PHOKHARA, the radiative return and the \((g - 2)_\mu\) puzzle

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The radiative return has proven to be a competitive method for the precise measurement of the hadronic cross section, detailed studies of hadronic interactions, and even discoveries of new resonances. The most recent and future developments of the Monte Carlo event generator PHOKHARA are highlighted, and the impact of the radiative return measurements on the \((g - 2)_\mu\) puzzle is discussed.

1. Introduction

Electron–positron annihilation into hadrons is one of the basic reactions of particle physics, relevant for the understanding of hadronic interactions. The low energy region is crucial for predictions of the hadronic contributions to \(a_\mu = (g - 2)_\mu/2\), the anomalous magnetic moment of the muon, and to the running of the electromagnetic coupling from its value at low energy up to \(M_Z\).

The traditional way of measuring the hadronic cross section, the energy scan, needs dedicated experiments. An alternative and advantageous way is the radiative return method. This method allows for a simultaneous extraction of the hadronic cross section over a wide energy range in an homogeneous data set, profiting from the data of all high luminosity meson factories.

The radiative return method relies on the observation that the cross section of the reaction \(e^+e^- \rightarrow \text{hadrons} + \text{photons}\), with photons emitted from the initial leptons (ISR), see Fig. 1(a), factorizes into a function \(H\), fully calculable within QED, and the cross section of the reaction \(e^+e^- \rightarrow \text{hadrons}\). Thus from the measured differential cross section of the reaction \(e^+e^- \rightarrow \text{hadrons} + \text{photons}\) as a function of the hadronic invariant mass one can evaluate \(\sigma(e^+e^- \rightarrow \text{hadrons})\) once the radiator function \(H\) is known. The radiative return method allows for the extraction of the hadronic cross section from the production energy threshold of a given hadronic channel almost to the nominal energy of the experiment. The smaller cross section of the radiative process as compared to the process without photon emission has to be compensated by higher luminosities. That requirement is met by meson factories (DAPHNE, CLEO, BABAR, BELLE). All of them were built for other purposes than the measurement of the hadronic cross section, but their huge luminosities provide with data samples large enough for very accurate measurements of interesting hadronic channels and/or give information on rare channels, which were not accessible in scan experiments.

Two representative examples of such measurements are the very accurate pion form factor extraction by the KLOE Collaboration [1], and the measurement of \(\sigma(e^+e^- \rightarrow 3\text{ pions})\) by the BABAR Collaboration [2], where it was shown that the old DM2 scan data were too low at high values of the invariant mass of the hadronic system. For a review of BABAR results see Ref. [3].

2. The PHOKHARA event generator

In realistic experimental situations, where sophisticated event selections are used, one needs a Monte Carlo event generator of...
the measured process. To meet that requirement the PHOKHARA event generator (http://cern.ch/german.rodrigo/phokhara/) was constructed. PHOKHARA started from the EVA generator [4], where the structure function method was used to model multi-photon emission. The physical accuracy of EVA was however far from the demanding experimental accuracy for the measurement of the pion form factor. PHOKHARA relies instead on exact matrix elements at next-to-leading order (NLO), namely it includes one loop radiative corrections to one-photon radiation and emission of two real hard photons. The accuracy of the simulation has been estimated to be of the order of 5 per mil from ISR [5].

The first version of PHOKHARA [5] was designed to run with tagged photon configurations. Radiative corrections necessary for photon emission at small angles were calculated afterwards [6] and implemented into the event generator [7]. The important issue of final state emission was addressed subsequently [8]. In parallel the generator was being extended to allow for the generation of more hadronic channels. The present version of the program simulates the production of not only a pair of pions or muons, but includes also the simulation of $K^+K^-$, $K^0\bar{K}^0$, events with three [9] and four pions [7], and nucleon pairs $p\bar{p}$ and $n\bar{n}$ [10].

All that allowed for building the state-of-the-art event generator. The proper implementation of the radiative corrections as well as the hadronic currents is guarantied by extensive tests. Comparisons with the KKMC [11] Monte Carlo event generator have been performed [12] leading to an excellent agreement. The comparison is however limited to muons in the final state. Higher order effects, that can be seen as a difference between exponentiated and non-exponentiated matrix elements reach at most 2 per mile with the exception of the region where the hadronic system has an invariant mass very close to the nominal energy of the experiment. There, soft multi-photon emission plays an important role and thus exponentiation is necessary. This region is however out of the region of interest for radiative return measurements [5].

The impressive amount of new data provided in particular by $B$-factories requires further efforts to improve the accuracy of the event generator, as well as to implement new hadronics channels. The latter requires a fairly good parametrization of various form factors.

3. Final state radiation and radiative $\phi$-decays

Final state radiation (FSR), see Fig. 1(b), is the main background for radiative return measurements. The situation at $B$-factories is however completely different from the one at the $\phi$-factory DAPHNE. In the former case the region of hadronic masses below 4 GeV, which is of the utmost physical interests, lays far from the nominal energy of the experiments. Thus an emission of a very hard photon is required to reach it. As a result the typical kinematic configuration of an event consists of a photon emitted back-to-back to the hadronic system. That provides a natural suppression of FSR contributions, which are large for photons emitted parallel to the direction of the charged hadrons in the final state.

At the $\phi$-factory the physically interesting region is not so far from the nominal energy of the experiment, and that natural suppression of FSR do not hold. Strategies should be established to either suppress FSR through kinematical cuts, or to control the uncertainty due to the model dependence of the simulation. The

Figure 1. Leading order contributions to the reaction $e^+e^- \rightarrow \pi^+\pi^-\gamma$ from ISR (a) and FSR (b).
KLOE analysis [1,13] of the pion form factor from events with untagged photons emitted at small polar angles follows the first strategy. When photons in the forward-backward directions and pions in the central region are selected FSR is easily reduced to less than 1%. The price to pay however is that the region close to threshold, $M_{\pi\pi} < 590$ MeV, is also suppressed as pions are produced in this kinematical region essentially back-to-back to the ISR photon and therefore at very small polar angles that scale from detection. For the case of untagged photons a specific background, $e^+e^- \rightarrow \pi^+\pi^-e^+e^-$, has to be also taken into account [14,15] as the final leptons are not vetoed.

KLOE is now performing a complementary analysis with photons tagged at large polar angles [13] that will provide accurate data in the threshold region. Due to the factor $1/s^2$ of the dispersion integral for $a_\mu$, that low energy region is highly enhanced; contributing to around 20% to the total integral. Therefore its relevance.

At large photon polar angles FSR and $\phi$ decays: $\phi \rightarrow \pi^+\pi^-\pi^0$ and $\phi \rightarrow f_0\gamma \rightarrow \pi^+\pi^-\gamma$ [16]; become important. The background channel $\phi \rightarrow \pi^+\pi^-\pi^0$ can be eliminated through dedicated selection cuts, but FSR and $\phi \rightarrow \pi^+\pi^-\gamma$ have to be subtracted relying on Monte Carlo. Another possibility to eliminate $\phi$ decays is to run off-peak. DAPHNE has taken data off-resonance at a center-of-mass energy of $\sqrt{s} = 1$ GeV that will allow to reduce the systematic errors in the threshold region [13].

The main tool to test the model dependence of photon emission from the final state pions and radiative $\phi$-decays is the charge asymmetry. For events with emission of one real photon the two-pion state is produced with charge conjugation $C = -1$ and odd orbital angular momentum when the real photon is emitted from the initial state, and with $C = +1$ and even orbital angular momentum when the real photon is emitted from the final state. As a result, the ISR-FSR interference is odd under the exchange $\pi^+ \leftrightarrow \pi^-$ and integrates to zero for charge blind event selections. At the same time it is the only source of the charge asymmetry and as such allow to test the FSR model.

As shown in [17], the charge asymmetry has large analyzing power and can provide information allowing for distinguishing between different models of the radiative $\phi \rightarrow \pi\pi\gamma$ decay [18]. PHOKHARA have adopted two models describing the decays $\phi \rightarrow \pi^+\pi^-\gamma$ and $\phi \rightarrow \pi^0\pi^0\gamma$: the “no structure” model [19] and the $K^+K^-$ model [20]. Again by appropriate event selections one can suppress those contributions or enhance them as for other sources of FSR emission. Other contributions that might be important in the threshold region beyond the model currently used in PHOKHARA to describe FSR (sQED + vector dominance + radiative $\phi$ decays) have been advocated in Ref. [21].

The reaction $e^+e^- \rightarrow \pi^+\pi^-\gamma$, with the photon emitted from the pions, does contribute also to dispersion integrals for the evaluation of $a_\mu$ and $\alpha_{QED}$. In the former case its theoretically estimated value [8] is of the size of the theoretical uncertainty and thus numerically important. As its theoretical estimations are not reliable it has to be measured. PHOKHARA also includes the simulation of events with simultaneous emission of one photon from the initial state and another from the final state.

4. Three-pion channel

The channel $e^+e^- \rightarrow \pi^+\pi^-\pi^0$ has been recently implemented in PHOKHARA [9]. The model for the form factor is based on generalized vector dominance with isospin $I = 0,1$ components:

$$J_{\mu}^{em,3\pi} = \langle \pi^+(q_+)\pi^-(q_-)\pi^0(q_0) | J_{\mu}^{em} | 0 \rangle = \epsilon_{\mu\alpha\beta\gamma} q_+^\alpha q_-^\beta q_0^\gamma \sum \tilde{F}_{3\pi}^{I=0,1} (q_+, q_-, q_0). \quad (1)$$

A global fit has been performed with contributions from $\omega(782)$, $\omega' = \omega(1420)$, $\omega'' = \omega(1650)$, $\phi(1020)$, $\rho(770)$, $\rho' = \rho(1450)$ and $\rho'' = \rho(1700)$ resonances. The model provides a very good description of the total cross section and also predicts in good agreement with experiment the decay width $\Gamma(\pi^0 \rightarrow \gamma\gamma)$, the slope parameter of $\pi^0 \rightarrow \gamma\gamma^*$, and the radiative vector meson decays $\rho \rightarrow \pi^0\gamma$ and $\phi \rightarrow \pi^0\gamma$, but is however in conflict with $\omega \rightarrow \pi^0\gamma$. There is still room from im-
provements once information on subdistributions is included in the fit.

5. Nucleon form factors

Another example of the power of the radiative return method is the measurement of the magnetic and the electric nucleon form factors in the time-like region. This measurement was first proposed in Ref. [10]. The recent measurement by the BABAR Collaboration [22] of the magnetic and electric proton form factor shows a clear evidence for a ratio $|G_E/G_M| > 1$ just above threshold with respect to to previous analysis. It is also interesting to note that the measurement of the relative phase between $G_E$ and $G_M$ requires access to the nucleon spin [10].

6. Production and decay of $J/\psi$ and other narrow resonances

$J/\psi$ resonances are copiously produced at B-factories. There is a strong demand from experimental groups to include this kind a new channels in our program, to analyse its reach phenomenology with high precision. Other narrow resonances are of interest as well.

The BABAR Collaboration [23] has reported the discovery of a new state, the Y(4260) resonance, by using radiative return events in the $\pi^+\pi^- J/\psi$ channel. These results have been confirmed by CLEO using energy-scan [24], and ISR [25] data, as well as by the BELLE Collaboration [26].

7. One-loop corrections to muon production and higher order radiative corrections

When the first version of PHOKHARA was constructed the two-pion channel was the most interesting as it enters the prediction for the anomalous magnetic moment of the muon. By that time ISR radiative corrections where introduced in PHOKHARA through a leptonic tensor:

$$L_{\text{ISR}}^{\mu\nu} = \alpha^2 \left[ \epsilon_0 0 g^{\mu\nu} + a_{11} p_1^\mu p_1^\nu + a_{22} p_2^\mu p_2^\nu ight.$$

$$+ a_{12}(p_1^\mu p_2^\nu + p_2^\mu p_1^\nu) + i\pi a_{-1}(p_1^\mu p_2^\nu - p_2^\mu p_1^\nu) \right],$$

(2)

where $p_1$ ($p_2$) are the four-momentum of the incoming positron (electron), and $a_{ij}$ are scalar coefficients where the antisymmetric imaginary contribution proportional to $a_{-1}$ appears first at NLO. While ISR corrections are independent of the hadronic channel, FSR and the ISR-FSR interference do depend and have to be calculated for each channel independently.

When more and more hadronic channels are of interest it seems more convenient to introduce radiative corrections at the amplitude level by using the helicity amplitude formalism. Interferences between different amplitudes are then obtained automatically without further analytical computations. The radiative one-loop amplitude can be factorized into three components:

$$|A\rangle = |A\rangle_{\text{ISR}} + |A\rangle_{\text{FSR}} + |A\rangle_{2\gamma^*},$$

(3)

where the last one steams from the exchange of two virtual photons between the initial and the final state.

Muon pair production is not only important for the normalization of the $R$-ratio, but being a very clean process that can be calculated in QED can be used for luminosity monitoring at $e^+e^-$ machines. Therefore the importance of having very accurate predictions. We are now completing the calculation of ISR radiative corrections at the amplitude level and FSR for the muon channel. The two-photon exchange amplitude will be calculated subsequently. Note that the latter is not even known for Bhabha scattering.

The estimated accuracy of PHOKHARA from ISR is of the order of 5 per mil. Although this is a very conservative estimate a better accuracy might be needed for future experiments; higher luminosity $B$-factories or even the International Linear Collider (ILC). Two loop corrections, at least in the leading log approximation, will reduce this uncertainty to at least 1 to 2 per mil.

8. Interplay between $e^+e^-$ and tau data

The hadronic vacuum polarization contribution to the SM prediction of the anomalous magnetic moment of the muon is obtained though a dispersion integral over the $e^+e^-$ hadronic cross section (once ISR and vacuum polarization correc-
A careful regard to these data tell us that the three experiments do overlap indeed only in the energy region above 600 MeV, while in the threshold region only data from the Novosibirsk experiments are available. As expected the statistical error of radiative return data is much better than of energy scan. CMD-2 has the better systematic error (0.6%), while the total systematic error of SND and KLOE are comparable (1.3%). The new KLOE analysis [13] will however reduce it to less than 1%. While SND and CMD-2 experiments agree to each other in the full energy range, they can not be considered completely independent as both analysis use the same radiative correction package. Furthermore, the new CMD-2 data do not cover the region between 520 and 600 MeV. This makes the ongoing KLOE large photon polar angle analysis particularly interesting because the agreement of the integral is very unlikely to happen in the threshold region. Excluding a set of data from the prediction of $a_\mu$, until these discrepancies are solved, might lead to a biased result.

### 9. Summary

The radiative return has proven to be a competitive method for the precise measurement of the hadronic cross section, detailed studies of hadronic interactions, and even discoveries of new resonances. Many interesting problems, for example a proper modeling of the hadronic current of multi-meson final states, FSR simulation for more than two-pions, modeling of narrow resonances and many others not mentioned await still for detailed theoretical investigations.

New data from the KLOE experiment at small polar photon angles with a total systematic uncertainty below 1% and in particular in the threshold region from the large photon polar angle analysis [13], the long awaited pion form factor measurement at $B$-factories, but also future new data from the energy scan [33] should help to clarify the discrepancies between different $e^+e^-$ data sets. To a great extent independent test of radiative corrections are also needed. One should remember that even the direct measurement of the ratio $\sigma_{\pi\pi}/\sigma_{\mu\mu}$, where most systematics are

### Table 1

<table>
<thead>
<tr>
<th>Energy (MeV)</th>
<th>SND</th>
<th>CMD-2</th>
<th>KLOE</th>
</tr>
</thead>
<tbody>
<tr>
<td>390-520</td>
<td>478.0 ± 17.3 ± 6.9(18.6)</td>
<td>461.7 ± 9.8 ± 3.2(10.3)</td>
<td></td>
</tr>
<tr>
<td>520-600</td>
<td>425.0(27)*</td>
<td></td>
<td></td>
</tr>
<tr>
<td>600-960</td>
<td>3768 ± 13 ± 47(48)</td>
<td>3771 ± 19 ± 27(33)</td>
<td>3756 ± 8 ± 49(50)</td>
</tr>
</tbody>
</table>

Contributions to $a_\mu^{\text{had,LO}}$ (in units of $10^{-11}$) from the different energy regions. Numbers from Ref. [33](∗ my own estimate).
expected to cancel, requires a careful treatment of radiative corrections and unfolding, due to the fact that radiative corrections affect differently the angular distributions of pions and muons [5], and because the $R$-ratio entering the dispersion integral for $a_\mu$ is normalized to the point-like cross section of the muon but not to the physical cross section. All together is necessary to confirm the discrepancy of the $(g-2)_\mu$ measurement with respect to the Standard Model prediction. Understanding tau data, whether the agreement with $(g-2)_\mu$ is by accident or not, is also crucial for this purpose.

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