On a class of supersoluble groups

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Abstract

A subgroup $H$ of a finite group $G$ is said to be $S$-semipermutable in $G$ if $H$ permutes with every Sylow $q$-subgroup of $G$ for all primes $q$ not dividing $|H|$. A finite group $G$ is an $MS$-group if the maximal subgroups of all the Sylow subgroups of $G$ are $S$-semipermutable in $G$. The aim of the present paper is to characterise the finite $MS$-groups.


Keywords and phrases: finite group, soluble PST-group, $T_0$-group, $MS$-group, BT-group.

1 Introduction

In the following, $G$ always denotes a finite group. Recall that subgroups $H$ and $K$ of $G$ is said to permute if $HK$ is a subgroup of $G$ and that a subgroup $H$ of $G$ is said to be permutable in $G$ if $H$ permutes with all subgroups of $G$.

Various generalisations of permutability have been defined and studied and, in particular, we mention the $S$-semipermutability. A subgroup $H$ is said to be $S$-semipermutable in $G$ if $H$ permutes with every Sylow $q$-subgroup of $G$ for all primes $q$ not dividing $|H|$. This subgroup embedding property

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has been extensively studied recently (see for instance [1, 4, 7, 9]). Most of these papers concern situations where many subgroups (for instance all maximal subgroups of the Sylow subgroups) have the stated property. Thus we say that a group $G$ is an MS-group if the maximal subgroups of all the Sylow subgroups of $G$ are S-semipermutable in $G$.

The main aim of this paper is to characterise the MS-groups.

2 Preliminary results

In this section, we collect the definitions and results which are needed to prove our main theorems.

We shall adhere to the notation used in [2]: this book will be the main reference for terminology and results on permutability.

A subgroup $H$ is permutable in a group $G$ if and only if $H$ permutes with every $p$-subgroup of $G$ for every prime $p$ (see for instance [2, Theorem 1.2.2]). A less restrictive subgroup embedding property is the S-permutability introduced by Kegel in 1962 [5] and defined in the following way:

Definition 1. A subgroup $H$ of $G$ is said to be S-permutable in $G$ if $H$ permutes with every Sylow $p$-subgroup of $G$ for every prime $p$.

Note that we are not considering all $p$-subgroups, but just the maximal ones, that is, the Sylow $p$-subgroups.

In recent years there has been widespread interest in the transitivity of normality, permutability and S-permutability.

Definition 2. 1. A group $G$ is a T-group if normality is a transitive relation in $G$, that is, if every subnormal subgroup of $G$ is normal in $G$.

2. A group $G$ is a PT-group if permutability is a transitive relation in $G$, that is, if $H$ is permutable in $K$ and $K$ is permutable in $G$, then $H$ is permutable in $G$.

3. A group $G$ is a PST-group if S-permutability is a transitive relation in $G$, that is, if $H$ is S-permutable in $K$ and $K$ is S-permutable in $G$, then $H$ is S-permutable in $G$.

If $H$ is S-permutable in $G$, it is known that $H$ must be subnormal in $G$ ([2, Theorem 1.2.14(3)]). Therefore, a group $G$ is a PST-group (respectively, a PT-group) if and only if every subnormal subgroup is S-permutable (respectively, permutable) in $G$. 
Note that $T$ implies $PT$ and $PT$ implies $PST$. On the other hand, $PT$ does not imply $T$ (non-Dedekind modular $p$-groups) and $PST$ does not imply $PT$ (non-modular $p$-groups).

A less restrictive class of groups is the class of $T_0$-groups which has been studied in [3, 6, 8].

**Definition 3.** A group $G$ is called a $T_0$-group if the Frattini factor group $G/\Phi(G)$ is a $T$-group.

The group in Example 13 below is a soluble $T_0$-group which is not a $PST$-group. Soluble $T_0$-groups are closely related to $PST$-groups as the following result shows.

**Theorem 4** ([6, Theorems 5 and 7 and Corollary 3]). Let $G$ be a soluble $T_0$-group with nilpotent residual $L = \gamma_\infty(G)$. Then:

1. $G$ is supersoluble.
2. $L$ is a nilpotent Hall subgroup of $G$.
3. If $L$ is abelian, then $G$ is a $PST$-group.

Here the nilpotent residual $\gamma_\infty(G)$ of a group $G$ is the smallest normal subgroup $N$ of $G$ such that $G/N$ is nilpotent, that is, the limit of the lower central series of $G$ defined by $\gamma_1(G) = G, \gamma_{i+1}(G) = [\gamma_i(G), G]$ for $i \geq 1$.

It is known that $S$-semipermutability is not transitive. Hence it is natural to consider the following class of groups:

**Definition 5.** A group $G$ is called a $BT$-group if $S$-semipermutability is a transitive relation in $G$, that is, if $H$ is $S$-semipermutable in $K$ and $K$ is $S$-semipermutable in $G$, then $H$ is $S$-semipermutable in $G$.

This class was introduced and characterised by Wang, Li and Wang in [9]. Further contributions were presented in [1].

**Theorem 6** ([9, Theorem 3.1]). Let $G$ be a group. The following statements are equivalent:

1. $G$ is a soluble $BT$-group.
2. Every subgroup of $G$ is $S$-semipermutable.
3. $G$ is a soluble $PST$-group and if $p$ and $q$ are distinct prime divisors of the order of $G$ not dividing the order of the nilpotent residual of $G$, then $[G_p, G_q] = 1$, where $G_p \in \text{Syl}_p(G)$ and $G_q \in \text{Syl}_q(G)$. 

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The group presented in Example 12 below is an MS-group which is not a soluble BT-group. Furthermore, Example 13 shows that the classes of $T_0$-groups and MS-groups are not closed under taking subgroups.

The first remarkable fact concerning the structure of an MS-group can be found in [7]. It is proved there that every MS-group is supersoluble.

**Theorem 7** ([7, Corollary 9]). Let $G$ be an MS-group. Then $G$ is supersoluble.

More recently, the second and fourth authors proved the following theorem.

**Theorem 8** ([4, Theorems A, B and C]). Let $G$ be an MS-group with nilpotent residual $L = \gamma_\infty(G)$. Then:

1. If $N$ is a normal subgroup of $G$, then $G/N$ is an MS-group;
2. $L$ is a nilpotent Hall subgroup of $G$;
3. $G$ is a soluble $T_0$-group.

It is well-known that the nilpotent residual of a supersoluble group is nilpotent. Hence the nilpotency of $L$ in Theorem 8 is a consequence of Theorem 7.

Let $G$ be a group whose nilpotent residual $L = \gamma_\infty(G)$ is a Hall subgroup of $G$. Let $\pi = \pi(L)$ and let $\theta = \pi'$, the complement of $\pi$ in the set of all prime numbers. Let $\theta_N$ denote the set of all primes $p$ in $\theta$ such that if $P$ is a Sylow $p$-subgroup of $G$, then $P$ has at least two maximal subgroups. Further, let $\theta_C$ denote the set of all primes $q$ in $\theta$ such that if $Q$ is a Sylow $q$-subgroup of $G$, then $Q$ has only one maximal subgroup, or equivalently, $Q$ is cyclic.

Throughout this paper we will use the notation presented above concerning $\pi, \theta = \pi', \theta_N,$ and $\theta_C$.

### 3 The main results

Our first main result is a characterisation theorem.

**Theorem 9.** Let $G$ be a group with nilpotent residual $L = \gamma_\infty(G)$. Then $G$ is an MS-group if and only if $G$ satisfies the following properties.

1. $G$ is a $T_0$-group.
2. $L$ is a nilpotent Hall subgroup of $G$. 

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3. If \( p \in \pi \) and \( P \in \text{Syl}_p(G) \), then a maximal subgroup of \( P \) is normal in \( G \).

4. Let \( p \) and \( q \) be distinct primes with \( p \in \theta_N \) and \( q \in \theta \). If \( P \in \text{Syl}_p(G) \) and \( Q \in \text{Syl}_q(G) \), then \( [P, Q] = 1 \).

5. Let \( p \) and \( q \) be distinct primes with \( p \in \theta_C \) and \( q \in \theta \). If \( P \in \text{Syl}_p(G) \) and \( Q \in \text{Syl}_q(G) \) and \( M \) is the maximal subgroup of \( P \), then \( MQ = \bigcup_{i=} \) is a nilpotent subgroup of \( G \).

Proof. Let \( G \) be an MS-group. By Theorems 7 and 8, \( G \) is a supersoluble \( T_0 \)-group whose nilpotent residual \( L \) is a nilpotent Hall subgroup of \( G \). Thus properties 1 and 2 hold.

Let \( \pi = \pi(L) \) and let \( p \in \pi \). Further, let \( P \) be a Sylow \( p \)-subgroup of \( G \) and let \( M \) be a maximal subgroup of \( P \). Then \( M \leq P \leq L \) and \( M \) is normal in \( L \) and subnormal in \( G \). Let \( q \in \theta = \pi' \) and note that \( MQ \) is a subgroup of \( G \) for a given Sylow \( q \)-subgroup \( Q \) of \( G \). Moreover \( M \) is a Sylow \( p \)-subgroup of \( MQ \) and so \( M \) is a normal subgroup of \( MQ \). Consequently \( M \) normalises \( P \) and each Sylow \( q \)-subgroup \( Q \) of \( G \), so \( M \) is a normal subgroup of \( G \) and property 3 holds.

Let \( X \) be a Hall \( \theta \)-subgroup of \( G \) and note that \( G = L \times X \), the semidirect product of \( L \) by \( X \), and \( X \) is nilpotent. Let \( t \) be a prime from \( \theta_N \) and \( r \) be a prime from \( \theta \). Also let \( T \in \text{Syl}_r(G) \) and \( R \in \text{Syl}_r(G) \). Let \( M_1 \) and \( M_2 \) be two distinct maximal subgroups of \( T = \langle M_1, M_2 \rangle \). Since \( G \) is an MS-group, \( M_1R = RM_1 \) and \( M_2R = RM_2 \). Applying [2, Theorem 1.2.2], we have \( TR = TR \). Observe that \( TR \) is a \( \theta \)-subgroup of \( G \) and so \( TR \) is nilpotent since \( TR \) is a subgroup of some conjugate of \( X \). Therefore, \( [T, R] = 1 \) and property 4 holds.

Let \( p \) and \( q \) be distinct primes with \( p \in \theta_C \) and \( q \in \theta \). Further, let \( P \in \text{Syl}_p(G) \) and \( Q \in \text{Syl}_q(G) \). If \( M \) is the maximal subgroup of \( P \), then \( MQ = \bigcup_{i=} \) is a nilpotent \( \theta \)-subgroup of \( G \). Thus property 5 holds.

Let \( G \) be a group satisfying properties 1–5. We are to show that \( G \) is an MS-group. By properties 1 and 2, \( G \) is a soluble \( T_0 \)-group, and by Theorem 4, \( G \) is thus supersoluble.

Let \( p \in \pi = \pi(L) \), let \( P \) be a Sylow \( p \)-subgroup of \( G \), and let \( M \) be a maximal subgroup of \( P \). Then \( M \) is a normal subgroup of \( G \) by property 3 and clearly \( P \) is a normal subgroup of \( G \). This means that \( M \) permutes with every Sylow subgroup of \( G \) and \( P \) permutes with every maximal subgroup of any Sylow subgroup of \( G \).

Let \( p \) and \( q \) be distinct primes from \( \theta \) and let \( P \in \text{Syl}_p(G) \) and \( Q \in \text{Syl}_q(G) \). We consider a maximal subgroup \( M \) of \( P \). Note that \( \theta = \theta_N \cup \theta_C \).
and \( \theta_N \cap \theta_C = \emptyset \), the empty set. If \( p \in \theta_N \), then by property 4, \([P,Q] = 1\), so that \( MQ = QM \). Hence assume \( p \in \theta_C \). Then, by property 5, \( MQ = QM \).

Therefore, every maximal subgroup of any Sylow subgroup of \( G \) is S-semipermutable in \( G \) and \( G \) is an MS-group.

The second and fourth authors in [4] posed the following two questions.

1. When is a soluble PST-group an MS-group?
2. When is a soluble PST-group which is also an MS-group a BT-group?

Using Theorem 9 we are able to answer the first question and provide a partial answer to the second.

**Theorem 10.** Let \( G \) be a soluble PST-group. Then \( G \) is an MS-group if and only if \( G \) satisfies 4 and 5 of Theorem 9.

**Proof.** Let \( G \) be a soluble PST-group with nilpotent residual \( L = \gamma_\infty(G) \). By [6, Lemma 5], \( G/\Phi(G) \) is a T-group and so \( G \) is a \( T_0 \)-group. Notice that 1, 2 and 3 of Theorem 9 are satisfied for the group \( G \).

Assume that \( G \) is an MS-group. By Theorem 9, 4 and 5 are satisfied by \( G \).

Conversely, assume that 4 and 5 of Theorem 9 are satisfied by \( G \). By Theorem 9, \( G \) is an MS-group.

This completes the proof.

The group given in Example 12 below is a soluble PST-group which is not an MS-group and the group given in Example 13 is an MS-group which is not a soluble PST-group.

**Theorem 11.** Let \( G \) be a soluble PST-group which is also an MS-group. If \( \theta_C \) is the empty set, then \( G \) is a BT-group.

**Proof.** Let \( G \) be a soluble PST-group which is also an MS-group. Let \( L = \gamma_\infty(G) \) be the nilpotent residual of \( G \). By the Theorem of Agrawal [2, Theorem 2.1.8], \( L \) is an abelian Hall subgroup of \( G \) on which \( G \) acts by conjugation as a group of power automorphisms. Recall that \( \theta = \pi'(\pi(L)) \). Moreover \( \theta = \theta_N \) if \( \theta_C \) is empty. Let \( p \) and \( q \) be distinct primes from \( \theta \) and let \( P \in \text{Syl}_p(G) \) and \( Q \in \text{Syl}_q(G) \). Note that since \( G \) is an MS-group, we have that \( G \) satisfies properties 4 and 5 of Theorem 9. Then \([G_p,G_q] = 1\) by property 4 of that theorem. Therefore, \( G \) is a BT-group by Theorem 6. This completes the proof of Theorem 11.

We remark that if \( \theta \) contains only one prime, then \( G \) is a BT-group by [9, Corollary 3.4].
4 Examples

The following examples appear in [4]. For the sake of completeness, we list them here.

Example 12. Let $G = \langle y, z, x \mid y^3 = z^2 = x^7 = 1, [y, z] = 1, x^y = x^2, x^z = x^{-1} \rangle$. Then $[(y)^x, z] \neq 1$ and $G$ is a soluble group which is not a BT-group. However, $G$ is an MS-group.

Example 13. Let $G = \langle a, x, y \mid a^2 = x^3 = y^3 = [x, y]^3 = [x, [x, y]] = 1, x^a = x^{-1}, y^a = y^{-1} \rangle$. Then $H = \langle x, y \rangle$ is an extraspecial group of order 27 and exponent 3. Let $z = [x, y]$, so $z^a = z$. Then $\Phi(G) = \Phi(H) = \langle z \rangle = Z(G) = Z(H)$. Note that $G/\Phi(G)$ is a T-group so that $G$ is a $T_0$-group. The maximal subgroups of $H$ are normal in $G$ and it follows that $G$ is an MS-group. Let $K = \langle x, z, a \rangle$. Then $\langle xz \rangle$ is a maximal subgroup of $\langle x, z \rangle$, the Sylow 3-subgroup of $K$. However, $\langle xz \rangle$ does not permute with $\langle a \rangle$ and hence $\langle xz \rangle$ is not an S-semipermutable subgroup of $K$. Therefore, $K$ is not an MS-subgroup of $G$. Also note that $\Phi(K) = 1$ and so $K$ is not a T-subgroup of $G$ and $K$ is not a $T_0$-subgroup of $G$. Hence the class of soluble $T_0$-groups is not closed under taking subgroups. Note that $G$ is a soluble group which is not a PST-group.

Example 14. Let $G = \langle y, z, x \mid y^9 = z^2 = x^{19^2} = 1, [y, z] = 1, x^y = x^{b_2}, x^z = x^{-1} \rangle$. Then the soluble group $G$ is a PST-group, but $G$ is not an MS-group since $[(y)^x, z] \neq 1$.

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