In this derivation and the one presented in a covariant way. We point out that Hawking radiation is immediately robust against an invariant Planck-scale cutoff. This important feature of Hawking radiation is relevant for a quantum gravity theory that preserves, in some way, the Lorentz symmetry.

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general, as \[ \mathbf{3, 12} \]

\[ \langle N_i \rangle = \hbar^{-1} \int d\Sigma_1^\mu d\Sigma_2^{\nu} [u_i^\alpha(x_1) \bar{\partial}_\mu] [u_i^\alpha(x_2) \bar{\partial}_\nu] G(x_1, x_2). \] (6)

After some algebra, one arrives at the expression

\[ \langle N_w \rangle = -\frac{1}{4\pi^2} \int_0^u dU_1 dU_2 \frac{e^{-i\omega(u(U_1) - u(U_2))}}{(U_1 - U_2 - i\epsilon)^2}. \] (7)

where \( U \) is the null Kruskal coordinate \( U = -\kappa^{-1} e^{-\kappa u} \) and \( u = t - r^* \) the corresponding retarded time of a Schwarzschild black hole. The double integral above is divergent, but this divergence is expected due to the infinite number of quanta emitted in the infinite amount of time involved in the formula. Restricting the computation to the mean particle number per unit time one gets the finite thermal result

\[ \langle \dot{N}_w \rangle = -\frac{1}{4\pi^2} \frac{d}{du} \int_0^\infty dU_1 dU_2 \frac{e^{-i\omega(u(U_1) - u(U_2))}}{(U_1 - U_2 - i\epsilon)^2} = \frac{1}{2\pi} e^{\pi\kappa^{-1}u} - 1. \] (8)

Again, the disturbing point in the above derivation is that a cutoff in distances requiring that

\[ (U_1 - U_2)^2 > \ell_P^2, \] (9)

turns the otherwise steady Hawking radiation into a transient phenomenon. One notices immediately that the common point in the cutoff \[ \mathbf{8} \] and that of \[ \mathbf{4} \] is that both are not Lorentz-invariant. Since we have put an upper limit, \( \omega' \sim 1/\ell_P \), on the early-time frequencies, the “in” modes remaining after this amputation are not sufficient to generate the radiated “out” modes at late times. This produces the described decay of Hawking radiation with time as a consequence of breaking the principle of relativity by means of a non-invariant cutoff.

It is possible, however, to introduce a cutoff in an invariant way. On dimensional grounds, one can demand that the two-point function \( G(x_1, x_2) \) that appears in our integrals does not exceed the inverse of Newton’s constant

\[ |G(x_1, x_2)| < \hbar \ell_P^2 \equiv G_N^{-1}. \] (10)

It is not difficult to show, as we will see, that this condition translates into a restriction in the integration range of the \( U_1, U_2 \) coordinates in \[ \mathbf{8} \] given by

\[ (U_1 - U_2)^2 > \ell_P^2 \hbar^2 U_1 U_2 / 4\pi^2. \] (11)

The factor \( \kappa^2 U_1 U_2 \) on the right-hand side of \[ \mathbf{11} \] is absent in Eq. \[ \mathbf{8} \]. This factor is required to have an invariant cutoff for all locally inertial observers and immediately ensures the robustness of Hawking radiation.

An understanding of how \[ \mathbf{11} \] follows from \[ \mathbf{10} \] can be obtained in a simple way by considering the Unruh effect \[ \mathbf{13} \]. A detector held at constant \( r \) just outside the horizon behaves like a uniformly accelerated detector in Minkowski space (equivalence principle). The thermal radiation detected by the accelerated observer can be related to the Hawking emission. The detector will have some internal energy states \( |E \rangle \) and it can interact with the field by absorbing or emitting quanta. The interaction can be modeled in the standard way by coupling the field \( \phi(x) \) along the detector trajectory \( x = x(\tau) \) \( (\tau \) is the detector proper time) to some operator \( m(\tau) \) acting on the internal detector eigenstates

\[ g \int d\tau \, m(\tau) \Phi(x(\tau)) \] (12)

where \( g \) is the strength of the coupling. The probability for the detector to make the transition from \( |E_i \rangle \) to \( |E_f \rangle \) is given by the expression \( P(E_i \rightarrow E_f) = \langle f | E_f | 0 \rangle^2 \langle 0 | | E_i \rangle P(\Delta E) \], where \( P(\Delta E) \) is the so-called response function

\[ F(\Delta E) = \int_{-\infty}^{+\infty} d\tau_1 d\tau_2 e^{-i\Delta E \Delta \tau / \hbar} \langle 0_M | \Phi(x(\tau_1)) \Phi(x(\tau_2)) | 0_M \rangle, \] (13)

where \( \Delta \tau = \tau_1 - \tau_2 \). For a massless field the Wightman two-point function in \[ \mathbf{13} \], where \( |0_M \rangle \) is the Minkowski vacuum, is given by

\[ \langle 0_M | \Phi(x_1) \Phi(x_2) | 0_M \rangle = \frac{\hbar}{4\pi^2 \sqrt{(\Delta t - i\epsilon)^2 - (\Delta x^2)^2}}. \] (14)

For trajectories having a proper-time translational symmetry under \( \tau \rightarrow \tau + \tau_0 \), it is natural to consider the constant transition probability per unit proper time and the corresponding response rate per unit proper time

\[ \hat{F}(\Delta E) = \int_{-\infty}^{+\infty} d\Delta \tau e^{-i\Delta E \Delta \tau / \hbar} \langle 0_M | \Phi(x(\tau_1)) \Phi(x(\tau_2)) | 0_M \rangle. \] (15)

Both the inertial detector and the uniformly accelerated detector possess proper-time translational symmetry. For an inertial detector trajectory, the response rate is given by

\[ \hat{F}(\Delta E) = - \int_{-\infty}^{+\infty} d\Delta \tau e^{-i\Delta E \Delta \tau / \hbar} \bigg[ \frac{\hbar}{4\pi^2 (\Delta \tau - i\epsilon)} \bigg]^2 \] (16)

in agreement with the principle of relativity. If the detector’s initial state is the ground state \( |E_i = E_0 \rangle \), then \( \Delta E > 0 \) and the probability for an inertial detector to be excited is exactly zero, irrespective of the velocity of the detector. (When \( \Delta E < 0 \) the result is non-vanishing and this leads to the expected non-zero probability for the spontaneous decay \( E_0 \rightarrow E_1 < E_0 \).)

For a uniformly accelerated trajectory in Minkowski spacetime

\[ t = \frac{1}{a} \sinh a \tau, \quad x = \frac{1}{a} \cosh a \tau, \] (17)
where $a$ is the acceleration, the response function is then
\[ F(\Delta E) = \int_{-\infty}^{+\infty} dt_1 dt_2 e^{-i \Delta E \Delta \tau} \frac{-\hbar (a/2)^2}{4\pi^2 \sinh^2 \left( \frac{\Delta \tau}{2} \right)} . \]
(18)

The corresponding response rate function turns out to be
\[ \dot{F}(\Delta E) = (\Delta E/2\pi)(e^{i \pi \Delta E/\hbar a} - 1)^{-1}, \]
which implies, via the detailed balance relation, \( \dot{P}(\Delta E) = \dot{P}(-\Delta E)e^{-2\pi \Delta E/\hbar a} \), that a uniformly accelerated observer in Minkowski space feels himself immersed in a thermal bath at the temperature $k_B T = \frac{\hbar a}{2\pi}$. 

Performing the change of variable
\[ U \equiv t - x = -a^{-1} e^{-a \tau}, \]
one can rewrite the integral (18) in the form
\[ F(\Delta E) = -\int_{-\infty}^{0} dU_1 dU_2 e^{-i \Delta E \Delta \tau/\hbar} \frac{\hbar}{4\pi^2 (U_1 - U_2 - i\epsilon)^2}. \]
(20)

The time derivative of this expression is exactly the same (up to the factor $1/h\pi$) as (13) obtained before in computing the expectation value of the number operator in the Hawking effect (identifying the acceleration $a$ with the surface gravity $\kappa$ and the coordinate $U$ with the corresponding Kruskal coordinate). It is now easy to see that the invariant cutoff condition
\[ \left| \frac{\hbar}{4\pi^2 (\Delta t)^2 - (\Delta \tau)^2} \right| < G_{\mathcal{N}}^{-1} \]
(21)
on the accelerated trajectory (17) becomes
\[ \frac{\hbar (a/2)^2}{4\pi^2 \sinh^2 \frac{a}{2} \Delta \tau} < G_{\mathcal{N}}^{-1}. \]
(22)

Expanding the denominator of (22) to lowest order in $\Delta \tau$ and using (19) to express $(\Delta \tau)^2$ in terms of $(\Delta U)^2 \equiv (U_1 - U_2)^2$, it is straightforward to show that this inequality is equivalent to (11). This confirms our statement that (10) implies (11).

The natural question now is to see if the invariant cutoff suffices to preserve the bulk of the Hawking effect. The answer is in the affirmative, but to see this requires an additional step [12]. Let us use again the Unruh effect to illustrate the argument. We want to take advantage of the fact that there is a state of the field, $|0_A\rangle$, for which the response function of the accelerated detector vanishes for $\Delta E > 0$
\[ F_A(\Delta E > 0) = \int_{-\infty}^{+\infty} dt_1 dt_2 e^{-i \Delta E \Delta \tau} \times \]
\[ \langle 0_A| \Phi(x(t_1)) \Phi(x(t_2)) |0_A\rangle = 0. \]
(23)

Taking this into account, it is possible to obtain an equivalent expression for the response function of the uniformly accelerating detector in the Minkowski vacuum, $|0_M\rangle$, by subtracting the previous quantity from the right-hand-side of equation (13)
\[ F(\Delta E > 0) = \int_{-\infty}^{+\infty} dt_1 dt_2 e^{-i \Delta E \Delta \tau} \times \]
\[ [(0_M| \Phi(x(t_1)) \Phi(x(t_2)) |0_M\rangle - (0_A| \Phi(x(t_1)) \Phi(x(t_2)) |0_A\rangle]. \]
(24)

This expression presents several advantages over (13). It explicitly shows that the difference between two-point correlation functions of the field in the vacuum states $|0_M\rangle$ and $|0_A\rangle$ is at the root of a non-vanishing response function. (Notice that although the integral of $\langle 0_A| \Phi(x(t_1)) \Phi(x(t_2)) |0_A\rangle$ in the response function is zero, the correlation function itself is not zero.) Moreover, the integrand is now a smooth and symmetric function, thanks to the universal short-distance behavior of the two-point functions. Thus, the usual "$i\epsilon$-prescription" in the two-point functions is now redundant and can be omitted. Additionally, expression (24) shows a remarkable fact when an invariant cut-off is considered. It manifestly produces a vanishing result in the limit $a \to 0$, respecting in that way the principle of relativity that we want to preserve.

Now, one can consistently implement the invariant and universal cutoff condition
\[ \langle |0_M| \Phi(x(t_1)) \Phi(x(t_2)) |0_M\rangle < G_{\mathcal{N}}^{-1}, \]
(25)
and
\[ \langle |0_A| \Phi(x(t_1)) \Phi(x(t_2)) |0_A\rangle < G_{\mathcal{N}}^{-1}. \]
(26)
in (24). The first inequality is equivalent to (22), and the second one to $\Delta \tau^2 > \ell_p^2/4\pi^2$. Moreover, both inequalities are essentially equivalent since all quantum states (in particular $|0_M\rangle$ and $|0_A\rangle$) have the same short distance behavior, as is seen explicitly from the short distance asymptotic form of (22).

In the black hole case, the same argument can be applied for the computation of the mean particle number [8, 12], and $G(x_1, x_2)$ in equation (6) can be substituted by
\[ G(x_1, x_2) = \langle \text{out} | \Phi(x_1) \Phi(x_2) | \text{out} \rangle, \]
(27)
where $|\text{out}\rangle$ is, as usual, the vacuum state defined by the modes $u_j^{\text{out}}(x)$. This leads to an expression for the mean particle number per unit time
\[ \langle \dot{N}_w \rangle = -\frac{1}{4\pi^2 w} \frac{d}{du} \left[ \int_{-\infty}^{0} dU_1 dU_2 \frac{e^{-i w (U_1 - U_2)}}{(U_1 - U_2)^2} \right] - \int_{-\infty}^{+\infty} du_1 dU_2 \frac{e^{-i w (u_1 - u_2)}}{(u_1 - u_2)^2}, \]
(28)
where now we want to restrict the range of integration, so $(U_1 - U_2)^2 > \ell_p^2/4\pi^2$ and $(u_1 - u_2)^2 > \ell_p^2/4\pi^2$. 
The explicit evaluation of these integrals, with the corresponding bounds for \((U_1 - U_2)^2\) and \((u_1 - u_2)^2\), leads to

\[
\langle \dot{N}_w \rangle \approx \frac{1}{2\pi} \frac{1}{e^{2\pi \kappa^{-1}w} - 1} - \frac{\kappa \ell_p}{96\pi^3(w/\kappa)} + O(\kappa \ell_p^3). \tag{29}
\]

For black hole radii much bigger than the planck length \((\kappa \ll \ell_p^{-1})\) and for reasonable values of the frequency, the correction terms are negligible, which shows the irrelevance of ultra-high energy physics in the derivation of the Hawking effect.

In summary, we have shown that a universal invariant cutoff condition for two-point functions is able to preserve the bulk of the thermal Hawking radiation.

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