

$B^0-\bar{B}^0$ Mixing and Decay Constants with the Non-Perturbatively Improved Action

D. Becirevic, D. Meloni and A. Retico

*Dip. di Fisica, Univ. di Roma "La Sapienza" and
 INFN, Sezione di Roma, P.le A. Moro 2, I-00185 Roma, Italy.*

V. Giménez

*Dep. de Física Teòrica and IFIC, Univ. de València,
 Dr. Moliner 50, E-46100, Burjassot, València, Spain.*

L. Giusti

*Department of Physics, Boston University
 Boston, MA 02215 USA.*

V. Lubicz

*Dipartimento di Fisica, Università di Roma Tre and INFN, Sezione di Roma Tre
 Via della Vasca Navale 84, I-00146 Rome, Italy.*

G. Martinelli

*Université de Paris Sud, LPT (Bât. 210)
 Centre d'Orsay, 91405 Orsay-Cedex, France.*

Abstract

Several quantities relevant to phenomenological studies of $B^0-\bar{B}^0$ mixing are computed on the lattice. Our main results are $f_{B_d}\sqrt{\hat{B}_{B_d}} = 206(28)(7)$ MeV, $\xi = f_{B_s}\sqrt{\hat{B}_{B_s}}/f_{B_d}\sqrt{\hat{B}_{B_d}} = 1.16(7)$. We also obtain the related quantities $f_{B_s}\sqrt{\hat{B}_{B_s}} = 237(18)(8)$ MeV, $f_{B_d} = 174(22)_{-0-0}^{+7+4}$ MeV, $f_{B_s} = 204(15)_{-0-0}^{+7+3}$ MeV, $f_{B_s}/f_{B_d} = 1.17(4)_{-1}^{+0}$, $f_{B_d}/f_{D_s} = 0.74(5)$. After combining our results with the experimental world average $\Delta m_d^{(\text{exp.})}$, we predict $\Delta m_s = 15.8(2.1)(3.3)$ ps^{-1} . We have also computed the relevant parameters for $D^0-\bar{D}^0$ mixing which may be useful in some extensions of the Standard Model. All the quantities were obtained from a quenched simulation with a non-perturbatively improved Clover action at $\beta = 6.2$, corresponding to a lattice spacing $a^{-1} = 2.7(1)$ GeV, on a sample of 200 gauge-field configurations. A discussion of the main systematic errors is also presented.

1 Introduction

Historically, the measurement by the UA1 Collaboration of a large value for the neutral meson mass-difference Δm_d was the first indication that top quark mass had to be very heavy [1]. This mass difference is induced by $B_d^0-\bar{B}_d^0$ oscillations. Since then, $B^0-\bar{B}^0$ mixing became one of the most important ingredients of current analyses of the unitarity triangle and of CP violation in the Standard Model [2]–[7] and beyond [8].

In the Standard Model the mixing, induced by the box diagrams with an internal top quark, is summarized by the next-to-leading order (NLO) formula [9, 10]

$$\Delta m_q = \frac{G_F^2}{6\pi^2} m_W^2 \eta_B S_0(x_t) |V_{tq} V_{tb}^*|^2 m_{B_q} f_{B_q}^2 \hat{B}_{B_q} \quad (q = s, d) , \quad (1)$$

where $S_0(x_t)$ is the Inami-Lim function [11], $x_t = m_t^2/M_W^2$ (m_t is the $\overline{\text{MS}}$ top mass $m_t^{\overline{\text{MS}}}(m_t^{\overline{\text{MS}}}) = 165(5)$ GeV) and $\eta_B = 0.55(1)$ is the perturbative QCD short-distance NLO correction. The remaining factor, $f_{B_q}^2 \hat{B}_{B_q}$, encodes the information of non-perturbative QCD and this is what can be computed on the lattice.

Traditionally, the B -parameter of the renormalized operator is defined as

$$\langle \bar{B}_q | Q_q^{\Delta B=2}(\mu) | B_q \rangle = \frac{8}{3} m_{B_q}^2 f_{B_q}^2 B_{B_q}(\mu) , \quad (2)$$

where $Q_q^{\Delta B=2} = (\bar{b}\gamma_\mu(1-\gamma_5)q)(\bar{b}\gamma_\mu(1-\gamma_5)q)$ and μ is the renormalization scale. This definition stems from the vacuum saturation approximation (VSA) in which $B_{B_q} = 1$. The renormalization group invariant, and scheme-independent, B parameter \hat{B}_{B_q} of eq. (1) is defined as

$$\hat{B}_{B_q} = \alpha_s(\mu)^{-\gamma_0/2\beta_0} \left\{ 1 + \frac{\alpha_s(\mu)}{4\pi} J \right\} B_{B_q}(\mu) , \quad (3)$$

where $\gamma_0 = 4$ and J depends on the scheme used for renormalizing $Q_q^{\Delta B=2}(\mu)$ (see below).

Besides the $B^0-\bar{B}^0$ amplitude, an important quantity for phenomenological applications is given by the ratio

$$\frac{\Delta m_s}{\Delta m_d} = \frac{|V_{ts}|^2 m_{B_s} \xi^2}{|V_{td}|^2 m_{B_d}} , \quad (4)$$

where $\xi = f_{B_s} \sqrt{\hat{B}_{B_s}} / f_{B_d} \sqrt{\hat{B}_{B_d}}$.

In this paper, we have computed the hadronic parameters appearing in eqs. (1) and (4), using a non-perturbatively improved action, and with operators renormalized on the lattice with the non-perturbative method of ref. [12], as implemented in [13, 14]. Our main results are

$$f_{B_d} \sqrt{\hat{B}_{B_d}} = 206(28)(7) \text{ MeV} , \quad \xi = \frac{f_{B_s} \sqrt{\hat{B}_{B_s}}}{f_{B_d} \sqrt{\hat{B}_{B_d}}} = 1.16(7) ,$$

$$\begin{aligned}
f_{B_s} \sqrt{\hat{B}_{B_s}} &= 237(18)(8) \text{ MeV} , & r_{sd} &= \xi^2 \left(\frac{m_{B_s}^2}{m_{B_d}^2} \right) = 1.40(18) , \\
f_{B_d} &= 174(22)_{-0-0}^{+7+4} \text{ MeV} , & f_{B_s} &= 204(15)_{-0-0}^{+7+3} \text{ MeV} , \\
\frac{f_{B_s}}{f_{B_d}} &= 1.17(4)_{-1}^{+0} , & \frac{f_{B_d}}{f_{D_s}} &= 0.74(5) .
\end{aligned} \tag{5}$$

From eq. (1) we write

$$\Delta m_d [ps^{-1}] = (0.153 \pm 0.010) |V_{td}|^2 f_{B_d}^2 \hat{B}_{B_d} [\text{MeV}^2] . \tag{6}$$

We stress that what is really relevant for $B^0-\bar{B}^0$ mixing, and can be directly computed on the lattice, is the physical amplitude, corresponding to $f_{B_d}^2 \hat{B}_{B_d}$, and not the decay constant f_{B_d} and \hat{B}_{B_d} separately. The value of $f_{B_d} \sqrt{\hat{B}_{B_d}}$ given in the first of eqs. (5) comes from the calculation of the amplitude, and thus it includes the correlation between the decay constant and the B parameter. Its final error is then smaller than that obtained by combining the results and errors obtained from f_{B_d} and \hat{B}_{B_d} from different calculations.

By taking $|V_{td}| \simeq A\lambda^3 \sqrt{(1-\rho)^2 + \eta^2} = 0.0080(5)$ (where A , ρ , η and λ are the Wolfenstein parameters) this gives

$$\Delta m_d = 0.42(12)(4) \text{ ps}^{-1} , \tag{7}$$

where the first error comes from the lattice uncertainty on $f_{B_d} \sqrt{\hat{B}_{B_d}}$ and the second from the error on m_t and on $|V_{td}|$. The most recent world average of experimental results is [15]:

$$\Delta m_d^{(\text{exp.})} = 0.473(16) \text{ ps}^{-1} . \tag{8}$$

To predict Δm_s it is convenient to use the above experimental information and eq. (4) written as

$$\Delta m_s = \frac{|V_{ts}|^2}{|V_{td}|^2} \xi^2 \left(\frac{m_{B_s}}{m_{B_d}} \Delta m_d \right)^{(\text{exp.})} . \tag{9}$$

Using our result for ξ^2 and $|V_{ts}|^2/|V_{td}|^2 \simeq 1/[\lambda^2((1-\rho)^2 + \eta^2)] = 24.4(5.0)$, we get

$$\Delta m_s = 15.8(2.1)(3.3) \text{ ps}^{-1} , \tag{10}$$

to be compared with the experimental lower bound

$$\Delta m_s > 12.4 \text{ ps}^{-1} . \tag{11}$$

In the above calculations, we have assumed the values of the relevant couplings, namely $|V_{td}|$ and $|V_{ts}|$, from the unitarity relations of the V_{CKM} matrix. Obviously, from the experimental determinations of Δm_q and the hadronic matrix elements we can, instead, constrain $|V_{tq}|$. This is what is usually done in the unitarity triangle analyses [2]–[8]. For the sake of comparison, we also give the values of the decay constants and their ratios from

our previous studies of these quantities, obtained using the same improved action, with a comparable statistics but on a larger lattice [16]

$$\begin{aligned} f_{B_d} &= 173(13)_{-2}^{+34} \text{MeV} , & f_{B_s} &= 196(11)_{-0}^{+42} \text{MeV} , \\ \frac{f_{B_s}}{f_{B_d}} &= 1.14(2)(1) , & \frac{f_{B_d}}{f_{D_s}} &= 0.72(2)_{-0}^{+13} . \end{aligned} \quad (12)$$

The last error in the figures above represents the uncertainty coming from the extrapolation (linear vs quadratic) in the inverse meson mass to the B mesons. This error is not given in eqs. (5), because in the present study only a linear extrapolations have been made for reasons discussed below.

Besides the quantities relevant to $B^0-\bar{B}^0$ mixing, we also give the corresponding quantities for $D^0-\bar{D}^0$ mixing. Although $D^0-\bar{D}^0$ mixing is expected to be well below the experimental limit in the Standard Model, it may be enhanced in its extensions [17]. We have obtained

$$\begin{aligned} f_D \sqrt{\hat{B}_D} &= 230(14)_{-8}^{+3} \text{MeV} , & f_D &= 207(11)_{-0-0}^{+3+3} \text{MeV} , \\ f_{D_s} &= 234(9)_{-0-0}^{+3+2} \text{MeV} , & \frac{f_{D_s}}{f_D} &= 1.13(3)_{-1}^{+0} . \end{aligned} \quad (13)$$

Reviews of recent results of quantities considered in this paper can be found in refs. [18].

The remainder of this paper is as follows: in sec. 2 we give the parameters of the lattice simulation, illustrate the calibration of the lattice spacing and of the quark masses and describe the calculation of the heavy meson spectrum and decay constants; in sec. 3 we discuss the renormalization of the relevant operators, the calculation of their matrix elements and the extrapolation to the physical points; in sec. 4 we give the physical results and discuss the statistical and systematic errors; sec. 5 contains a comparison of our results with other calculations of the same quantities as well as our conclusions.

2 Lattice Calibration and Decay Constants

In this section we give some details about our lattice simulation and the calculation of the decay constants, which are a byproduct of this study.

2.1 Lattice Setup

The results presented in this study have been obtained on a $24^3 \times 48$ lattice, using the non-perturbatively improved Clover action at $\beta = 6.2$ with the Clover coefficient $c_{sw} = 1.614$, as computed in ref. [19]. The statistical sample consists of 200 independent gauge-field configurations. Statistical errors have been estimated by using the familiar jackknife procedure with 40 jacks, each obtained by decimating 5 configurations from the whole ensemble. The following values for the heavy- and light-quark hopping parameters, κ_Q and κ_q respectively, have been used:

- 0.1352 (κ_{q_1}); 0.1349 (κ_{q_2}); 0.1344 (κ_{q_3}),

- 0.1250 (κ_{Q_1}); 0.1220 (κ_{Q_2}); 0.1190 (κ_{Q_3}) .

Although we are not discussing light mesons, it is important to mention some results which will be useful in the analysis of heavy-light meson physics, for example for the extrapolation/interpolation in light-quark masses. More details on the procedures used to calibrate the lattice spacing and fix the light-quark masses can be found in previous publications of our group [16, 20].

- The ratio of the masses of the light pseudoscalar (P) and vector (V) mesons are:

$$\frac{m_P}{m_V} = \{0.597(22)_{q_1}, 0.682(15)_{q_2}, 0.761(8)_{q_3}\} . \quad (14)$$

- We use the method of physical lattice planes [21] to fix the value of the inverse lattice spacing ¹. This method consists in the following. First we fit the vector meson mass to the form

$$M_V(m_q, m_q) = \alpha_0 + \alpha_1 (M_P(m_q, m_q))^2 , \quad (15)$$

obtaining $\alpha_0 = 0.286(16)$, and $\alpha_1 = 1.24(13)$. Then we fix m_V/m_P to the physical kaon ratio (m_{K^*}/m_K) and compare M_{K^*} to the experimentally measured $m_{K^*} = 894$ MeV. In this way we obtain

$$a^{-1} = 2.72(13) \text{ GeV} . \quad (16)$$

This is the value of the inverse lattice spacing which has been used throughout this study.

- For a generic physical quantity in the heavy-light meson sector, $\mathcal{F}(m_Q, m_q)$, the interpolation/extrapolation in the light quark to the strange/up-down (s/d) mass is performed through the fit

$$\mathcal{F}(m_Q, m_q) = \alpha_0^Q + \alpha_1^Q (M_P(m_q, m_q))^2 , \quad (17)$$

where $\alpha_{0,1}^Q$ are the fitting parameters. The light-quark mass corresponds either to the d -quark, when we extrapolate to $M_P(m_d, m_d) \equiv M_\pi = 5.1(3) \cdot 10^{-2}$ (as obtained from the lattice-plane method from m_ρ/m_π), or to the strange quark, when we extrapolate to $M_P(m_s, m_s) \equiv M_{\eta_{ss}} = 0.266(17)$ (as inferred from eq. (15) by fixing $M_V(m_s, m_s)$ to $m_\phi = 1020$ MeV).

2.2 Heavy-light Decay Constants

We now discuss the heavy-light meson decay constants, which are important ingredients for physical predictions related to $B^0-\bar{B}^0$ mixing. Since this has been extensively discussed in the literature (see for example ref. [16]), we only recall here the essential steps.

¹ As in previous publications, we use small-case letters for referring to quantities in physical units (*e.g.* m_P in MeV), and capital letters for denoting the same quantities in lattice units (*e.g.* $M_P = m_P a$).

- As usual, hadron masses are extracted by fitting two-point correlation functions. For mesons, the standard form is

$$\mathcal{C}_{JJ}(t) = \sum_{\vec{x}} \langle 0 | J(\vec{x}, t) J^\dagger(0) | 0 \rangle \xrightarrow{t \gg 0} \frac{\mathcal{Z}_J}{M_J} e^{-M_J T/2} \cosh \left[M_J \left(\frac{T}{2} - t \right) \right], \quad (18)$$

where for $J(x) = P_5(x) = i\bar{Q}(x)\gamma_5 q(x)$, we choose $t \in [16, 23]$, whereas for $J(x) = V_i(x) = \bar{Q}(x)\gamma_i q(x)$, we choose $t \in [19, 23]$. The time intervals are established in the standard way (*i.e.* after inspection of the corresponding effective masses). The resulting masses of the pseudoscalar and vector heavy-light mesons, with the light quark interpolated/extrapolated to s/d (see eq. (17)), are listed in tab. 1. We also checked that on the same interval ($t \in [16, 23]$), the masses of pseudoscalar mesons extracted from the correlation function $\mathcal{C}_{AP}(t)$ ($A_0(x) = \bar{Q}(x)\gamma_0\gamma_5 q(x)$), are indeed the same.

In physical units, the masses directly accessed from our simulation, are:

$$\begin{aligned} m_{P_d} &= \{1.75(8) \text{ GeV}, 2.02(9) \text{ GeV}, 2.26(11) \text{ GeV}\} , \\ m_{P_s} &= \{1.85(7) \text{ GeV}, 2.11(9) \text{ GeV}, 2.38(10) \text{ GeV}\} . \end{aligned} \quad (19)$$

- The pseudoscalar meson (P) decay constant is defined as

$$\langle 0 | \hat{A}_0 | P(\vec{p} = 0) \rangle = i\hat{F}_P M_P , \quad (20)$$

where the ‘hat’ symbol means that the appropriate renormalization constant Z_A is taken into account. To determine the decay constant, one computes the ratio

$$\frac{\sum_{\vec{x}} \langle \hat{A}_0(\vec{x}, t) P_5(0) \rangle}{\sum_{\vec{x}} \langle P_5(\vec{x}, t) P_5(0) \rangle} \simeq \hat{F}_P \frac{M_P}{\sqrt{\mathcal{Z}_P}} \tanh \left(M_P \left(\frac{T}{2} - t \right) \right) . \quad (21)$$

When working with improved Wilson fermions, the axial current is improved by adding to A_μ the operator $\partial_\mu P$ with a suitable coefficient

$$\langle 0 | \hat{A}_0 | P \rangle = Z_A \left(\langle 0 | A_0 | P \rangle + c_A \langle 0 | a \partial_0 P_5 | P \rangle \right) = iM_P (\hat{F}_P^{(0)} + c_A a \hat{F}_P^{(1)}) , \quad (22)$$

where the constant, $c_A = -0.037$, was computed in ref. [22]. It is actually easy to see that

$$aF_P^{(1)} = \frac{\sqrt{\mathcal{Z}_P}}{M_P} \sinh(M_P) . \quad (23)$$

The size of the correcting term $c_A a \hat{F}_P^{(1)} / \hat{F}_P^{(0)}$ for our data is in the range $(5 \div 7)\%$.

- As far as the normalization constant is concerned, its improved value is given by (choice 1.)

$$Z_A(m_Q, m_q) = Z_A(0) (1 + b_A a \bar{m}) , \quad (24)$$

where $\bar{m} = (m_Q + m_q)/2$. In the above equation, we have taken

$$am_{Q,q} = \frac{1}{2} \left(\frac{1}{\kappa_{Q,q}} - \frac{1}{\kappa_c} \right). \quad (25)$$

The value of the critical hopping parameter, $\kappa_c = 0.13580(1)$, is obtained from the linear fit $M_P^2 \simeq m_q \rightarrow 0$, whereas the value of improvement coefficient $b_A = 1.24$ in (26) is taken from boosted perturbation theory [23]² For a recent non-perturbative determination of b_A , see ref. [24]. .

In our numerical estimates we have also used

$$Z_A(m_Q, m_q) = Z_A(0) \left[\frac{\sqrt{1 + am_Q} \sqrt{1 + am_q}}{1 + a\bar{m}} \right] (1 + b_A a\bar{m}), \quad (26)$$

This equation (choice 2.) differs from the choice 1. in that it contains higher order tree-level mass corrections through the so-called KLM factor [25]. As for the value of $Z_A \equiv Z_A(0)$ in the chiral limit, we take the nonperturbative determination $Z_A(0) = 0.80$ from ref [26]. This value has been also non-perturbatively computed in refs. [24] ($Z_A = 0.82$) and [27] ($Z_A(0) = 0.81$), using different approaches. The values of the lattice decay constants \hat{F}_P for different heavy-quark masses are listed in the third column of tab. 1.

“Flavor” content ($Q - q$)	M_P	\hat{F}_P	M_V
$Q_3 - s$	0.866(4)	0.0884(21)	0.898(4)
$Q_3 - d$	0.832(5)	0.0781(35)	0.869(7)
$Q_2 - s$	0.777(4)	0.0858(20)	0.812(4)
$Q_2 - d$	0.742(3)	0.0761(29)	0.782(6)
$Q_1 - s$	0.681(5)	0.0853(19)	0.721(5)
$Q_1 - d$	0.643(3)	0.0755(24)	0.690(5)

Table 1: Mass spectrum of heavy-light pseudoscalar and vector mesons, with the light-quark masses extrapolated/interpolated to the d/s quark, using eq. (17). The meson decay constants are also given. All quantities are in lattice units.

To estimate the values of the physical quantities related to B -mesons ($m_{B_d} = 5.28$ GeV, and $m_{B_s} = 5.38$ GeV), the extrapolations to the physical heavy-quark masses necessarily

² By working with three light quark species only, we were unable to make a quadratic fit in the quark masses as done in ref. [20].

f_{D_d} $207(11)_{-0}^{+3} {}_{-0}^{+3} \text{ MeV}$	f_{D_s} $234(9)_{-0}^{+3} {}_{-0}^{+2} \text{ MeV}$	f_{D_s}/f_{D_d} $1.13(3)_{-0}^{+0} {}_{-1}^{+0}$
f_{B_d} $174(22)_{-0}^{+7} {}_{-0}^{+4} \text{ MeV}$	f_{B_s} $204(15)_{-0}^{+7} {}_{-0}^{+3} \text{ MeV}$	f_{B_s}/f_{B_d} $1.17(4)_{-0}^{+0} {}_{-1}^{+0}$

$\Phi_0^{(q=d)}$ $0.46(7) \text{ GeV}^{3/2}$	$\Phi_1^{(q=d)}$ $-0.73(13) \text{ GeV}$
$\Phi_0^{(q=s)}$ $0.56(5) \text{ GeV}^{3/2}$	$\Phi_1^{(q=s)}$ $-0.81(7) \text{ GeV}$

Table 2: Summary of physical results for the decay constants. The parameters of eq. (28) are also listed. Our value for Φ_1 agrees well with QCD sum rule estimates, namely $\Phi_1 = -0.9(2) \text{ GeV}$ [28].

rely on the heavy-quark symmetry. Accordingly, the decay constants scale as

$$f_P = \frac{\Phi(m_P)}{\sqrt{m_P}} \left(1 + \frac{\Phi'(m_P)}{\Phi(m_P)} \frac{1}{m_P} + \dots \right). \quad (27)$$

Apart from an overall (mild) logarithmic correction, the coefficients $\Phi(m_P)$ and $\Phi'(m_P)$, are non-perturbative quantities which we extract from a fit to the lattice data. Since we work with three heavy quarks (three values of κ_Q), we have only included the leading $1/m_P$ correction by fitting the decay constant to the expression

$$\hat{F}_P \sqrt{M_P} = \Phi_0 \left(1 + \frac{\Phi_1}{M_P} \right). \quad (28)$$

In tab. 2, we give the physical results, and the values of parameters $\Phi_{0,1}$, in physical units. The statistical errors are given in parentheses. The systematic errors require some explanation: the first one is obtained from the differences between results obtained by using Z_A with choice 1. or 2; the second is obtained by comparing the central values of the procedure described above to those extracted from the study of the ratio \hat{F}_P/M_V .

We have not included in the systematic errors: *i*) the uncertainty due to quenching. Pioneering attempts for estimating quenching errors [29]–[31] show an increase of about $20 \div 30 \text{ MeV}$ of the value of the decay constants. *ii*) the error due to the truncation of the $1/m_P$ expansion in eq. (28). The latter is usually estimated by considering also a quadratic term ($\propto 1/m_P^2$). With only three heavy-quark masses, we have not done such a fit. From our previous experience [16], we know that this error has a negligible effect on the value

of the D -meson decay constants, whereas for the B meson it represents a major source of systematics (~ 30 MeV).

Finally, we present an estimate of f_{B_d} obtained using the experimental measurement of $f_{D_s} = 241(32)$ MeV [5]. With the ratio f_B/f_{D_s} obtained in our simulation, table 2, we get

$$\begin{aligned} \left(\frac{f_{B_d}}{f_{D_s}}\right)^{(\text{latt.})} &= 0.74(4)_{-0}^{+2} {}_{-0}^{+2} = 0.74 \pm 0.05 \\ \implies f_B &= 184 \pm 24(\text{exp.}) \pm 12(\text{theo.}) \text{ MeV} \end{aligned} \quad (29)$$

3 Matrix Elements of the Renormalized Operators

In this section, we discuss the renormalization of the relevant four-fermion operators and describe the calculation of the matrix elements introduced in eq. (2). Since we are interested to the $B^0-\bar{B}^0$ mixing amplitude, only the parity-even part of the operator, $Q = (\bar{b}\gamma_\mu q)(\bar{b}\gamma_\mu q) + (\bar{b}\gamma_\mu\gamma_5 q)(\bar{b}\gamma_\mu\gamma_5 q)$, will be discussed in the following.

3.1 Renormalization of the Four-fermion Operators

On the lattice, the renormalized operator $Q(\mu)$ takes the form [32]

$$Q(\mu) = Z(\mu, g_0^2) \left(O_1 + \sum_{i=2}^5 \Delta_i(g_0^2) O_i \right), \quad (30)$$

where the O_i denotes a bare operator and we choose to work in the following parity-even basis

$$\begin{aligned} O_1 &= \bar{b}\gamma_\mu q \bar{b}\gamma_\mu q + \bar{b}\gamma_\mu\gamma_5 q \bar{b}\gamma_\mu\gamma_5 q \\ O_2 &= \bar{b}\gamma_\mu q \bar{b}\gamma_\mu q - \bar{b}\gamma_\mu\gamma_5 q \bar{b}\gamma_\mu\gamma_5 q \\ O_3 &= \bar{b}q \bar{b}q - \bar{b}\gamma_5 q \bar{b}\gamma_5 q \\ O_4 &= \bar{b}q \bar{b}q + \bar{b}\gamma_5 q \bar{b}\gamma_5 q \\ O_5 &= \bar{b}\sigma_{\mu\nu} q \bar{b}\sigma_{\mu\nu} q. \end{aligned} \quad (31)$$

To obtain the renormalized operator, two steps are then necessary. First, one has to subtract operators of the same dimension, which (on the lattice) mix with O_1 (this is the consequence of the presence of the Wilson term in the fermion action, which explicitly breaks chiral symmetry). This means that one has to compute the subtraction constants $\Delta_i(g_0^2)$. After an appropriate subtraction of the operators O_i , with $i \neq 1$, the matrix elements still need (to be finite) an overall renormalization constant provided by $Z(\mu, g_0^2)$. A detailed discussion of the mixing matrix can be found in ref. [14]. In this work, we use the method of refs. [13, 14] to calculate the matching and the renormalization constants non-perturbatively. The computation, at three different values of the scale ($\mu a = 0.71, 1.00, 1.42$), is performed in the RI-MOM renormalization scheme, in Landau gauge. The results are listed in table 3. For sufficiently large μ , we can use the available NLO anomalous

Scale $a\mu$	$Z(\mu, g_0^2)$	Δ_2	Δ_3	Δ_4	Δ_5
0.71	0.630(11)	-0.052(3)	-0.016(2)	0.014(2)	0.001(1)
1.01	0.583(7)	-0.055(2)	-0.017(2)	0.016(2)	0.002(2)
1.42	0.601(8)	-0.069(2)	-0.022(1)	0.016(1)	0.007(2)

Table 3: Summary of the results for renormalization and subtraction constants (eq. (30)), evaluated non-perturbatively at $\beta = 6.2$ at three different scales μ . $Z(\mu, g_0^2)$ is the renormalization constant in the RI-MOM scheme.

dimension of the operator $\langle Q(\mu) \rangle^{\text{RI-MOM}}$ [33], to define the renormalization group invariant (RGI) operator:

$$\langle \hat{Q} \rangle = w^{-1}(\mu) \langle Q(\mu) \rangle \equiv \alpha_s(\mu)^{-\gamma_0/2\beta_0} \left\{ 1 + \frac{\alpha_s(\mu)}{4\pi} J \right\} \langle Q(\mu) \rangle. \quad (32)$$

where

$$\gamma_0 = 4; \quad J_{\text{RI-MOM}} = -\frac{17397 - 2070n_f + 104n_f^2}{6(33 - 2n_f)^2} + 8 \log 2. \quad (33)$$

We also give the NLO part relevant in the $\overline{\text{MS}}$ scheme [9]

$$J_{\overline{\text{MS}}} = \frac{13095 - 1626n_f + 8n_f^2}{6(33 - 2n_f)^2}. \quad (34)$$

By using $\alpha_s(m_Z) = 0.118$, and setting $n_f = 4$, we obtain

$$\begin{aligned} \mu &= \{1.9 \text{ GeV}, 2.73 \text{ GeV}, 3.85 \text{ GeV}\}, \\ w(\mu)_{\text{RI-MOM}}^{-1} &= \{1.408, 1.453, 1.489\}. \end{aligned} \quad (35)$$

In quenched calculations, there is always the embarrassing question whether to use the physical value of α_s , with the complete formulae for the anomalous dimensions and the β -function, or a “quenched value” of the coupling constant (which suffers from intrinsic ambiguities), together with anomalous dimensions and the β -function computed for $n_f = 0$. The present case is particularly lucky, since by computing $w(\mu)$ in the unquenched ($n_f = 4$) and quenched cases (with $\Lambda_{\text{QCD}}^{n_f=0} = 250 \text{ MeV}$), one finds values of \hat{B}_{B_d} which differ by less than 2%.

We give in passing, $w(m_b)_{\overline{\text{MS}}}^{-1} = 1.477$, which will be used later on for a comparison with the results of other authors. In the perturbative calculation of $w(m_b)_{\overline{\text{MS}}}^{-1}$, we have used $m_b = 4.25 \text{ GeV}$ [34].

3.2 Matrix elements

We have computed the relevant three-point correlation functions:

$$\mathcal{C}_Q^{(3)}(\vec{p}, \vec{q}; t_{P_1}, t_{P_2}) = \int d\vec{x}d\vec{y} e^{i(\vec{p}\cdot\vec{y}-\vec{q}\cdot\vec{x})} \langle 0 | P_5(\vec{x}, t_{P_2}) Q(\vec{0}, 0) P_5^\dagger(\vec{y}, t_{P_1}) | 0 \rangle. \quad (36)$$

When the sources are sufficiently separated and far away from the origin, the lowest pseudoscalar mesons are isolated and one has

$$\mathcal{C}_Q^{(3)}(\vec{p}, \vec{q}; t_{P_1}, t_{P_2}) \xrightarrow{t_{P_1}, t_{P_2} \gg 0} \frac{\sqrt{\mathcal{Z}_P}}{2E_P(\vec{q})} e^{-E_P(\vec{q})t_{P_1}} \langle \bar{P}(\vec{q}) | Q | P(\vec{p}) \rangle \frac{\sqrt{\mathcal{Z}_P}}{2E_P(\vec{p})} e^{-E_P(\vec{p})t_{P_2}}, \quad (37)$$

where $\mathcal{Z}_P \equiv \langle 0 | P_5 | P \rangle$. In order to eliminate irrelevant factors and to extract the B parameter we form the following ratio

$$\mathcal{R}(\vec{p}, \vec{q}; \mu) = \frac{3}{8} \frac{\mathcal{C}_Q^{(3)}(\vec{p}, \vec{q}; t_{P_1}, t_{P_2}; \mu)}{\mathcal{C}_{AP}^{(2)}(\vec{p}; t_{P_2}) \mathcal{C}_{AP}^{(2)}(\vec{q}; t_{P_1})}, \quad (38)$$

where μ indicates the scale dependence. The two point correlation functions are both multiplied by the axial-current renormalization constant in the chiral limit $Z_A(0)$. We have not used the improved renormalization constant $Z_A(m_Q, m_q)$ since we have not attempted to improve the operator Q . This is beyond the scope of the present study and would demand the inclusion of many operators of dimension seven, whose coefficients are presently unknown. Our hope is that a part of the uncertainties of $\mathcal{O}(\bar{m}a)$ cancel in the ratio (38). This has been computed at $|\vec{p}| = |\vec{q}| \equiv 0$, corresponding to

$$\mathcal{R}(\mu) \rightarrow \frac{3}{8} \frac{\langle \bar{P} | Q(\mu) | P \rangle}{|\langle 0 | \hat{A}_0 | P \rangle|^2} \equiv B_{P_q}(\mu). \quad (39)$$

In this way we obtain the main results of this work. We also considered reasonably low momentum injections to the sources but the additional noise makes these data not useful in practice. For this reason these data will be ignored in the following.

In order to show the quality of the signal, in fig. 1 we show the ratio $\mathcal{R}(\mu)$ defined in eq. (38) at $\mu = 3.8$ GeV ($\mu a = 1.42$), for a specific combination of the heavy and light hopping parameters ($\kappa_Q = \kappa_{Q_2} = 0.1220$ and $\kappa_q = \kappa_{q_2} = 0.1349$). The ratio is plotted as a function of the time-distance of one of the two sources. The other has been fixed to $t_{P_1} = 16$. We also performed the simulation by fixing $t_{P_1} = 12$ and $t_{P_1} = 20$. From the study of the two-point correlation functions, one isolates the lightest pseudoscalar state around $t = 16$. That makes $t_{P_1} = 12$ rather small for our purposes. For $t_{P_1} = 20$, the sources are not far enough from each other because of the periodic boundary conditions, and this may spoil the signal. After inspection of the ratios $\mathcal{R}(\mu)$ (for each combination of the heavy and light quarks), a stability plateau is found in the range $28 \leq t \leq 33$. From this plateau we extract the values of $\mathcal{R}(\mu)$, that is the B parameters, for each combination of quark masses. At this point, we extrapolate the light quark to the s - and d -quark mass by fitting the B parameter to eq. (17). This is illustrated in fig. 2 for $\kappa_{Q_2} = 0.1220$ at $\mu = 3.8$ GeV. In tab. 4, we give a detailed list of results for the RI-MOM $B_{P_q}(\mu)$, as well as for the RGI \hat{B}_{P_q} (see eq. (32)).

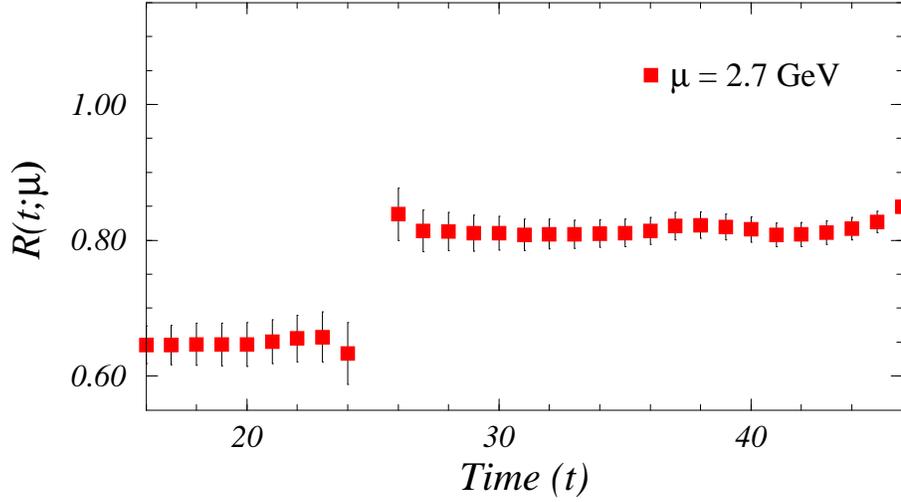


Figure 1: Ratio $\mathcal{R}(\mu)$ at $\mu a \simeq 1.42$, which corresponds to $\mu \simeq 3.8$ GeV. The values of the heavy and light hopping parameters are $\kappa_q = 0.1349$ and $\kappa_Q = 0.1220$, respectively.

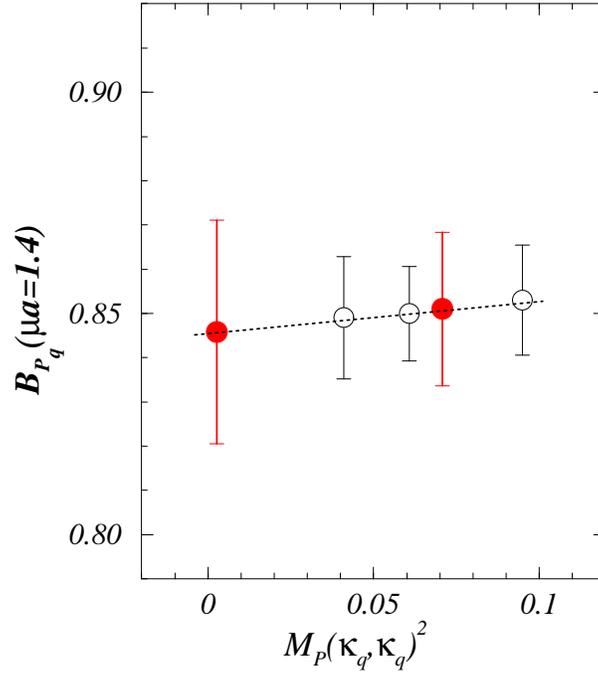


Figure 2: Fit of $\mathcal{R}(\mu) \equiv B_{P_q}(\mu)$ in the light-quark mass according to eq.(17). Empty symbols denote the data obtained from our simulation ; filled symbols represent $B_{P_s}(\mu)$ and $B_{P_d}(\mu)$, obtained after interpolation to the strange and extrapolation to the down quark mass. The heavy quark corresponds to $\kappa_{Q_2} = 0.1220$.

Scale (μ)	1.9 GeV		2.7 GeV		3.8 GeV	
Light quark	$q = s$	$q = d$	$q = s$	$q = d$	$q = s$	$q = d$
$B_{P_q}(\mu; \kappa_Q = 0.1250)$	0.853(16)	0.842(23)	0.789(15)	0.779(21)	0.831(15)	0.821(21)
$\hat{B}_{P_q}(\kappa_Q = 0.1250)$	1.201(23)	1.186(32)	1.148(21)	1.132(30)	1.237(21)	1.222(29)
$B_{P_q}(\mu; \kappa_Q = 0.1220)$	0.875(19)	0.867(27)	0.810(18)	0.803(25)	0.851(18)	0.846(25)
$\hat{B}_{P_q}(\kappa_Q = 0.1220)$	1.233(26)	1.222(39)	1.178(25)	1.167(37)	1.264(25)	1.259(38)
$B_{P_q}(\mu; \kappa_Q = 0.1190)$	0.880(18)	0.879(29)	0.815(17)	0.814(27)	0.855(16)	0.854(27)
$\hat{B}_{P_q}(\kappa_Q = 0.1190)$	1.239(25)	1.238(41)	1.184(23)	1.182(39)	1.273(24)	1.272(39)

Table 4: B_{P_q} for the three values of κ_Q and with the light-quark mass extrapolated/interpolated to the d/s quarks using eq. (17). The results are given for both the RI-MOM scheme and the RGI case.

It is interesting to check whether the scale dependence of the extracted $B_{P_q}(\mu)$ is well described by the NLO anomalous dimension (eq.(32)). In fig. 3, we plot the evolution of $B_{P_q}^{\text{RI-MOM}}(\mu)$. The three curves (dashed, full and dotted) have been normalized to the B parameter at the scale $\mu a = 0.71, 1.01$ and 1.42 respectively, as given in table 4. We note that the point at $\mu a = 1.01$ is sensibly lower than what expected on the basis of the perturbative evolution. This effect is a consequence of a fluctuation of the value of $Z(\mu)$ at this value of the scale as shown in table 3. We checked that the fluctuation is enhanced by the extrapolation of $Z(\mu)$ to the chiral limit and consider it as part of the uncertainty in the determination of the matrix element. The central values of \hat{B}_{P_q} quoted below correspond to the result obtained converting the RI-MOM B parameters at $\mu a = 1.42$ to the RGI one. The difference with the results obtained by using the other two scales, namely $\mu a = 1$ and $\mu a = 0.7$, is included in the estimate of the systematic uncertainty.

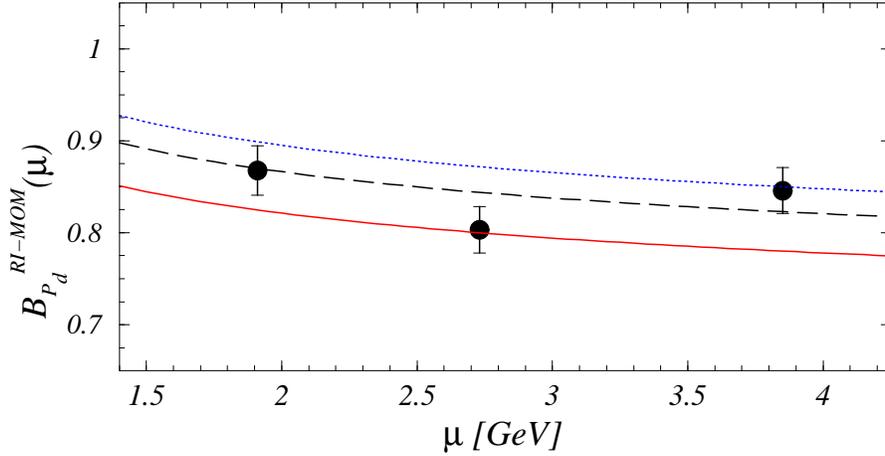


Figure 3: NLO evolution of $B_{P_d}^{\text{RI-MOM}}(\mu)$ ($\kappa_Q = 0.1220$). The dashed, solid and dotted lines are obtained by starting the evolution from the lattice result (filled circles) at $\mu a = 0.71, 1.01, 1.42$, respectively.

4 Physical results

In this section, we derive the physical amplitudes from the B parameters computed at fixed m_Q . Before discussing the extrapolation in the heavy-quark mass, let us summarize our results. We have computed the matrix elements of the operator $Q(\mu)$ in the RI-MOM scheme at three different scales μ . The dependence on light quark is rather smooth, and we have easily performed the extrapolation/interpolation to the d and s quarks. By using the NLO evolution, we have checked the stability of our results. For each value of the heavy-quark mass, from the B parameter in the RI-MOM scheme we have computed \hat{B}_{P_q} by using the result at $\mu \simeq 3.8$ GeV. This is the only step where perturbation theory comes in the game. At this point we dispose of the following values:

$$\begin{aligned}
 \kappa_Q &= \{ 0.1250, 0.1220, 0.1190 \} \\
 \hat{B}_{P_s} &= \{ 1.24(2)_{-0.09}^{+0.00}, 1.26(2)_{-0.09}^{+0.00}, 1.27(2)_{-0.09}^{+0.00} \} \\
 \hat{B}_{P_d} &= \{ 1.22(3)_{-0.09}^{+0.00}, 1.26(4)_{-0.09}^{+0.00}, 1.27(4)_{-0.09}^{+0.00} \}
 \end{aligned} \tag{40}$$

In order to extrapolate the above results into the b quark, we rely on the heavy-quark symmetry as in the case of decay constants and fit the \hat{B} parameters to the expression

$$\hat{B}_{P_q} = c_0^{(q)} \left(1 + \frac{c_1^{(q)}}{M_P} \right). \tag{41}$$

The result is shown in fig. 4, and the main numbers are listed in tab. 5. We stress again that we are not able to include terms of order $1/m_P^2$ in our fit, because we work with

three values of the heavy-quark mass only. Note, however, that our results for the slope $c_1^{(q)}$ are much smaller than the predictions of ref. [35], where the heavy quark was treated non-relativistically. We will comment on this in the next section. These results, when

\hat{B}_{D_d} 1.24(4) $^{+0.00}_{-0.09}$	\hat{B}_{B_d} 1.38(11) $^{+0.00}_{-0.09}$	\hat{B}_{B_s} 1.35(5) $^{+0.00}_{-0.08}$	$(\hat{B}_{B_s}/\hat{B}_{B_d})$ 0.98(5)
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$c_0^{(q=d)}$ 1.46(2)	$c_1^{(q=d)}$ -0.28(16) GeV
$c_0^{(q=s)}$ 1.41(6)	$c_1^{(q=s)}$ -0.24(6) GeV

Table 5: Physical results for \hat{B}_{P_q} parameters obtained in this work. In addition, we give the parameters $c_{0,1}^{(q)}$ from eq. (41).

combined with those in tab. 2, give the following final numbers

$$\begin{aligned}
f_{B_d}\sqrt{\hat{B}_{B_d}} &= 206(28)(7) \text{ MeV}, & f_{B_s}\sqrt{\hat{B}_{B_s}} &= 237(18)(8) \text{ MeV}; \\
\xi \equiv \frac{f_{B_s}}{f_{B_d}}\sqrt{\frac{\hat{B}_{B_s}}{\hat{B}_{B_d}}} &= 1.16(7), & r_{sd} = \xi^2 \left(\frac{m_{B_s}^2}{m_{B_d}^2}\right)^{(\text{exp.})} &= 1.40(18).
\end{aligned} \tag{42}$$

In practice, for the reasons discussed in the introduction, we have extracted the value of $f_{B_d}\sqrt{\hat{B}_{B_d}}$, as well as the other quantities appearing in the above equation, directly from the calculation of the mixing amplitude.

For completeness, and for comparison with other calculations, we also give the B parameter in the $\overline{\text{MS}}$ scheme

$$\begin{aligned}
\hat{B}_{B_d}^{\overline{\text{MS}}}(m_b) &= 0.93(8)_{-0.6}^{+0.0}, \\
\hat{B}_{B_s}^{\overline{\text{MS}}}(m_b) &= 0.92(3)_{-0.6}^{+0.0}.
\end{aligned} \tag{43}$$

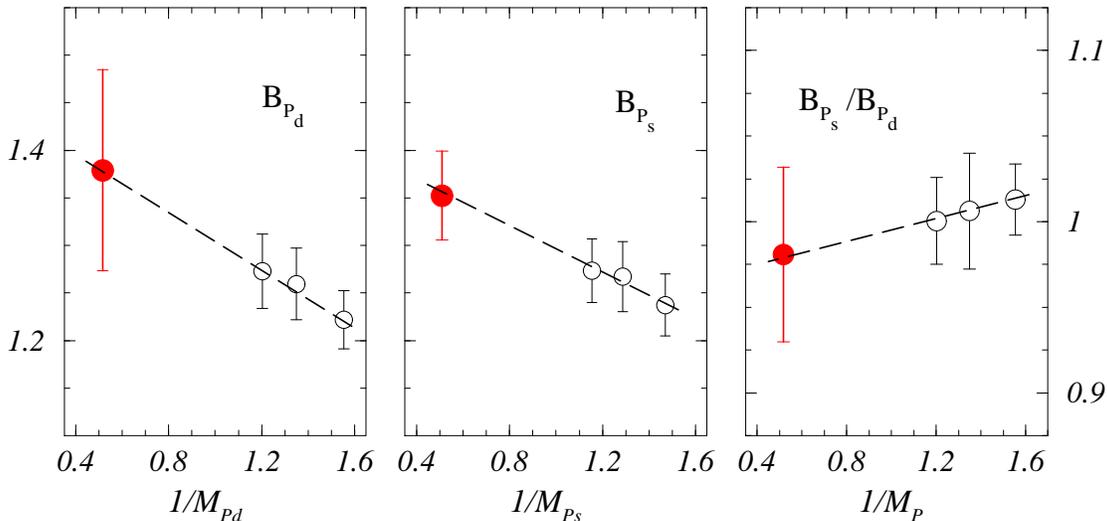


Figure 4: Filled circles correspond to the B -meson sector. We see the $1/m_P$ negative slope for \hat{B}_{B_d} and \hat{B}_{B_s} . Third picture shows the $SU(3)$ breaking for B -parameter, which is clearly compatible with 1.0 for all heavy-light mesons.

5 Comparison with Other Calculations and Conclusions

In this section we compare our results with those obtained with different lattice calculations or with other methods.

On the lattice, two approaches have been used so far to compute $B^0-\bar{B}^0$ mixing: the direct one (as done in this work) which extrapolates in the heavy-quark mass and the one where the heavy quark is treated with an effective theory. In the latter, either the Heavy Quark Effective Theory (HQET) or Non-Relativistic QCD (NRQCD) were used. There is a large number of lattice studies in which the HQET was invoked. Unfortunately, most of them were plagued by a mistake in the calculation of the renormalization constant. Only recently, this has been corrected in ref. [36], and the data of refs. [37, 38] have been reanalyzed³. On the other hand, very recently, the NRQCD treatment of the heavy quark has been employed to compute B_{B_q} [35]. The authors also calculated the relevant $\mathcal{O}(1/m_P)$ corrections. The slope in $1/m_P$ that they observe for $B_{B_d}^{\overline{\text{MS}}}(m_b)$ is roughly a factor of 3 larger than ours (or the one by ref. [39]). We stress that there is a general argument which shows that the uncertainty on the $1/m_P$ corrections can be easily as large as the corrections themselves unless perturbation theory on the leading term is not pushed to very high orders [40]. This argument found confirmation by the explicit calculation of the $\mathcal{O}(\alpha_s^2)$ perturbative corrections to the heavy-quark mass in the HQET [41]. Moreover, even the lowest order result of ref. [35] in $1/m_P$ is suspicious, since they combine the renormalization constants computed in the HQET with the computation of the matrix elements of NRQCD.

³This is the reason why we quote results from these three papers only.

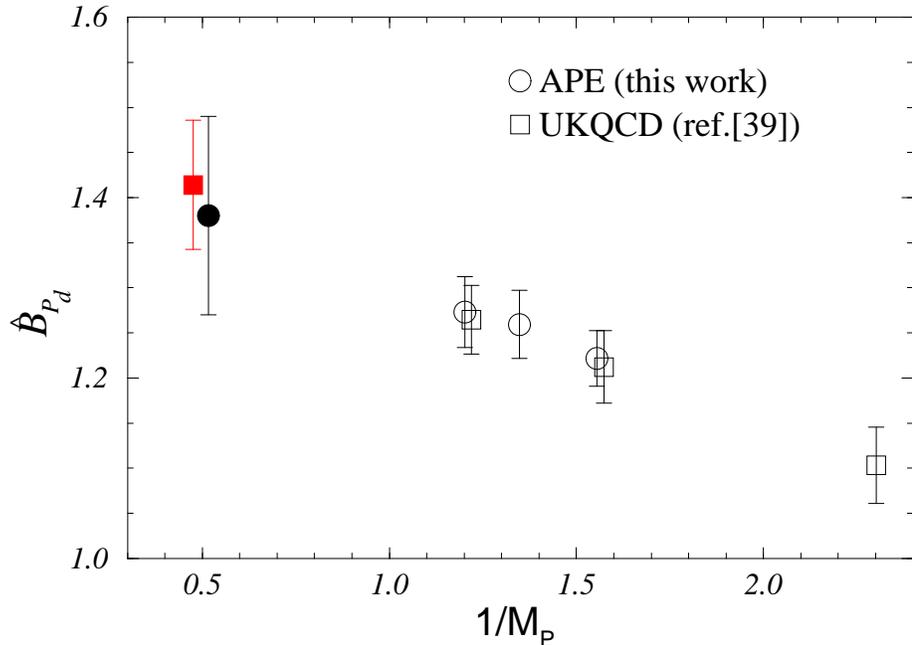


Figure 5: Comparison of our results with those of ref. [39] as a function of the inverse heavy-meson mass in lattice units.

Until the renormalization of the relevant operators will not be completed in a consistent theoretical framework, the results of ref. [35] should not be used for comparisons with other approaches.

In the direct approaches, a full relativistic treatment is given to both heavy and light quarks [18]. In this case a major source of systematic error comes from the extrapolation of the results to the physical B mesons. Recent results for $B^0-\bar{B}^0$ mixing can be found in refs. [39] and [43]. In the second paper, an extrapolation to the continuum ($a \rightarrow 0$) obtained using non-improved Wilson fermions, has been attempted. They obtain a large central value for the ratio r_{sd} ($r_{sd} = 1.76(10)_{-42}^{+57}$), albeit with large errors. The central value is difficult to reconcile with the value of ξ and of the decay constants produced by the same authors with the same set of data. In the present work and the one by UKQCD [39], improved fermions are used but without extrapolation to the continuum. The differences between the present study and that of [39] is that in our case the action is improved non-perturbatively whereas UKQCD uses the mean-field improved action, and that we renormalize non-perturbatively the relevant four-fermion operators. The overall agreement is excellent. This is true also at the values of the heavy-quark masses where lattice measurements have been actually made as shown in fig. 5. Their extrapolated value has an error smaller than ours, probably because they also use a point corresponding to a small value of m_P , for which the heavy-quark expansion may be questionable. In table 6, for comparison, we list several recent results of the direct and the HQET approaches.

For completeness, let us mention that also QCD sum rules were employed to compute

Reference	\hat{B}_{B_d}	$\hat{B}_{B_s}/\hat{B}_{B_d}$
This work	$1.38(11)_{-0.09}^{+0.00}$	0.98(5)
[39]	$1.41(6)_{-0.00}^{+0.05}$	0.98(3)
[43]	1.41(18)	$\simeq 1$
[36]+[37]	1.29(8)(6)	—
[36]+[38]	1.26(6)(6)	—

Table 6: *Recent lattice results on \hat{B}_{P_d} (at NLO accuracy). The upper part of the table refer to results obtained by extrapolation to the B -mesons, whereas in the lower part, the HQET was used. The $SU(3)$ breaking ratios are also listed (when available).*

B_{B_d} [45]. Radiative corrections were also included in ref. [46], where the value for $B_{B_d}^{\overline{\text{MS}}}(m_b)$ was found to be compatible with 1 (VSA value). The most recent QCD sum rule estimate has been given by Chernyak [47], who quotes a value lower than ours and closer to the results obtained from the HQET at lowest order in $1/m_P$: $B_{B_d}^{\overline{\text{MS}}}(5.04 \text{ GeV}) = 0.82$, corresponding to $\hat{B}_{B_d} = 1.26$. The effect of $SU(3)$ breaking in B_{B_s}/B_{B_d} , was studied in the framework of the HQET, combined with chiral perturbation theory [44]. Their result suggests that $SU(3)$ breaking for the B_B parameter is practically negligible as confirmed by many lattice calculations. The approach has been extended to quenched chiral perturbation theory [42], where a pessimistic estimate of quenching errors ($\sim 10\text{--}20\%$ for the matrix element) was quoted.

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