1.5)$$

²This formula differs slightly from that given in ref. 2), where the factor $\exp 2\lambda$ in Ω was inadvertently missed. The arguments presented there did not specify the integration measure in λ , which can be fixed only by appealing to $SL(2, R)$ invariance of the amplitude, leading then to the $\exp 2\lambda$ factor above. This error went undetected in ref. 2) because it was cancelled by a similar factor omitted by mistake in eq. (37) relating the vertex function to the string wave functional.

Since A, B, C, D in (1.1) can also be understood as the wave functionals of the incoming and outgoing full-string states, the expression can also be regarded as an overlap integral in which the arguments of different string wave functionals are identified and integrated over.

The initial purpose of the present investigation is to generalise the above result not only to tree amplitudes of closed strings, but also to loop amplitudes involving both closed and open strings. The extension to closed string tree amplitudes is intuitively straightforward. That the result can be generalised also to loop amplitudes of higher topologies may at first sight be surprising but can readily be seen from the following example. We recall first that an open string wave functional, $\Psi[\mathbf{X}]$ say, can be regarded as a Feynman or Polyakov integral over all semi-infinite strips bounded by $\mathbf{X}(\sigma)$ at $\tau = 0$ and satisfying open boundary conditions at all τ ⁵⁾. Thus, the expression (1.1) for the tree amplitude may be considered as the overlap integral in which the planar strips representing the wave functionals $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ are sewn together as shown in Fig. 1, (a) and (b). [That there are two graphically different cases is a consequence of ‘duality’ which is encoded in the operator $e^{-\lambda M}$ in (1.2) which reparametrises σ , shifting in particular the mid-point $\sigma = \pi/2$ of the string to the left or right depending on the sign of λ .] Suppose now, instead, we sew together $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$, not as in Fig. 1, but by sewing first the extremities of their boundaries at $\tau = 0$ to one another giving fig. 2(a), then the middle segment of the boundary of $\mathbf{A}$ to the middle segment of the boundary of $\mathbf{C}$ giving Fig. 2(b). If we then finally sew together the remaining segments of the boundaries of $\mathbf{B}$ and $\mathbf{D}$ represented by the two little loops in Fig. 2(b), we would obtain no longer a planar surface but a surface with a handle as depicted in Fig. 2(c). Building on this example, it is then not hard to see that one can construct even more complicated topologies as overlap integrals in a similar fashion.

Alternatively, if one prefers rather to generalise Witten’s matrix notation adopted in eq. (1.1), one may regard a wave functional, $\Psi[\mathbf{X}]$ say, as an ∞ -rank tensor with a continuum of tensor indices $\mathbf{X}(\sigma)$ labelled by $\sigma = 0 \rightarrow \pi$. The tree amplitude in (1.1) may then be considered as a scalar quantity obtained by saturating the indices of four such tensors A, B, C, D in a particular fashion, namely according to the rules of matrix multiplication and trace defined in (1.4) and (1.5). Similarly, the loop amplitude in Fig. 2 may be regarded as a scalar quantity obtained from the same tensors but by saturating their indices in a different manner. We shall have occasion to refer to this interpretation again in the future, although most of the time we find it more convenient to think of amplitudes as overlap integrals.

Working along these lines, one will arrive at new expressions for string loop amplitudes as overlap integrals of string wave functionals, which, apart from being theoretically interesting, may conceivably be more convenient in certain applications. In addition, one deduces from these expressions a consequence, which we regard as even more important, namely, a new physical conception of string dynamics of the utmost simplicity. It will appear that the new expressions for string amplitudes are all of the form of exchange integrals, quite analogous to those occurring, say, in deuteron - deuteron scattering. There, each deuteron being made up of a proton and a neutron, a two-deuteron wave function ought to be antisymmetrised between the two protons as well as between the two neutrons. Therefore, in evaluating the overlap integral between the two-deuteron incoming and outgoing states, there will be terms in which the proton and neutron from one single incoming deuteron are contracted with a proton and a neutron from two different outgoing deuterons. Although these represent no more than Fermi statistical effects of the constituents they will appear as effective interactions between the deuterons when regarded as elementary particles. Suppose now we consider strings not to be elementary but composite objects, being made up of, say, Bose-Einstein segments. Then in evaluating overlap integrals, there will also be exchange terms in which the segments from one single string are contracted with segments from several different other strings. Again these terms will appear as interaction between strings if the strings are considered as elementary, even though they represent no more than effects of Bose-Einstein symmetrisation between the constituent segments. Indeed, our discovery is that the new expressions we found for string amplitudes are just exchange integrals of this general type. However, because of parametrisation independence, the specification of exactly what constitutes a string segment is rather delicate, and it is this which gives rise to the intricate complexity of string interactions. The surprising conclusion one obtains then is that string theory can be regarded as a completely free theory, and all the effects we used to interpret as string interactions can be traced to the assertion that strings are composite objects with Bose string segments as constituents. Such effects include in particular transitions between initial and final states with different number of strings, i.e. string creation and annihilation, so that there is in principle no need even to second quantise the theory. We believe that the realisation of this fact may be of use to the long-standing search for a more basic understanding of string theory.

2 Initial Feasibility Study

Our first concern is to see whether surfaces of any topology can indeed be obtained as overlap integrals of the wave functionals of external string states. Consider first surfaces representing amplitudes of open strings which are simpler to visualise. Recalling that wave functionals $\Psi[\mathbf{X}]$ of open strings can be regarded as semi-infinite strips bounded by $\mathbf{X}(\sigma), \sigma = 0 \rightarrow \pi$, at $\tau = 0$, one sees that the question of whether a given loop amplitude is expressible as an overlap integral is the same as that of whether the Riemann surface representing that amplitude can be cut into planar pieces each corresponding to the wave functional of an external string state.

Let us examine first the simplest loop diagram, represented by the cylinder of Fig. 3, say, with the external open strings attached as shown in Fig. 3(a). It can be cut into four planar strips, each corresponding to an external state, as indicated. [This is not the only way in which the four wave functionals can be sewn together to give the cylinders, other ways being obtainable from it by applying ‘duality’ as illustrated in Fig. 1, but this need not bother us at present since we are interested now only in ascertaining the feasibility.] A similar conclusion attains even when the external states are attached to the cylinder differently, as depicted in Fig. 3(b), or as in our old example of Fig. 2(c) where all four external states are attached to one single boundary.

Next, consider the torus diagram obtained by adding a handle to the cylinder. Again, it can be cut into four planar strips, each corresponding to the wave functional of an external string state, the conclusion remaining valid when the external states are differently attached, as e.g. in Fig. 4(a) and (b). Having constructed these examples, it is then easily seen that the assertion can be generalised to diagrams with any number of handles, as illustrated e.g. in Fig. 5. We are thus led to the conclusion that Riemann surfaces of any topology can always be cut into planar pieces, each attached to one external string, which conclusion is necessary if all string loop amplitudes were to be obtained as overlap integrals of external string wave functionals.

Amplitudes with external closed strings attached can be similarly studied. The wave functionals $\Phi[\mathbf{X}]$ should now be regarded as a Feynman integral over all semi-infinite cylinders ending at $\tau = 0$ in the closed string $\mathbf{X}(\sigma), \sigma = 0 \rightarrow \pi$. The condition that a certain amplitude be obtainable as the overlap integral of external state wave functionals requires now that the Riemann surface representing the amplitude can be cut up into cylinders, each corresponding to an external state.

That the tree diagram can be so dissected is obvious, as seen in Fig. 6. The case of the torus is not much more difficult, and is indicated in Fig. 7. The generalisation to higher loop diagrams is similar. Again, one comes to the conclusion that all surfaces of whatever topology can be cut up into simple cylindrical pieces each attached to an external string state, as we expected.

3 Construction of the Closed String Tree Amplitude

The arguments presented in the last section validated a necessary condition for expressing all loop amplitudes as overlap integrals, but were insufficient to show us, given the wave functionals of external string states, how exactly to sew them together to obtain the scattering amplitudes. In the case of the open string tree amplitude²⁾, we were able to guess the correct formula simply by drawing on the analogy with the group theory trace factor for internal symmetry, and by an insistence on duality. We plan therefore to follow a similar procedure here for the closed string tree amplitude, but using now the open string tree amplitude itself as an analogue.

The open string tree amplitude was written basically just as a trace of Witten half-string matrices, only with the insertion of the ‘dualising operator’ (1.2) everywhere appropriate to remove the unsatisfactory dependence of the matrices on the choice of the strings’ mid-points. Suppose we now conjecture that closed string tree amplitudes can also be expressed in a similar fashion. In order to do so, we shall first have to suggest a criterion for interpreting closed string wave functionals as matrices, and then to define the trace and the product for these matrices.

Following the standard convention, we parametrise a closed string by a function $\mathbf{X}(\sigma)$, $\sigma = 0 \rightarrow \pi$, satisfying the cyclic boundary condition $\mathbf{X}(\sigma) = \mathbf{X}(\sigma + \pi)$. A closed string state is then represented by a wave functional, say $\Phi[\mathbf{X}]$, of the function $\mathbf{X}$. The most obvious generalisation of Witten’s matrices from open to closed strings is again to chop the string into two halves, say the first half: $\sigma = 0 \rightarrow \pi/2$, and the second half: $\sigma = \pi/2 \rightarrow \pi$, and to regard the functional $\Phi[\mathbf{X}]$ as a matrix $\Phi[X_1, X_2]$ with:

$$\begin{aligned} X_1(\sigma) &= \mathbf{X}(\sigma), \quad \sigma = 0 \longrightarrow \pi/2, \\ X_2(\sigma) &= \mathbf{X}(\pi - \sigma), \quad \sigma = \pi/2 \longrightarrow \pi, \end{aligned} \tag{3.1}$$

acting respectively as row and column indices. One can then proceed to define the product and the trace in exactly the same way as (1.4) and (1.5) respectively.

Thus, for example, the product can be visualised as in Fig. 8. Notice that so long as one keeps to a given σ -parametrisation so that the points $\sigma = 0$ and $\pi/2$ are both fixed, the product so defined is, at present level of sophistication, associative.

Suppose, by analogy with the open string, we propose that the tree amplitude be expressed again as the trace of a product of these matrices. It is clear that if we were to take simply the product of four such matrices, say A, B, C, D in that order, and take the trace, recalling at the same time that each of them can be regarded as a cylinder, with the string occurring in its argument as the cylinder's boundary at $\tau = 0$, we would obtain Fig. 6 as the result, namely the surface for the four-closed-string tree amplitude we desire. The only problem is again that the answer depends unsatisfactorily on the choice of the special points $\sigma = 0, \pi/2$ and violates therefore duality. Let us try to cure this defect as before by inserting 'dualising operators' between the matrices.

Let then $V_{i_r}, r = 1, \dots, N$ be the matrices representing the wave functionals of the r th external closed string state. Our conjecture for the tree amplitude is:

$$\mathcal{A}_N = Tr[V_{i_1} \cdot \Omega V_{i_2} \cdot \Omega \dots \Omega V_{i_N}], \quad (3.2)$$

where Ω is some 'dualising operator' yet to be specified which is to play the role of Ω in (1.2) for the open string. What should we choose for Ω ?

Presumably, Ω for closed strings has to be constructed from the six Virasoro operators, for $n = 0, \pm 1$:

$$L_n = \int_0^\pi d\sigma e^{-2in\sigma} T_{--} = \int_0^\pi d\sigma e^{-2in\sigma} \left\{ -i\pi \frac{\delta}{\delta \mathbf{X}(\sigma)} + \mathbf{X}'(\sigma) \right\}^2, \quad (3.3)$$

$$\tilde{L}_n = \int_0^\pi d\sigma e^{2in\sigma} T_{++} = \int_0^\pi d\sigma e^{2in\sigma} \left\{ -i\pi \frac{\delta}{\delta \mathbf{X}(\sigma)} - \mathbf{X}'(\sigma) \right\}^2, \quad (3.4)$$

generating the $sl(2, C)$ algebra under which closed string tree amplitudes are invariant. Of these the following three:

$$L = L_0 - \tilde{L}_0, \quad (3.5)$$

$$M = M_1 + \tilde{M}_1, \quad (3.6)$$

$$K = K_1 - \tilde{K}_1, \quad (3.7)$$

with

$$M_1 = L_1 - L_{-1}; \quad \tilde{M}_1 = \tilde{L}_1 - \tilde{L}_{-1}, \quad (3.8)$$

$$K_1 = L_1 + L_{-1}; \quad \tilde{K}_1 = \tilde{L}_1 + \tilde{L}_{-1}, \quad (3.9)$$

generate an $sl(2, R)$ subalgebra which reparametrises σ only leaving $\tau = 0$ unchanged. In particular, the operator M in (3.6) which is explicitly:

$$M = \int_0^\pi d\sigma (-4\pi) \sin 2\sigma \mathbf{X}'(\sigma) \frac{\delta}{\delta \mathbf{X}(\sigma)}, \quad (3.10)$$

reparametrises σ in such a way as to shrink one-half of the string, namely the segment between $\sigma = \pi/4$ and $\sigma = 3\pi/4$, while expanding its complement, and is thus similar in effect to M of the open string in (1.2). A possible suggestion for Ω for the closed string is thus simply to replace M in (1.2) by the M in (3.6). At first sight, this may not seem enough, since it still leaves the points $\sigma = \pi/2$ and $\sigma = 0, \pi$ fixed. However, one recalls that $\mathbf{X}(\sigma)$ is cyclic, so that functionals of $\mathbf{X}$ are annihilated by the operator L in (3.5) which generates ‘rigid rotations’ of the string, so that one is free to insert it at every stage to shift $\sigma = \pi/2$ or $\sigma = 0, \pi$ to any other point. This means that choosing Ω as in (1.2) may already be sufficient, apart from the integration measure in λ , which cannot be determined by the present arguments. An *a posteriori* appeal to $SL(2, C)$ invariance of the amplitude fixes the measure as $d\lambda \exp 4\lambda$, which gives then for closed strings the formula:

$$\Omega = \int_{-\infty}^{\infty} d\lambda e^{\lambda(M+4)} \quad (3.11)$$

with M as given in (3.6).

The proof that the suggested formula (3.2) is indeed equivalent to the standard formula for the closed string tree amplitude follows closely that in ref 2) for the tree amplitude of open strings. We note first that although the dualising operator Ω in (3.2), being not a half-string but a full-string operator, acts on the whole quantity to its right, it nevertheless still satisfies the identity:

$$Tr(A.(\Omega B)) = Tr((\Omega A).B), \quad (3.12)$$

as for open strings. Further, again as for open strings, the vertex function in the conventional full-string formulation is related to the half-string quantities here via the formula:

$$\langle j_r \parallel \mathcal{V}_{ir}(\sigma) \parallel j_{r+1} \rangle = \lim_{\substack{\alpha \rightarrow \infty \\ \beta \rightarrow \infty}} Tr((e^{-\alpha M} A_{j_r}) V_{ir}(e^{\beta M} A_{j_{r+1}})) \quad (3.13)$$

for any half-string matrices A_{j_r} and $A_{j_{r+1}}$ corresponding to the wave functionals of full-string states $\parallel j_r \rangle$ and $\parallel j_{r+1} \rangle$. The reason for the presence in (3.13) of the operators $e^{-\alpha M}$ and $e^{\beta M}$ with $\alpha, \beta \rightarrow \infty$, which shrink one half-string to a point is similar to that in the open string case, namely that in the conventional formalism, closed strings are meant to interact in the manner illustrated in Fig. 9(b), and

not in the manner 9(a) implied by the product depicted in Fig. 8. With these two observations, one can take over formally all the arguments from equation (29) to (39) of ref. 2) although the symbols there represent different quantities, and rewrite the expression (3.2) here also as:

$$\mathcal{A}_N \cong \int_0^\infty d\lambda'_2 \dots d\lambda'_N < 0 \parallel \mathcal{V}_{i_1}(0) e^{\lambda'_2(M+4)} \mathcal{V}_{i_2}(0) \dots \mathcal{V}_{i_N}(0) \parallel 0 > . \quad (3.14)$$

where for convenience of exposition, a constant (infinite) factor pertaining to the normalisation of the vacuum state has been omitted.

Effect next a transformation of $\mathcal{A}_N$ by $\exp \frac{\pi}{4}(K_1 + \tilde{K}_1)$, for K_1 and $\tilde{K}_1$ defined as in (3.9). The point $z = 0$ is mapped to $z = 1$ and M is changed to $-2(L_0 + \tilde{L}_0)$. We obtain then instead,

$$\mathcal{A}_N \cong \int_0^\infty d\lambda'_2 \dots d\lambda'_N < 0 \parallel \mathcal{V}_{i_1}(1) e^{-2\lambda'_2(L_0 + \tilde{L}_0 - 2)} \mathcal{V}_{i_2}(1) \dots \mathcal{V}_{i_N}(1) \parallel 0 > . \quad (3.15)$$

Next, since the vacuum state $\parallel 0 >$ is annihilated by $L = L_0 - \tilde{L}_0$, and since the operators $\mathcal{V}_{i_r}$ (being the standard closed (full)-string vertex function) commutes with L , we may insert in front of each $\mathcal{V}_{i_r}$ in (3.15) an integral:

$$\int_0^{2\pi} d\phi_r e^{i\phi_r(L_0 - \tilde{L}_0)} \quad (3.16)$$

without changing the value of the expression. Hence defining,

$$\rho_r = e^{-2\lambda'_r}, \quad (3.17)$$

we may write (3.15) as

$$\begin{aligned} \mathcal{A}_N &\cong \int_0^1 \frac{d\rho_2 \dots d\rho_N}{\rho_2 \dots \rho_N} \int_0^{2\pi} d\phi_2 \dots d\phi_N \\ &< 0 \parallel \mathcal{V}_{i_1}(1) \rho_2^{(L_0 + \tilde{L}_0 - 2)} e^{i\phi_2(L_0 - \tilde{L}_0)} \mathcal{V}_{i_2}(1) \dots \mathcal{V}_{i_N}(1) \parallel 0 >, \end{aligned} \quad (3.18)$$

or, equivalently, in terms of the complex variable:

$$\zeta_r = \rho_r e^{i\phi_r} \quad (3.19)$$

as:

$$\mathcal{A}_N \cong \int_{|\zeta_r| \leq 1} \frac{d^2\zeta_2 \dots d^2\zeta_N}{|\zeta_2|^4 \dots |\zeta_N|^4} < 0 \parallel \mathcal{V}_{i_1}(1) \zeta_2^{L_0} \bar{\zeta}_2^{\tilde{L}_0} \mathcal{V}_{i_2}(1) \dots \mathcal{V}_{i_N}(1) \parallel 0 > . \quad (3.20)$$

Finally, changing the variables of integration to z_r , with

$$\zeta_r = z_r / z_{r-1}, \quad (3.21)$$

reduces the expression in (3.20) to the form:

$$\mathcal{A}_N \cong \int_{|z_1| < \dots < |z_N|} \frac{d^2 z_2 \dots d^2 z_N}{|z_2|^2 \dots |z_N|^2} \left| \frac{z_1}{z_N} \right|^2 < 0 \parallel \mathcal{V}_{i_1}(z_1) \mathcal{V}_{i_2}(z_2) \dots \mathcal{V}_{i_N}(z_N) \parallel 0 >, \quad (3.22)$$

where we have used the fact that:

$$\mathcal{V}_{i_r}(z_r) = z_r^{L_0} \bar{z}_r^{\bar{L}_0} \mathcal{V}_{i_r}(1) z_r^{-L_0} \bar{z}_r^{-\bar{L}_0}, \quad (3.23)$$

and the fact that both L_0 and $\bar{L}_0$ annihilate the vacuum, so that:

$$z_N^{L_0} \bar{z}_N^{\bar{L}_0} \parallel 0 > = z_1^{L_0} \bar{z}_1^{\bar{L}_0} \parallel 0 > = \parallel 0 >. \quad (3.24)$$

The expression (3.22) is the standard formula⁵⁾ for the closed string tree amplitude apart from some (infinite) constant factors due to $SL(2, C)$ invariance which can be handled in the usual manner.

4 Loop Amplitudes

Consider for simplicity only loop amplitudes with closed strings as external states. The diagram in Fig. 7 is an example of how a surface of nontrivial topology can be built up by sewing simple cylindrical pieces together. From this illustration, one can imagine constructing the torus amplitude from the wave functionals of four external states as follows. We join first strings **X** and **Y** together to form string **S**, and **Z** and **W** together to form **T**. We then squeeze **S** in the middle to give two smaller loops, say, **S**₁ and **S**₂, and again **T** to give two other smaller loops **T**₁ and **T**₂. Finally, we match **S**₁ to **T**₁ and **S**₂ to **T**₂. The steps are illustrated in Fig. 10. Recalling that the strings **X**, **Y**, **Z** and **W** are the $\tau = 0$ edges of respectively the cylinders **A**, **B**, **C** and **D** representing the wave functionals of the corresponding states, one sees that one obtains the diagram of fig. 7 as desired.

Notice that what we have designated above in a rather colourful language as joining, squeezing and matching, are all basically the same process, namely the identification of two segments from either the same string or two different strings, and then integrating over all the co-ordinates of the identified segments. However, this statement is not as innocent as it may sound, for a string segment is a priori just a continuous collection of points, and in saying that we identify two segments, it is not clear which point in one segment is to be identified to which in the other segment. We have to be more precise.

Now, from the experience gained in section 3, we know in certain cases how this may be managed. First, in defining the product between functionals of two

closed strings, as in Fig. 8, we have learnt how to identify the second half of a string $\mathbf{X}$, i.e. $\mathbf{X}(\sigma), \sigma = \pi/2 \rightarrow \pi$, to the first half of another string $\mathbf{Y}$, i.e. $\mathbf{Y}(\sigma), \sigma = 0 \rightarrow \pi/2$, namely by putting the point $\mathbf{Y}(\sigma)$ equal to the point:

$$\bar{\mathbf{X}}(\sigma) = \mathbf{X}(\pi - \sigma) \quad (4.1)$$

for all $\sigma = 0 \rightarrow \pi/2$. Second, since by duality we are allowed to reparametrise the strings by operating with $\exp i\phi L$ and $\exp \lambda M$ for any ϕ and λ without changing the amplitude, we can in fact transform any segment on a string into the ‘standard segment’ $\sigma = 0 \rightarrow \pi/2$. That this is so can be seen as follows. The operator $\exp i\phi L$ rigidly rotates the parameter σ of a string by an amount ϕ , thus $\sigma \rightarrow \sigma + \phi$. Hence given any segment, say $\sigma = \sigma_1 \rightarrow \sigma_2$, we can always transform the point $\sigma = \sigma_1$ to the point $\sigma' = (\sigma_1 - \sigma_2)/2$ and, at the same time, $\sigma = \sigma_2$ to $\sigma' = (\sigma_2 - \sigma_1)/2$. Next, the operator $\exp \lambda M$ changes σ' into σ'' , where σ' and σ'' are related by:

$$\sin 2\sigma'' = \frac{\sin 2\sigma'}{\cosh 2\lambda \cos 2\sigma' - \sinh 2\lambda \sin 2\sigma'}, \quad (4.2)$$

so that by a judicious choice of λ , one can transform again σ'_1 to $\sigma''_1 = 3\pi/4$ and σ'_2 to $\sigma''_2 = \pi/4$. Finally, via another rigid rotation by $\pi/4$, we can turn our segment into the standard segment as desired. Given then any two segments on two different strings, say $\mathbf{X}$ and $\mathbf{Y}$, one can always transform first the segment on $\mathbf{X}$ to the ‘conjugate standard segment’ $\bar{\mathbf{X}}(\sigma), \sigma = 0 \rightarrow \pi/2$, and the segment on $\mathbf{Y}$ into the standard segment $\mathbf{Y}(\sigma), \sigma = 0 \rightarrow \pi/2$. Having done so, we can now identify the two standard segments as above by putting $\bar{\mathbf{X}}(\sigma) = \mathbf{Y}(\sigma)$ for all σ . Hence, we now know how to identify any two segments from two different strings.

What is not clear yet is how to identify two segments from the same string, as was implied in the process we called ‘squeezing’ above, which was represented by the step (b) to (c) in Fig. 10. Here, by operating only with $\exp i\phi L$ and $\exp \lambda M$, we cannot transform two arbitrary segments both into standard segments. We have thus to devise some other means for defining such a process.

To do so, we note that ‘squeezing’ occurred above actually only as an intermediate step. We are not therefore really interested in ‘squeezing’ as such, so long as we are able to reproduce its effect in constructing the amplitude. That being the case, we suggest the following procedure which involves only known processes, but will be shown to incorporate already the desired effect. As before, we first multiply strings $\mathbf{X}$ and $\mathbf{Y}$ together to form $\mathbf{S}$, and $\mathbf{Z}$ and $\mathbf{W}$ together to form $\mathbf{T}$. However, rather than ‘squeezing’ $\mathbf{S}$ and $\mathbf{T}$ each to form two smaller loops, we introduce instead two closed strings, say $\mathbf{U}_1$ and $\mathbf{U}_2$, whose product is $\mathbf{X}$, and similarly two further closed strings $\mathbf{V}_1$ and $\mathbf{V}_2$ whose product is $\mathbf{T}$. We then match $\mathbf{U}_1$ to $\mathbf{V}_1$

and U_2 to V_2 . Here, for expediency, we use also the term ‘product’ of two strings, say X and Y , to mean the string obtained by identifying the second half of X to the first half of Y in accordance to (4.1) and then removing the identified halves; namely, it is the string which occurs as the argument in the product (1.4) of two wave functionals A and B of respectively X and Y .

Now although it is clear that the new procedure involves only known processes, namely closed string products and traces as defined in (3.1), (1.4) and (1.5), it is not a priori obvious that it is equivalent to ‘squeezing’, or that it will yield a torus as the result. We shall now demonstrate that this is indeed the case. Instead of proving our assertion algebraically, which will be cumbersome, we prefer to do so diagrammatically, as follows. The internal strings involved are depicted in Fig. 11, where $S = U_1.U_2$, and $T = V_1.V_2$. U_1 and V_1 are really the same strings, apart from a reparametrisation by $\exp \lambda M \exp i\phi L$, because they have in the end to be matched to each other. The same holds for U_2 and V_2 . Because of the reparametrisation, however, what we call the standard segment $\sigma = 0 \rightarrow \pi/2$ in $\bar{U}_1$ will be different in general from the standard segment in $\bar{V}_1$, and similarly in U_2 and V_2 . Let us assume for the moment that the standard segment in $\bar{U}_1$ is disjoint from the standard segment in $\bar{V}_1$, and also that in U_2 is disjoint from that in V_2 , as illustrated in Fig. 12(a) and (b), where dots denote the standard segment in $\bar{U}_1$ and triangles the standard segment in $\bar{V}_1$, and similarly for U_2 and V_2 . Recall now, however, that the product of U_1 and U_2 is by definition S , which means that the standard segment of $\bar{U}_1$ must be identical to the standard segment of U_2 . Similarly, the product of V_1 and V_2 being T , the standard segment of $\bar{V}_1$ must be the same as that of V_2 . The situation is thus as depicted in Fig. 13. The string S being the product of U_1 and U_2 means that it is the string obtained from Fig. 13 by removing the standard segments of $\bar{U}_1$ and U_2 between the dots, namely the diagram (a) in Fig. 14. Similarly, T is the string obtained from Fig. 13 with the segments between the triangles removed, namely diagram (b) in Fig. 14. One sees therefore that each of S and T is indeed in the form of two loops joined by a narrow neck, i.e. ‘squeezed’ in our language above. If we prefer to see a picture more directly comparable to Fig. 10(c), we can imagine pulling out the inside loop in Fig. 13 to obtain Fig. 15; S and T are then as depicted in Fig. 16(a) and (b) respectively. Hence, we have shown that in the case where the standard segments are all disjoint we do obtain a torus from our new procedure.

What happens if the standard segments of $\bar{U}_1$ and $\bar{V}_1$ overlap while those of U_2 and V_2 remain disjoint? The diagrams corresponding to Fig. 13 and 14 are then respectively as shown in Fig. 17 and 18. Since the arguments leading to them are very similar to, though simpler than, the third case where both pairs of

standard segments overlap which we shall next consider, we shall not reproduce the arguments here. We note only that, as in the case above where standard segments are disjoint, both $\mathbf{S}$ and $\mathbf{T}$ are ‘squeezed’ and a torus is obtained.

Finally, consider the situation depicted in Fig. 19, where again dots demarcate the standard segments in $\bar{\mathbf{U}}_1$ and $\mathbf{U}_2$, and triangles those in $\bar{\mathbf{V}}_1$ and $\mathbf{V}_2$, and both pairs of segments overlap. Suppose we wish now to multiply $\mathbf{V}_1$ to $\mathbf{V}_2$ to obtain $\mathbf{T}$, as required in constructing the amplitude, we see that the diagram as drawn in Fig. 19 will not do at all for the following reason. The top empty triangle on the dashed curve is already identified in the figure with the opposite point on the full curve because $\mathbf{U}_1.\mathbf{U}_2 = \mathbf{S}$. Now, we wish further to identify it with the top full triangle since we also want $\mathbf{V}_1.\mathbf{V}_2 = \mathbf{T}$. This is possible only if the top full triangle is itself coincident with the point on the full curve opposite the top empty triangle. In other words, the top full triangle is a double point on the full curve. The same conclusion obtains for all the points on the full curve following the top full triangle until the bottom full dot, so that the full curve must double back on itself in the manner illustrated in Fig. 20. Next, we have to match the dashed curve on to the full curve in such a way that the segment on the dashed curve between the two empty dots matches the segment on the full curve between the two full dots, while the segment on the dashed curve between the two empty triangles matches the segment on the full curve between the two full triangles. The result is the rather complicated diagram of Fig. 21. Now $\mathbf{S}$ is by definition the string obtained by removing from Fig. 21 the segments in both the curves between the dots, while $\mathbf{T}$ is that obtained by removing the segments between the triangles. We arrive therefore at Fig. 22. Again, both $\mathbf{S}$ and $\mathbf{T}$ consist of two closed loops connected by a neck, i.e. ‘squeezed’, and when we match $\mathbf{S}$ to $\mathbf{T}$, we obtain a torus.

One sees therefore that in all three cases, one can avoid the ambiguous process of ‘squeezing’ altogether and replace it by the above procedure involving known operations with standard segments only. Apart from being necessary for the explicit construction of loop amplitudes, this clarification is technically rewarding. In attempting to go beyond our present functional treatment of limited rigour, we are also in the process of developing an oscillator formalism which is reported elsewhere⁶). The fact now that the whole construction of a loop amplitude is reduced to operations with standard segments means that one needs to develop an oscillator formalism only for the standard segment (i.e. the half-string) and not for an arbitrary string segment, which would otherwise have been a formidable task.

Returning then to the torus amplitude, we are now able to write it explicitly as:

$$\begin{aligned} \mathcal{A}_{torus} = \int & \delta \mathbf{S} \delta \mathbf{T} \delta \mathbf{U}_1 \delta \mathbf{U}_2 \delta \mathbf{V}_1 \delta \mathbf{V}_2 \Omega_{\mathbf{S}}[A.B] \delta[\mathbf{S}_L - \mathbf{U}_{1L}] \delta[\mathbf{S}_R - \mathbf{U}_{2R}] \\ & \delta[\bar{\mathbf{U}}_{1R} - \mathbf{U}_{2R}] \Omega_{\mathbf{T}}[C.D] \delta[\mathbf{T}_L - \mathbf{V}_{1L}] \delta[\mathbf{T}_R - \mathbf{V}_{2R}] \\ & \delta[\bar{\mathbf{V}}_{1R} - \mathbf{V}_{2L}] \Omega_{\mathbf{V}_1} \delta[\mathbf{U}_1 - \mathbf{V}_1] \Omega_{\mathbf{V}_2} (\delta[\mathbf{U}_2 - \mathbf{V}_2]), \end{aligned} \quad (4.3)$$

where the products $(A.B)$ and $(C.D)$ between wave functionals are defined as in (1.4), L and R denote the left (first) and right (second) halves of the strings, and the subscript added to the dualising operator Ω defined in (1.2) and (3.6) specifies on which string variable it is to be applied. The rest of the formula is, we believe, self-explanatory, and reproduces what we have stated above in words.

That the expression (4.3) is equivalent to the conventional formula for the torus amplitude will be postponed to the next section. Here we notice only that in (4.3), there are altogether 4 dualising operators Ω , each of which is an integral over a real parameter we called λ in eq. (1.2). In addition, for closed string wave functionals, which are by definition invariant under rigid rotations ($L_0 - \tilde{L}_0$), we can insert after each Ω an integral of the form (3.16), as explained in section 3. The expression in (4.3) therefore contains, apart from the functional integrals displayed, an implicit integral over 8 real parameters, 2 for each Ω . These correspond exactly to the 4 complex moduli of the 4-closed-string torus amplitude in conventional language.

Although we have discussed here in detail only the torus attached to 4 external strings, it is clear that the procedure can be extended to amplitudes of higher topology with any number of external strings attached. For example, to construct an amplitude with one more handle from the torus of Fig. 7, all we need is to take $\mathbf{S}_2$ and $\mathbf{T}_2$ in Fig. 10, ‘squeeze’ each again as in Fig. 10(c), giving each two smaller loops, say, $\mathbf{S}_{21}, \mathbf{S}_{22}$ and $\mathbf{T}_{21}, \mathbf{T}_{22}$, then match $\mathbf{S}_{21}$ to $\mathbf{T}_{21}$, and $\mathbf{S}_{22}$ to $\mathbf{T}_{22}$. The act of ‘squeezing’ can again be implemented by the procedure detailed above by reducing it to known operations with standard segments. One notes, however that by ‘squeezing’ once more, one will need to introduce more dualising operators Ω . To ensure that we do not prejudice any particular choice of what we call the standard segments, we need to apply an Ω to each of $\mathbf{S}_2$ and $\mathbf{T}_2$ before ‘squeezing’ them, and then one more Ω each to $\mathbf{T}_{21}$ and $\mathbf{T}_{22}$ before ‘matching’ to $\mathbf{S}_{21}$ and $\mathbf{S}_{22}$; i.e. altogether 4 Ω ’s. From these, we have to subtract one for the Ω operator which was already present in the torus, acting on $\mathbf{T}_2$ before matching to $\mathbf{S}_2$. We have gained therefore 3 more Ω ’s, i.e. 6 more real integration parameters in adding a handle, corresponding exactly to the 3 extra complex moduli for each increment in genus deduced from the Riemann–Roch Theorem.

5 Explicit Construction of the Torus Amplitude

To show that our procedure gives indeed the equivalent to the conventional torus amplitude, we find it easier to work with a different formula than the one in eq. (4.3), although the latter is more convenient for illustrative purposes. The particular construction described in the section above is clearly not the only way in which a torus can be obtained. We can imagine, for example, that instead of proceeding as in Fig. 10, we first multiply together $A.B.C.D$ in that order, obtaining a functional of say a string $\mathbf{R}$, then ‘squeeze’ $\mathbf{R}$ to form two smaller loops, $\mathbf{R}_1, \mathbf{R}_2$ which we match to each other. The result is again, of course, a torus. Proceeding in this manner, one will arrive at a formula which looks different from, but is in fact the same as, (4.3), apart from an interchange in order of some of the functional integrations. This assertion can be illustrated as in Fig. 23, where the construction of (4.3) is depicted on the left and the new construction on the right. The two constructions differ only in the intermediate step, where in the old construction, $\mathbf{S}$ and $\mathbf{T}$ are squeezed first before being multiplied, while in the new, $\mathbf{S}$ and $\mathbf{T}$ are first multiplied and then ‘squeezed’.

That being the case, we shall work rather with the new construction. As before, however, we have first to specify what ‘squeezing’ means by reducing it to operations on standard segments. Introduce therefore two closed strings $\mathbf{P}_1$ and $\mathbf{P}_2$ such that $\mathbf{P}_1.\mathbf{P}_2 = \mathbf{R}$ and then match $\mathbf{P}_1$ to $\mathbf{P}_2$; we claim that this is equivalent to ‘squeezing’ $\mathbf{R}$ into $\mathbf{R}_1$ and $\mathbf{R}_2$, and then ‘matching’ $\mathbf{R}_1$ to $\mathbf{R}_2$. The argument is similar to that in the old construction though somewhat harder to illustrate. By ‘squeezing’ $\mathbf{R}$, we obtain Fig. 24(a), and then by ‘matching’ $\mathbf{R}_1$ to $\mathbf{R}_2$ after applying to $\mathbf{R}_2$ a dualising operator, we obtain Fig. 24(b). On the other hand, suppose we start with $\mathbf{P}_1$ and $\mathbf{P}_2$ whose product is $\mathbf{R}$, as illustrated in Fig. 25 (a), where full and open dots demarcate the standard segments of $\bar{\mathbf{P}}_1$ and $\mathbf{P}_2$ respectively. After applying a dualising operator to $\mathbf{P}_2$, its standard segment will be changed. Let us, for simplicity of illustration, assume that the new standard segment is disjoint from the old one, so that it is now the segment marked by triangles in the same figure. We are next required to match this new standard segment of $\mathbf{P}_2$ to the standard segment of $\bar{\mathbf{P}}_1$, taking care, in particular, that the directions of the arrows on the two segments are opposite. This is possible - since the standard segment of $\bar{\mathbf{P}}_1$ is itself coincident with the old standard segment of $\mathbf{P}_2$ - only if the string $\mathbf{P}_2$ doubles back on itself in the form of a figure-of-eight. Finally, matching the rest of $\mathbf{P}_1$ on to the rest of $\mathbf{P}_2$, we obtain Fig. 25 (b). Now the string $\mathbf{R}$ is by definition that obtained from Fig. 25 (b) by omitting the segments between the dots on both $\mathbf{P}_1$ and $\mathbf{P}_2$, namely Fig. 25 (c), which

is the same as that in Fig. 24 (b) obtained previously from ‘squeezing’, hence justifying our claim above. From an earlier experience in our old construction, we expect that a similar conclusion will obtain even when the standard segments of $\mathbf{P}_2$ before and after dualisation are not disjoint; this is indeed seen to be correct but the picture is then so involved as to defy simple presentation.

Putting the above statement together into a formula, we have:

$$\mathcal{A}_{torus} = \int \delta \mathbf{P}_1 \delta \mathbf{P}_2 \delta [\bar{\mathbf{P}}_1 - \mathbf{P}_2] Tr[\Delta_1 \Omega(A \Omega(B \Omega(C \Omega(D \Delta_2))))] \quad (5.1)$$

where

$$\Delta_i[\mathbf{X}_i] = \delta[\mathbf{X}_i - \mathbf{P}_i] \quad (5.2)$$

is a functional of the full-string $\mathbf{X}_i$ but considered here as a Witten half-string matrix Δ_i , and the trace Tr occurring in (5.1) is the half-string trace defined in eq. (1.5). Our task now is to show that this is equivalent to the standard formula for the torus amplitude.

To do so, one notes first that the trace occurring in the integral of (5.1) is in fact very similar to the expression (3.2) for the tree amplitude, apart from the replacement of the first and last factors V_{i_1} and V_{i_N} in (3.2) by respectively Δ_1 and Δ_2 in (5.1). Using the same arguments, therefore, one can reduce (5.1) into the following form in terms of full-string quantities, as we did in casting (3.2) into the form (3.14):

$$\begin{aligned} \mathcal{A}_{torus} = & \int \delta \mathbf{P}_1 \delta \mathbf{P}_2 \delta [\bar{\mathbf{P}}_1 - \mathbf{P}_2] \int_0^\infty d\lambda'_1 d\lambda'_2 d\lambda'_3 d\lambda'_4 \\ & \langle \mathbf{P}_1 \parallel e^{\lambda'_1(M+4)} \mathcal{V}_A e^{\lambda'_2(M+4)} \mathcal{V}_B e^{\lambda'_3(M+4)} \mathcal{V}_C e^{\lambda'_4(M+4)} \mathcal{V}_D \parallel \mathbf{P}_2 \rangle, \end{aligned} \quad (5.3)$$

where, instead of $\mathcal{V}_{i_1} \parallel 0 \rangle$ and $\mathcal{V}_{i_N} \parallel 0 \rangle$ in (3.14), which were respectively the full-string states labelling the vertex functions $\mathcal{V}_{i_1}$ and $\mathcal{V}_{i_N}$, we have inserted above in (5.3) the corresponding full-string states $\parallel \mathbf{P}_1 \rangle$ and $\parallel \mathbf{P}_2 \rangle$ represented by the wave functionals Δ_1 and Δ_2 in (5.2). Equivalently, since $\{\parallel \mathbf{P} \rangle\}$ form a complete set of full-string states, we can rewrite (5.3) as:

$$\begin{aligned} \mathcal{A}_{torus} = & \int_0^\infty d\lambda'_1 d\lambda'_2 d\lambda'_3 d\lambda'_4 \text{Tr}[e^{\lambda'_1(M+4)} \mathcal{V}_A(0) e^{\lambda'_2(M+4)} \mathcal{V}_B(0) \\ & e^{\lambda'_3(M+4)} \mathcal{V}_C(0) e^{\lambda'_4(M+4)} \mathcal{V}_D(0)], \end{aligned} \quad (5.4)$$

where Tr is the trace over all full-string states, in contrast to Tr in (5.1) which was a trace over half-string states.

As in section 3, effect next the transformation by $\exp \frac{\pi}{4}(K_1 + \hat{K}_1)$ to turn (5.4) into:

$$\mathcal{A}_{torus} = \int_0^\infty d\lambda'_1 d\lambda'_2 d\lambda'_3 d\lambda'_4 \text{Tr}[e^{-2\lambda'_1(L_0+\tilde{L}_0-2)} \mathcal{V}_A(1) e^{-2\lambda'_2(L_0+\tilde{L}_0-2)} \mathcal{V}_B(1) e^{-2\lambda'_3(L_0+\tilde{L}_0-2)} \mathcal{V}_C(1) e^{-2\lambda'_4(L_0+\tilde{L}_0-2)} \mathcal{V}_D(1)]. \quad (5.5)$$

Then again, since the operators $\mathcal{V}_A$ etc. commute with $(L_0 - \tilde{L}_0)$, we may insert in front of each the integral (3.16) to turn (5.5) into:

$$\mathcal{A}_{torus} = \int_{|\zeta_r| \leq 1} \frac{d^2\zeta_1 d^2\zeta_2 d^2\zeta_3 d^2\zeta_4}{|\zeta_1|^4 |\zeta_2|^4 |\zeta_3|^4 |\zeta_4|^4} \times \text{Tr}[\zeta_1^{L_0} \bar{\zeta}_1^{\tilde{L}_0} \mathcal{V}_A(1) \zeta_2^{L_0} \bar{\zeta}_2^{\tilde{L}_0} \mathcal{V}_B(1) \zeta_3^{L_0} \bar{\zeta}_3^{\tilde{L}_0} \mathcal{V}_C(1) \zeta_4^{L_0} \bar{\zeta}_4^{\tilde{L}_0} \mathcal{V}_D(1)], \quad (5.6)$$

for

$$\zeta_r = e^{-2\lambda_r} e^{i\phi_r}. \quad (5.7)$$

Changing the intergration variables to z_r , where

$$\zeta_1 = z_1, \quad \zeta_r = z_r/z_{r-1}, \quad r = 2, 3, 4. \quad (5.8)$$

we have then

$$\mathcal{A}_{torus} = \int \frac{d^2w}{|w|^4} \int \prod_{r=1}^3 \frac{d^2z_r}{|z_r|^2} \theta(|z_r| - |z_{r+1}|) \text{Tr}[\mathcal{V}_A(z_1) \mathcal{V}_B(z_2) \mathcal{V}_C(z_3) \mathcal{V}_D(z_4) w^{L_0} \bar{w}^{\tilde{L}_0}] \quad (5.9)$$

which is a standard form of the torus amplitude, with as usual:

$$\mathcal{V}_A(z_1) = z_1^{L_0} \bar{z}_1^{\tilde{L}_0} \mathcal{V}_A(1) z_1^{-L_0} \bar{z}_1^{-\tilde{L}_0}, \dots \quad (5.10)$$

The only remaining point to discuss is the integration region, given only symbolically above in (5.9). The integration region of three of the integrals are correctly prescribed. For the remaining variable, say $\tau = \ln \omega / 2\pi$, however, the integral was given above as over the upper half-plane, but we know, of course, that its range is actually limited by unitarity to only one fundamental region, usually taken to be $-\frac{1}{2} \leq \text{Re}\tau \leq \frac{1}{2}, \text{Im}\tau > 0, |\tau|^2 > 1$, because the amplitude has an $SL(2, Z)$ invariance. This means that the upper τ -half-plane is divided into an ∞ number of fundamental regions which are all equivalent and should be counted only once. In our present language, this means that in implementing the contraction of string segments for calculating overlap integrals by the ‘standard segment prescription’, some contractions are counted more than once. That this is so is clear for those repetitions due to one of the $SL(2, Z)$ generators: $\tau \rightarrow \tau + 1$. We recall that $\tau \rightarrow \tau + \psi/2\pi$ represents a rigid rotation of σ in $\mathbf{P}_2$ by ψ before matching to

$\mathbf{P}_1$. Clearly, a rotation of $\mathbf{P}_2$ by 2π makes no difference at all and must therefore represent the same contraction of string segments after the rotation as before the rotation. The other generator $\tau \rightarrow -1/\tau$ will no doubt similarly lead to repeated contractions of string segments but is harder, we find, to appreciate. Apart from this, one has shown that the formula (5.1) does indeed reproduce the standard torus amplitude.

6 String Interactions as Exchange Effects

The most interesting aspect of the expressions obtained above for string loop amplitudes is perhaps not the formulae themselves. Although any new expressions for even well-known quantities are always useful, we do not regard our particular approach as an advance in calculational technique over the several elegant methods already available. What is intriguing, however, is that the new expressions are basically just exchange integrals in the sense we shall explain, and the realisation of this fact gives one a new and quite surprisingly simple conception of the physics underlying string dynamics.

In order to elucidate our observation, let us first remind ourselves of what happens in particle dynamics for comparison. Transition amplitudes there can be written as overlap integrals between initial and final state wave functions, symbolically thus:

$$\langle f|i \rangle = \int dx dy \delta(x - y) \psi_f^*(x) \psi_i(y), \quad (6.1)$$

where the integral is to be taken over the co-ordinates of all the particles involved. In case the particles are identical, say bosonic, then the wave functions ψ_i and ψ_f have both to be totally symmetric in the co-ordinates of the identical particles. Sometimes, however, it is inconvenient to work always only with symmetric wave functions. In that case, one can still evaluate the transition amplitude as an overlap integral of the unsymmetrised wave functions provided that one takes only the symmetrised overlap. Say, suppose $y_1, \dots, y_n$ are the co-ordinates of the n particles in the initial state and $x_1, \dots, x_n$ those of the particles in the final state. By the symmetrised overlap of the unsymmetrised wave functions $\psi_i(y_1, \dots, y_n)$ and $\psi_f(x_1, \dots, x_n)$, we mean:

$$\begin{aligned} \langle f|i \rangle = & \int dx_1 \dots dx_n dy_1 \dots dy_n \prod_r \delta(x_r - y_r) \\ & \psi_f^*(x_1, \dots, x_n) \sum_P \psi_i(y_1, \dots, y_n), \end{aligned} \quad (6.2)$$

where P is a permutation of the n variables $y_1, \dots, y_n$, and the sum is taken over all permutations. Another way of making the same statement is to say we identify each $y_s, s = 1, \dots, n$ in the initial state wave function ψ_i with an $x_r, r = 1, \dots, n$ in the final state wave function ψ_f , and then we integrate over all $x_r, r = 1, \dots, n$ and sum over all possibilities.

Suppose we deal now instead with strings. In evaluating transition amplitudes between string states, we would of course symmetrise (for bosonic strings) between all identical strings as we did for particles. However, this may not be sufficient, for strings are presumably composite objects. If we were to regard a string as being made up of segments, and if the segments themselves can in some sense be considered as identical, would it not be necessary then to symmetrise between these segments as well? If so, we shall have either to work only with string wave functionals symmetrised over all identical string segments, or else to calculate transition amplitudes only as symmetrised overlap integrals of unsymmetrised wave functionals.

In a sense, closed string wave functionals are already symmetrised between string segments to a certain extent. The operators $L_0 - \tilde{L}_0$ annihilate closed string states, so that the wave functionals are invariant under the operator $\exp i\phi(L_0 - \tilde{L}_0)$, representing for any ϕ a rigid rotation of the string: $\sigma \rightarrow \sigma + \phi$. If we were to regard the string as being made up of little segments each corresponding to an elemental range of σ , then a rigid rotation may also be interpreted as a cyclic permutation of such segments, under which the wave functionals remain invariant, or in other words, are symmetrised.

It is, however, not easy to go beyond this towards a total symmetrisation of the wave functionals for the following reason. Unlike the case for a discrete collection of points, for which there is no ambiguity on what is meant by total symmetrisation, for the continuum collection of points on a string, it is not even clear what total symmetrisation means. Given any parametrisation, say $\sigma = 0 \rightarrow \pi$, of the string, we can of course mark off various segments corresponding to various ranges of σ , but it is not at all obvious what the criterion is for deciding which of the segments so specified can be considered identical, and which not. Further, since the parametrisation of the string is non-unique, we shall be at a loss how to justify that any chosen criterion in one parametrisation will still be valid in some other parametrisation. It appears difficult, therefore, to start by working only with wave functionals which are totally symmetric under the interchange of all 'identical' segments. Let us see instead whether we may fare better by leaving the wave functionals themselves unsymmetrised, and only evaluating transition

amplitudes as symmetrised overlap integrals.

In evaluating the overlap of string functionals, there are two preliminary points we need to clarify. First, a point we already mentioned: the arguments of wave functionals are parametrised strings, or, in other words, functions of a parameter σ , say $\sigma = 0 \rightarrow \pi$. Hence, in putting the argument of one wave functional equal to that of another, say $\mathbf{X} = \mathbf{Y}$, it is insufficient to require just that the points on the string $\mathbf{X}$ are the same as the points on the string $\mathbf{Y}$; we have yet to specify exactly which point on string $\mathbf{X}$ labelled by a certain value of σ , corresponds to which point on string $\mathbf{Y}$. The same is true also when contracting string segments. It is natural, of course, to suggest that we equate the points on $\mathbf{X}$ and $\mathbf{Y}$ for the same value of σ , i.e. $\mathbf{X}(\sigma) = \mathbf{Y}(\sigma)$, for a prescribed range of σ , and write the overlap between $\Phi_1[\mathbf{X}]$ and $\Phi_2[\mathbf{Y}]$ as:

$$\int \delta\mathbf{X}\delta\mathbf{Y}\delta[\mathbf{X} - \mathbf{Y}]\Phi_1^*[\mathbf{X}]\Phi_2[\mathbf{Y}], \quad (6.3)$$

where $\delta\mathbf{X}$ and $\delta\mathbf{Y}$ denote ordinary functional integrations, but it has to be remembered that this specification depends on the parametrisation.

Second, we recall that our wave functionals satisfy the reality condition:³⁾

$$\Phi^*[\mathbf{X}] = \Phi[\bar{\mathbf{X}}], \quad \bar{\mathbf{X}}(\sigma) = \mathbf{X}(\pi - \sigma), \quad (6.4)$$

so that (6.3), for example, can also be written as:

$$\int \delta\mathbf{X}\delta\mathbf{Y}\delta[\bar{\mathbf{X}} - \mathbf{Y}]\Phi_1[\mathbf{X}]\Phi_2[\mathbf{Y}]. \quad (6.5)$$

This allows one to generalise to overlap integrals between wave functionals which are both incoming or both outgoing; instead of taking the complex conjugation of the outgoing wave functionals, we just reverse the parametrisation of one string relative to the other before contraction. That overlaps exist also between two incoming wave functionals is because our strings are their own conjugates, and string anti-segments are again just string segments, only with their parametrisation reversed. If we had taken antiparticles into account in (6.2), there would be overlap also between two wave functions both in the initial (final) states.

Suppose now we are given the wave functionals of N free string states and we are asked to calculate their symmetrised overlap. The first thing we shall need to specify is what are the fundamental entities which are to be regarded as identical. If strings are considered as elementary, and not as composite objects made up of segments, then all we shall need is to symmetrise between all the strings. The result is then trivial, there being no overlap unless the initial and final states

are the same. The transition amplitude is zero, and there will be no interaction between the free string states. However, if instead we regard each string to be made up of segments, and the constituent segments as identical objects, then we shall need to symmetrise the overlap between the segments, in particular between segments from two different strings. For example, suppose we start with four string states $\mathbf{A}[\mathbf{X}], \mathbf{B}[\mathbf{Y}], \mathbf{C}[\mathbf{Z}], \mathbf{D}[\mathbf{W}]$, and regard each string to be composed of two constituent segments, say, the standard segment X_1 etc., and its complement, X_2 etc. Then in the symmetrised overlap between $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$, there will be a ‘direct’ term in which X_i is contracted with $\bar{Z}_i$, and Y_i with $\bar{W}_i$, for $i = 1, 2$ namely:

$$Tr[A.C]Tr[B.D] \quad (6.6)$$

in half-string matrix notation. This will vanish unless the states A and B are the same as the states C and D . However, there will also be an ‘exchange’ term in which X_1 is contracted with $\bar{Z}_1$ and Y_1 with $\bar{W}_1$ but X_2 with $\bar{W}_2$ and Y_2 with $\bar{Z}_2$, giving instead of (6.6) the overlap:

$$Tr[A.C.B.D] \quad (6.7)$$

The overlap need not then vanish even when the string states A, B are not the same as the states C, D , so that when viewed from the standpoint of the full strings, there will be an effective interaction between them. As far as the segments are concerned, however, this seeming interaction has only a kinematic origin coming just from the standard symmetrisation procedure of Bose-Einstein statistics.

The example above showed that regarding strings as being made up of Bose segments could in general lead to interactions, but it did not yield the usual string transition amplitudes as the result. The reason is that we were a little naive in specifying which were the strings’ Bose-Einstein constituents. The designation of the so-called standard segment, say $\mathbf{X}(\sigma), \sigma = 0 \rightarrow \pi/2$, depends on the parametrisation. Furthermore, it seems somewhat arbitrary to cut a string into just two halves at the particular values $\sigma = 0, \pi/2$. We know from previous arguments that we are allowed to reparametrise a string by a rigid rotation $\exp i\phi(L_0 - \tilde{L}_0)$ and by a distortion of the form $\exp \lambda(M_1 + \tilde{M}_1)$ for any ϕ and λ without changing the physics. Now, by applying $\omega(\phi, \lambda) = \exp \lambda M \exp i\phi L$, one sees that one can transform a segment with any end points, say $\sigma = \sigma_1 \rightarrow \sigma_2$, into a standard segment $\sigma = 0 \rightarrow \pi/2$. It would appear therefore that all segments in a string are in fact equivalent. This suggests that each string can be chopped up arbitrarily into any number of segments, and all of them are to be considered as identical. Hence, in evaluating the overlap between string wave functionals, one will have to symmetrise between all segments of all strings, or, since string

segments are self-conjugated by reversing the parametrisation, one will need to sum over all possibilities in contracting pairs of all such segments. Alternatively, if we revert to the notation mentioned at the end of section 1 and regard each string functional $\Phi[\mathbf{X}]$ as an infinite-rank tensor with the string co-ordinates $\mathbf{X}(\sigma)$ as a continuum of tensor indices, then the symmetrised overlap of string wave functionals will be a sum over all possibilities of saturating these tensor indices.

As an example, consider the simple case with four string functionals $\mathbf{A}[X], \mathbf{B}[Y], \mathbf{C}[Z], \mathbf{D}[W]$. If each string is chopped up into 3 segments, then by contracting pairs of these segments, we can obtain in particular the diagrams in Fig. 26, where the 3 segments of $\mathbf{X}$ are contracted respectively with a segment each from $\mathbf{Y}, \mathbf{Z}$ and $\mathbf{W}$. The two diagrams are related by duality and both occur in the four-string tree amplitude. Consistency with duality requires in addition the diagrams in Fig. 27, but (a), for example, can be obtained by chopping $\mathbf{X}$ and $\mathbf{Z}$ each into 4 segments and $\mathbf{Y}$ and $\mathbf{W}$ each into 2, and contracting the appropriate pairs. There is thus no difficulty in seeing that all tree diagrams can be obtained in this way. Further, since a segment can also be contracted with another from the same string, we can ‘squeeze’ a closed string and construct loop diagrams also with topologies other than that of a sphere. It is clear then that all string amplitudes can be obtained as overlap integrals simply as consequences of the Bose-Einstein symmetrisation of constituent segments. Conversely, it is also clear that every possible way of contracting all pairs of segments, or, if one prefers, of saturating tensor indices, will lead to a diagram representing an amplitude.

There remains then only the questions of how explicitly to contract segments and of what weight to assign to each contraction. These questions, however, cannot be answered by an appeal to analogy with the case for particles as we have done so far. For a system of identical particles, there was never any ambiguity of how to ‘contract indices’ or of how to normalise a symmetrised overlap - one just contracts by putting the co-ordinates of one particle equal to those of another, and normalise by dividing the sum over all permutations by the number of permutations occurring in the sum. To contract one string segment with another, however, one would need to specify which point on one segment is to be equated to which point on the other. Furthermore, since any segment can be subdivided into any number of equally elementary segments, there are no fundamental objects one could count to fix the normalisation of one contraction relative to another. It would appear, therefore that an appeal to the particle analogue is of no avail, and the prescription of exactly how to proceed will have to be imposed as a new postulate. What we have shown in the last two sections is that the proposed prescription we called ‘reduction to operations with standard segments’ leads to

the correct answer for string amplitudes.

7 Concluding Remarks

The results reported above have two complementary aspects, one technical and one conceptual.

Technically, one has obtained new expressions for string loop amplitudes as overlap integrals of state wave functionals. Although new expressions are always useful, we are not particularly hopeful that ours will improve on, or even compete with, the elegant techniques which have already been developed for computing string amplitudes. The reason is the following. The formalism in terms of functionals that we have used throughout this article is obviously insufficiently precise for actual calculations. For this reason, half-string oscillator formalism has been developed in parallel⁽⁶⁾. It gives the whole approach a much firmer and more rigorous foundation than our functional formalism here can offer. However, although we can see no difficulty in principle in developing the formalism further as a calculational technique, it seems to be of dubious efficacy. The main trouble is the so-called ‘dualising operator’ Ω in (1.2) which is basically a full-string concept. Changing as it does the definition of the half-string continuously from the whole string to a point, it runs contrary in spirit to the half-string formalism on which the rest of the approach is based. In terms of half-string oscillators, it mixes the two halves of the string in a rather complicated fashion, and makes a calculation of amplitudes entirely in terms of half-string oscillators difficult. In the end, the right tactic might be to use both the full-string and half-string formalisms in conjunction, keeping the full-string apparatus for ‘dualising’ but using the half-string formalism for the rest, and changing from one to the other as the need arises. Since the transformation between full- and half-string co-ordinates are now explicitly known in terms of Neumann functions, this ought to be manageable, but needs to be further explored.

One special technical feature of the present approach is perhaps worth noting. In evaluating string amplitudes as Feynman integrals following Polyakov, one has to integrate over surfaces of all topologies. The topological classification of two-dimensional surfaces, however, is known only when they are Euclidean, which fact forces us, when dealing with the actual physical Minkowskian case, to rely on dubious arguments based on Wick rotations etc., mostly impossible at present to justify. In this approach, on the other hand, topologies arise only from the different contractions of wave functionals all taken at the same τ . Superficially, therefore,

it would appear that the formalism does not depend on whether the surfaces are Euclidean or Minkowskian. In our discussion above, we did of course refer to surfaces as spheres and tori etc., implicitly using the usual Euclidean terminology, but this was only for easy comparison with the conventional approaches. A formulation in this approach which avoids Euclideanisation altogether looks possible in principle, although it may not really work out in practice.

Conceptually, on the other hand, the results are more interesting. It pictures string theory as completely free, and all that appear to us as string interactions are but exchange effects due to the Bose-Einstein symmetrisation of the strings' constituent segments. Also, the strings are merely first quantised - nowhere did we need to introduce the concept of a quantised string field, since the exchange terms in the overlap integral between free string wave functionals already connect initial and final states with different numbers of strings. Viewed in certain ways, of course, these statements are not too surprising. Take, for example, the Polyakov viewpoint, where amplitudes are given as Feynman integrals. The string action providing the measure for the integration there is just the Nambu-Goto action for a free string with never an interaction term. The interaction arose merely from the non-trivial topologies of the surfaces integrated over. Moreover, the formulation is, in a sense, also only first quantised since the Feynman integrals were only over world sheets, namely the co-ordinates $\mathbf{X}(\sigma, \tau)$ of the strings, and not over fields in string space as one expects a normal second quantised formulation would be. Of course, even for particles, one can formulate a second quantised interacting theory similarly as a 'first quantised free theory' if one allows oneself to integrate over all graphs instead of over just world paths. But then, one has to introduce interactions surreptitiously as a vertex in the graphs, whereas in the case of the Polyakov string, the interaction is inherent already in the natural topology of 2-dimensional surfaces. In the present case, what we seem to have done is to go one step further in interpreting this interaction - inducing topology as exchange effects due just to the Bose symmetry of identical string segments. Cast in these terms, it is the simplest conception of string dynamics that we have yet come across.

In view of the fact that string theory was historically discovered in the 'wrong order' - in contrast to, say, gauge theory or relativity, where transition amplitudes were deduced as results following from some basic principles - there has long been an urge among theoreticians to discover the fundamental physical concepts on which string theory is based. The realisation, promulgated above, that the theory can be reduced to extremely simple terms will, we believe, help in the right direction.

In particular, it has given us a bit more insight into an alternative formulation of string theory as a generalised gauge theory following Witten and others⁴⁾⁷⁾. There, it appeared that the gauge group is the group of unitary half-string functional transformations:

$$\begin{aligned}
U : \Psi[X] &\longrightarrow \int \delta Y U[X, Y] \Psi[Y] \\
U^\dagger U &= \int \delta Y U^*[Y, X] U[Y, Z] = \delta[X - Z] = 1 \\
X &= X(\sigma), \quad \sigma = 0 \rightarrow \pi/2, \text{ etc.},
\end{aligned} \tag{7.1}$$

while the space-time translation algebra of ordinary gauge theories is replaced by the Virasoro algebra. The generalisation of the gauge group to the enormous group of (7.1), though drastic in practice, is conceptually not difficult. The replacement of the translation by the Virasoro algebra, on the other hand, is a little hard to appreciate. In ordinary gauge theories, translations come into the statement of local gauge invariance, which may be interpreted as physics being invariant under independent phase rotations before and after the action of a translation. It was a little difficult physically, therefore, to envisage why in string theory there should be a generalised ‘local’ gauge invariance in terms of a Virasoro instead of the usual translation algebra. The picture becomes somewhat clearer when viewed from the angle of the new conception of string interactions as mere wave function overlaps. In overlap integrals, the phases of wave functions cancel, which explains therefore phase invariance. For string theory, this means invariance under the half-string unitary group (7.1), whose definition, however, depends on the choice of how one divides the string into two halves, which is arbitrary. To be consistent, one needs to generalise to gauge invariance under other definitions of the half-strings, and the allowed reparametrisations which changes the definition are generated by the operator M , which is part of the Virasoro algebra. That is why the Virasoro algebra enters instead of the usual translation algebra. The reason why only this subalgebra generated by M comes in is, we believe, due to the fact that our formulation is not completely Lorentz covariant since our overlaps are always taken at $\tau = 0$. We speculate that in a fully covariant treatment, the role played here by the operator M will be extended to the whole Virasoro algebra.

Be that as it may, the above example serves to illustrate our point. We think that the ultra-simple physical picture proposed in this paper for string dynamics is both conceptually interesting in its own right and likely to be of use in elucidating the physical principles underlying the theory in other, more conventional, formulations.

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Figure Captions

- Figure 1. Sewing 4 open-string wave functionals together to give a ‘planar’ tree diagram.
- Figure 2. Sewing 4 open-string wave functionals together to give a loop diagram with a handle.
- Figure 3. Cutting a cylinder into 4 planar pieces.
- Figure 4. Cutting a cylinder with handle into 4 planar pieces.
- Figure 5. Cutting a cylinder with 2 handles into 4 planar pieces.
- Figure 6. Cutting a sphere with 4 holes into 4 cylinders.
- Figure 7. Cutting a torus with 4 holes into 4 cylinders.
- Figure 8. The ‘product’ of 2 closed strings.
- Figure 9. The 3-closed-string vertex in
(a) Witten matrix notation,
(b) conventional notation.
- Figure 10. Construction of the torus by ‘joining’, ‘squeezing’ and ‘matching’.
- Figure 11. Construction of the torus reduced to operations with ‘standard segments’.
- Figure 12. Disjoint ‘standard segments’ from two different parametrisations.
- Figure 13. Construction of the torus: the case when all ‘standard segments’ are disjoint.
- Figure 14. Effective ‘squeezing’ of strings in Fig. 13.
- Figure 15. Construction of torus: the case when all ‘standard segments’ are disjoint (second view).
- Figure 16. Effective ‘squeezing’ of strings in Fig. 15.
- Figure 17. Construction of torus: the case when one pair of ‘standard segments’ are disjoint.
- Figure 18. Effective ‘squeezing’ of strings in Fig. 17.

Figure 19. Both pairs of ‘standard segments’ overlap.

Figure 20. String doubling back on itself.

Figure 21. Construction of torus: case when both pairs of ‘standard segments’ overlap.

Figure 22. Effective ‘squeezing’ of strings in Fig. 21.

Figure 23. Alternative construction of torus.

Figure 24. Alternative construction of torus by ‘squeezing’ and ‘matching’ strings.

Figure 25. Alternative construction of torus reduced to operations with ‘standard segments’.

Figure 26. String amplitudes obtained as Bose-Einstein symmetrisation effects by contracting string segments.

Figure 27. Further examples of string amplitudes obtained as Bose-Einstein symmetrisation effects by contracting string segments.

Fig. 1

Fig. 2

(a)

(b)

Fig. 3

(a)

(b)

Fig. 4

Fig. 5

Fig. 6

Fig. 7

Fig. 8

(a)

(b)

Fig. 9

Fig. 10

Fig. 11

Fig. 12

Fig. 13

Fig. 14

Fig. 15

Fig. 16

Fig. 17

Fig. 18

Fig. 19

Fig. 20

Fig. 21

Fig. 22

Fig. 23

(a)

(b)

Fig. 24

(a)

(b)

(c)

Fig. 25

(a)

(b)

Fig. 26

(a)

(b)

Fig. 27